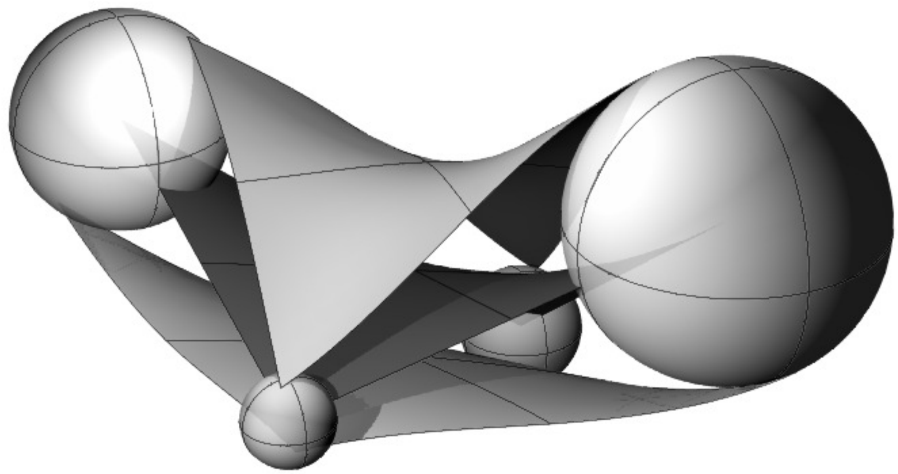
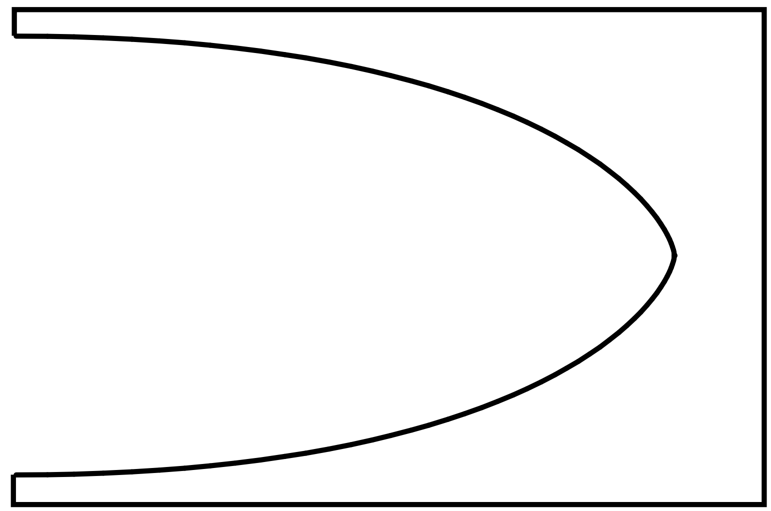
Medial Object and Rational Offsets



Jiří Kosinka

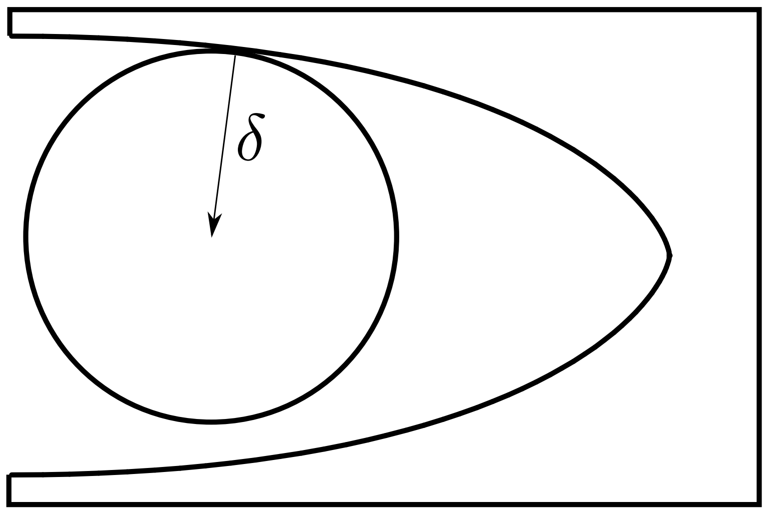
Medial Object Workshop
Cambridge
October 10, 2014

Motivation



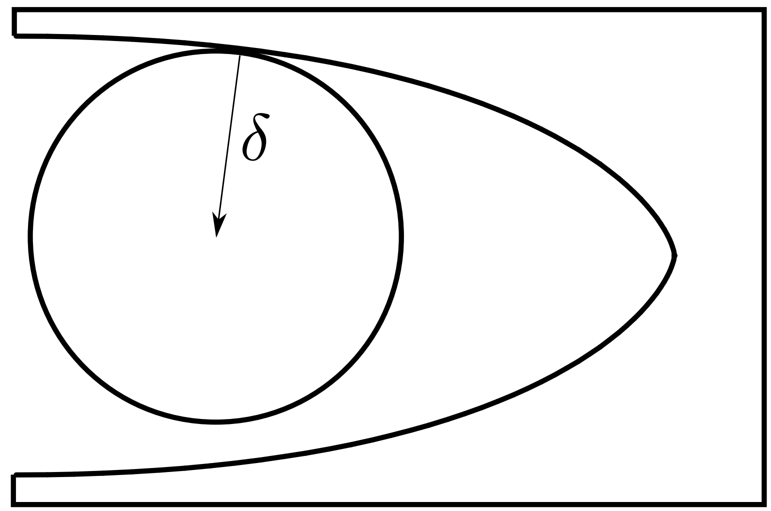
- NURBS curve $\mathbf{c}(t) = (x(t), y(t))^\top$

Motivation



- NURBS curve $\mathbf{c}(t) = (x(t), y(t))^\top$, tool radius δ

Motivation



- NURBS curve $\mathbf{c}(t) = (x(t), y(t))^\top$, tool radius δ
- offset $\mathbf{o}_\delta(t) = \mathbf{c}(t) + \delta\mathbf{n}(t)$

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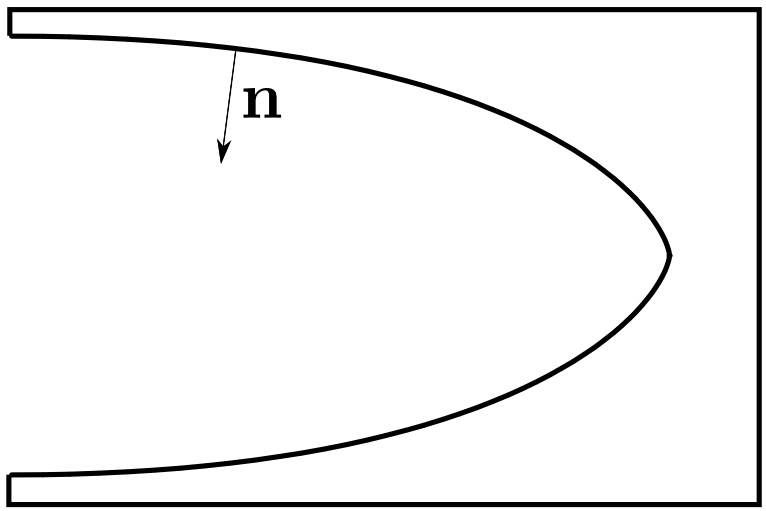

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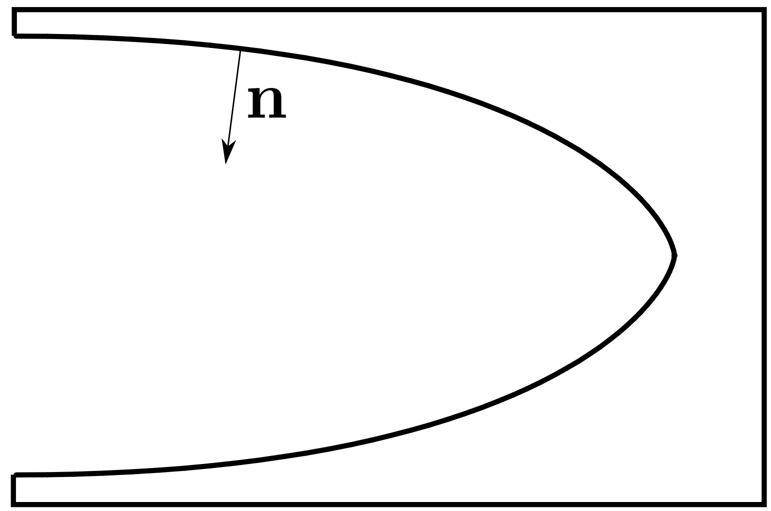
Motivation



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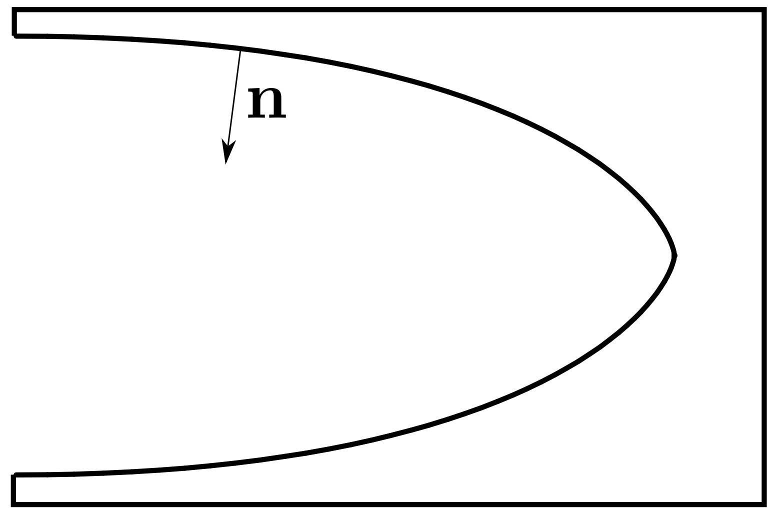
Motivation



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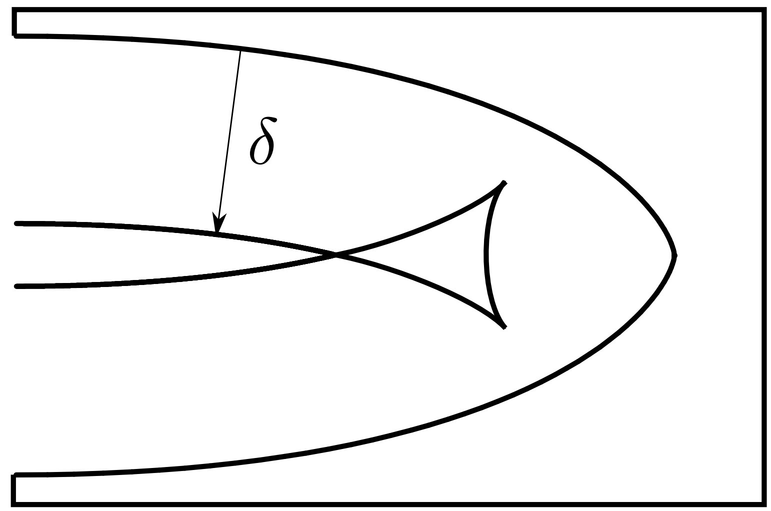

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- approximated offset $\hat{\mathbf{o}}_\delta(t)$

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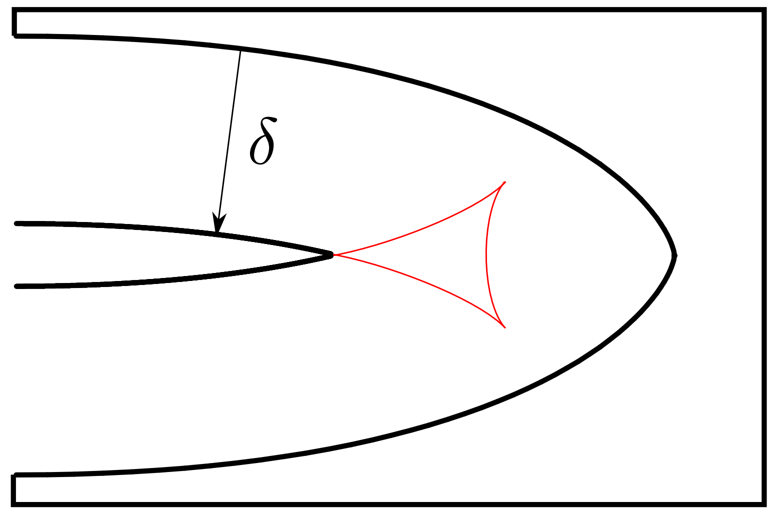

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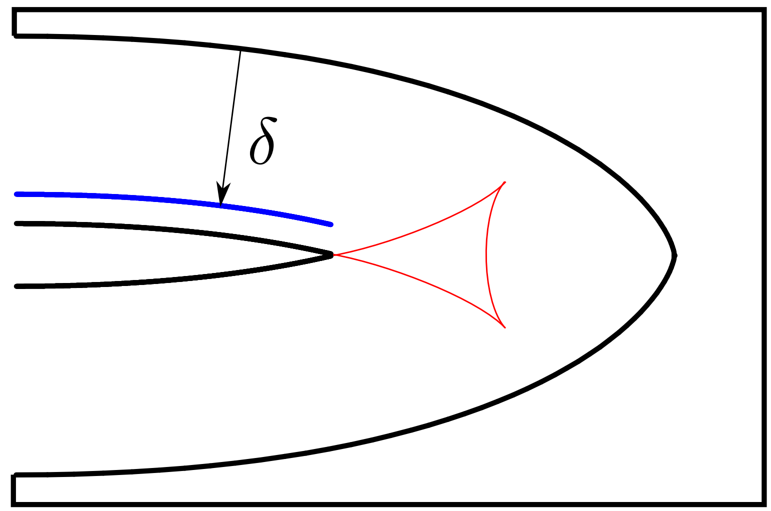
- trimming

Motivation



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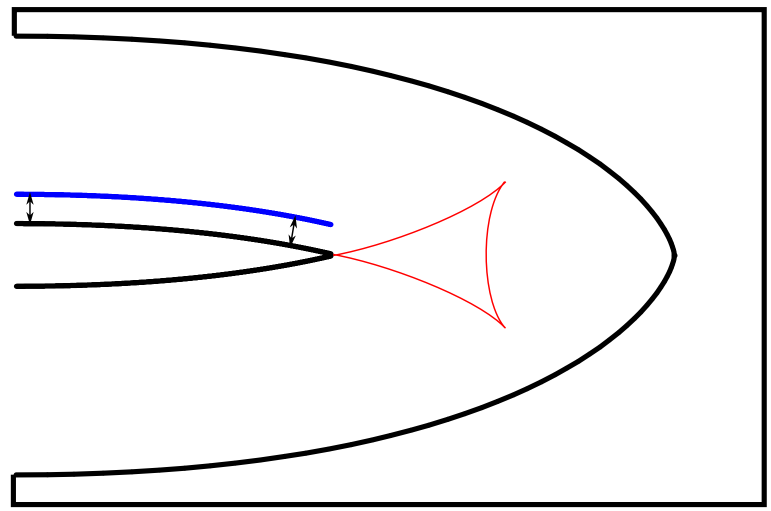

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Polynomial Curves

- The arc-length function of a planar curve $\mathbf{c}(t) = (x(t), y(t))^\top$:

$$s(t) = \int \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t$$

- The δ-offsets of a planar curve:

$$\mathbf{o}_\delta(t) = \left(x(t) \pm \delta \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, y(t) \mp \delta \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right)^\top$$

- The unit tangent vector of a spatial curve $\mathbf{c}(t) = (x(t), y(t), z(t))^\top$:

$$\mathbf{T} = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}} (x'(t), y'(t), z'(t))^\top$$

Pythagorean Hodograph Curves

- A polynomial curve is said to be a Pythagorean hodograph (PH) curve, if the norm of its first derivative (or hodograph) is (piecewise) polynomial

  - **(Euclidean) planar PH curve** – if there exists a polynomial $\sigma(t)$ such that $x'(t)^2 + y'(t)^2 = \sigma^2(t)$ Farouki, Sakkalis (1990)

  - **(Euclidean) spatial PH curve** – if there exists a polynomial $\sigma(t)$ such that $x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma^2(t)$ Farouki, Sakkalis (1994)

Polynomial Curves

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PH Curves

- The arc-length function of a planar curve $\mathbf{c}(t) = (x(t), y(t))^\top$:

$$s(t) = \int |\sigma| \, \mathrm{d}t$$

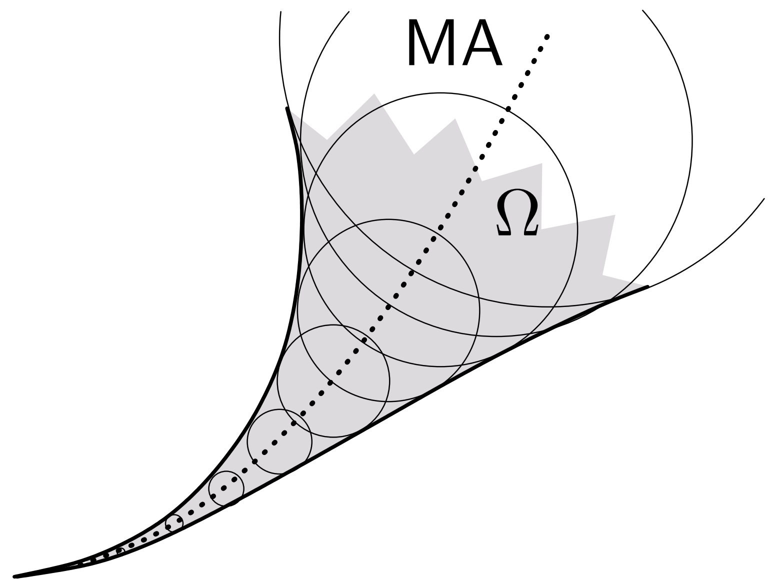
- The δ-offsets of a planar curve:

$$\mathbf{o}_\delta(t) = \left(x(t) \pm \delta \frac{y'(t)}{|\sigma|}, y(t) \mp \delta \frac{x'(t)}{|\sigma|} \right)^\top$$

- The unit tangent vector of a spatial curve $\mathbf{c}(t) = (x(t), y(t), z(t))^\top$:

$$\mathbf{T} = \frac{1}{|\sigma|} (x'(t), y'(t), z'(t))^\top$$

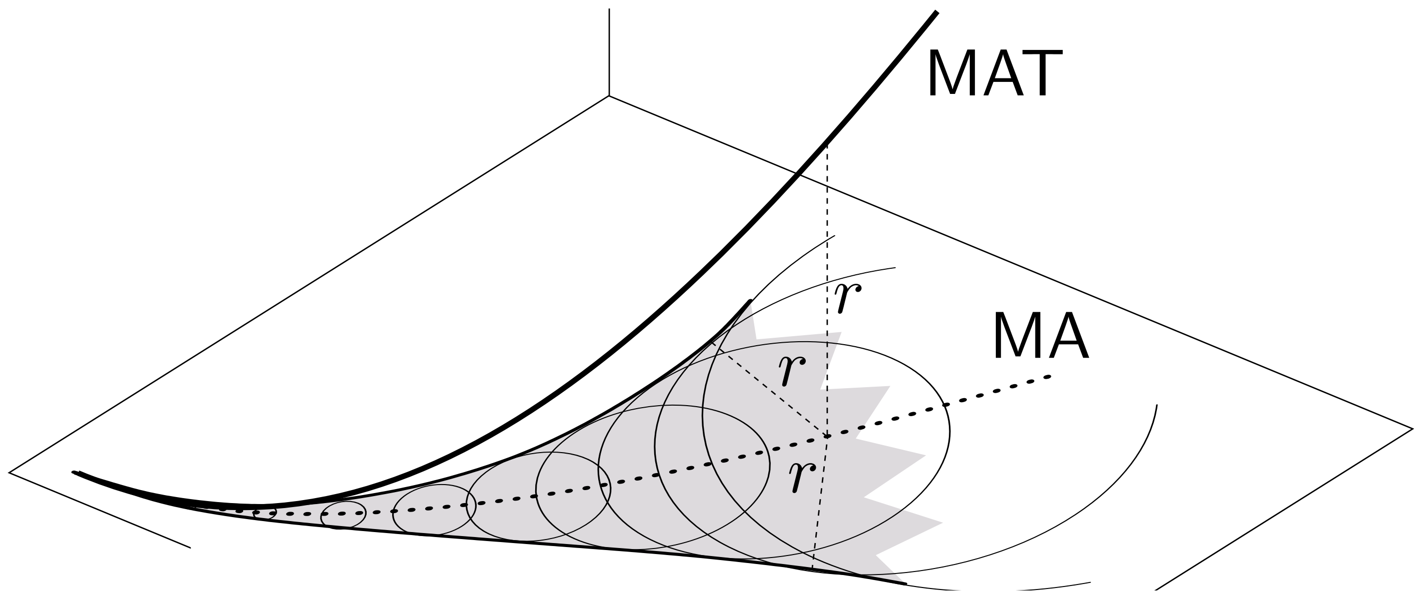
Medial Axis Transform



- Find all maximal inscribed discs $D_r(x, y)$ touching $\partial\Omega$ at at least two points

- The medial axis (MA) of Ω is the locus of all centres of $D_{r(t)}(x(t), y(t))$, i.e., $\mathbf{f}(t) = (x(t), y(t))^\top$

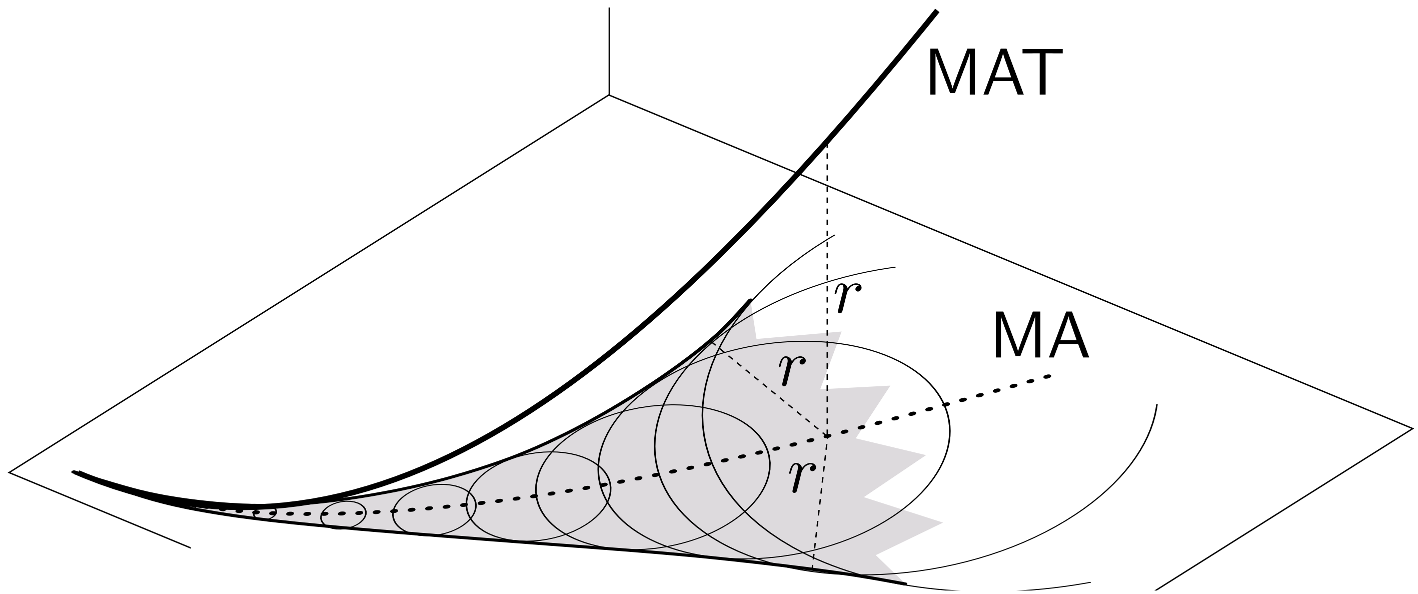
Medial Axis Transform



- Find all maximal inscribed discs $D_r(x,y)$ touching $\partial\Omega$ at at least two points
- The medial axis (MA) of Ω is the locus of all centres of $D_{r(t)}(x(t),y(t))$, i.e., $\mathbf{f}(t) = (x(t),y(t))^\top$
- The medial axis transform (MAT) of Ω is the curve

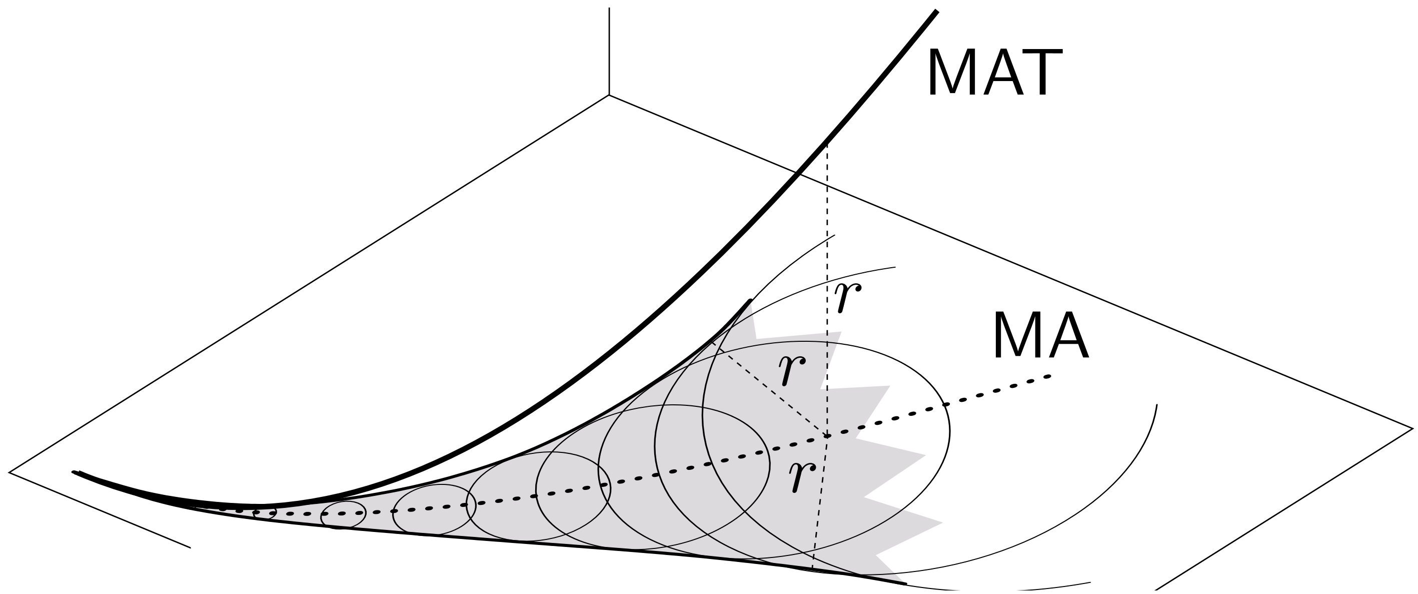
  $\mathbf{g}(t) = (x(t),y(t),r(t))^\top$

Medial Axis Transform



- Conversely, given a segment $\mathbf{g}(t) = (x(t), y(t), r(t))^\top$ of MAT
  $\Rightarrow \Omega = \bigcup_{t \in I} D_{r(t)}(x(t), y(t))$

Medial Axis Transform



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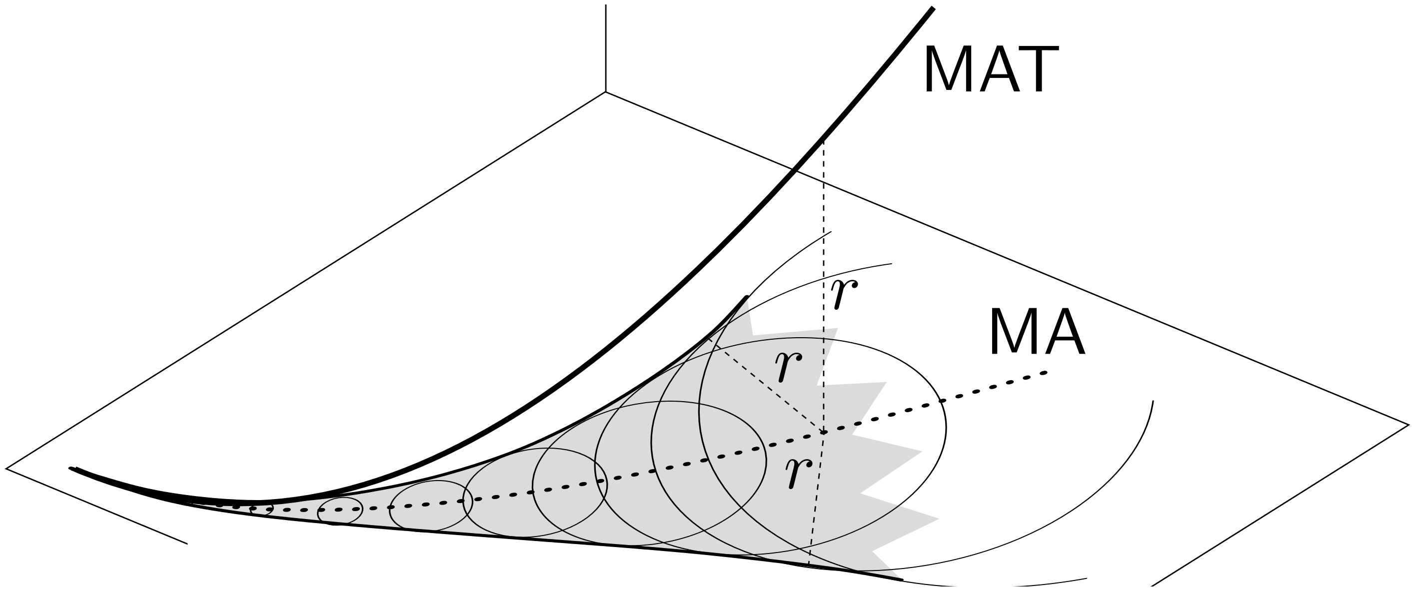
- The boundary $\partial\Omega$ can be obtained from the envelope of the MAT discs

$$\mathbf{b}^{(\pm)}(t) = \begin{pmatrix} x \\ y \end{pmatrix} - \tfrac{r}{x'^2+y'^2} \left[r' \begin{pmatrix} x' \\ y' \end{pmatrix} \pm \sqrt{x'^2 + y'^2 - r'^2} \begin{pmatrix} -y' \\ x' \end{pmatrix} \right]$$

Medial Axis Transform



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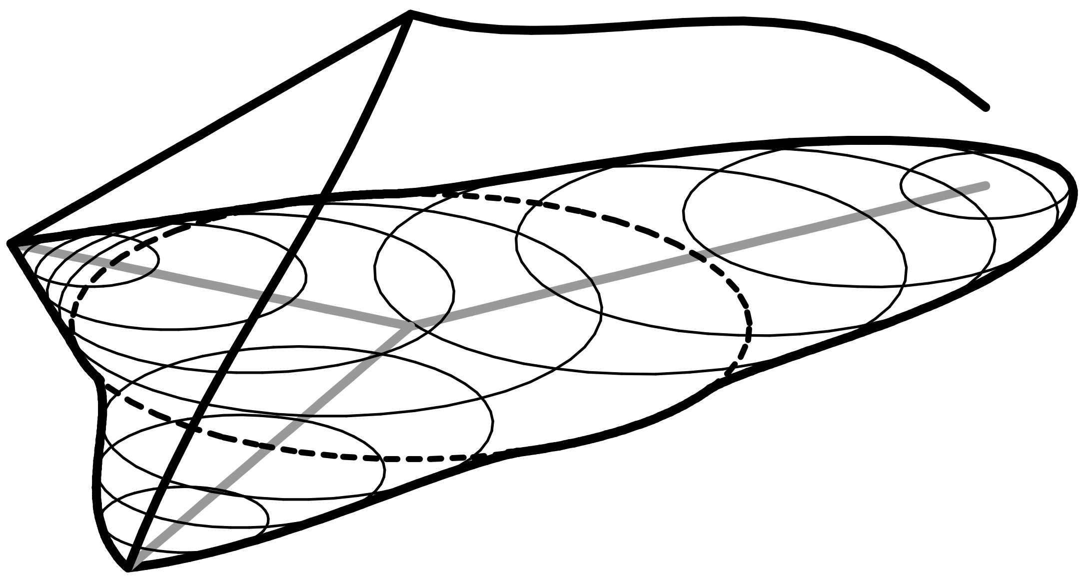
'PH' condition: $x'^2 + y'^2 - r'^2 = \sigma^2$

Medial Axis Transform

- An example of a 'branched' planar domain



- Domain decomposition lemma (Choi et al., 1997)

Pythagorean Hodograph Curves

- A polynomial curve $\mathbf{c} = (x, y, z)^\top \in \mathbb{R}^3$ is called a PH curve if there exists a polynomial σ such that ${x'}^2 + {y'}^2 + {z'}^2 = \sigma^2$

Farouki, Sakkalis (1994)

Minkowski Pythagorean Hodograph Curves

- A polynomial curve $\mathbf{c} = (x, y, r)^\top \in \mathbb{R}^{2,1}$ is called an MPH curve if there exists a polynomial σ such that $x'^2 + y'^2 - r'^2 = \sigma^2$

Moon (1999)

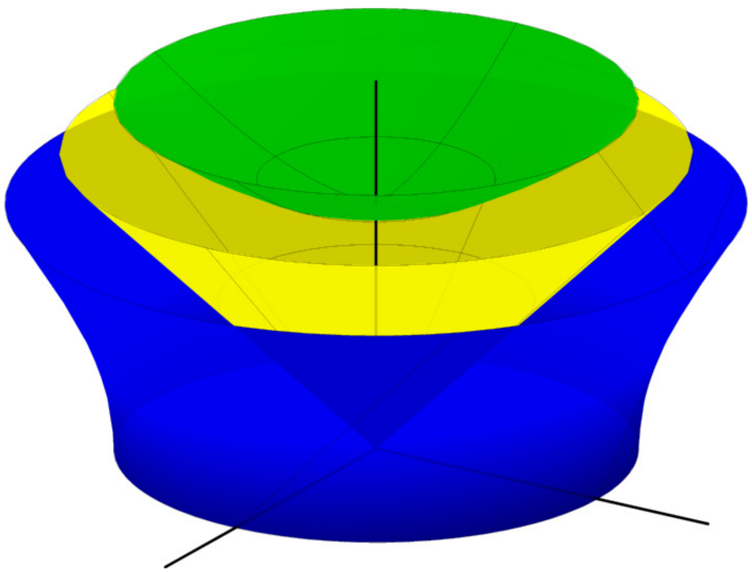
Minkowski Space

- $\mathbb{R}^{2,1}$ – affine space, inner product given by $G = \text{diag}(1, 1, -1)$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top G \mathbf{v} = u_1 v_1 + u_2 v_2 - u_3 v_3$$

- $||\mathbf{u}||^2 = \langle \mathbf{u}, \mathbf{u} \rangle \Rightarrow$ *'causal character'*
  - $||\mathbf{u}||^2 > 0 \Rightarrow$ *space-like*
  - $||\mathbf{u}||^2 = 0 \Rightarrow$ *light-like* (or *isotropic*)
  - $||\mathbf{u}||^2 < 0 \Rightarrow$ *time-like*

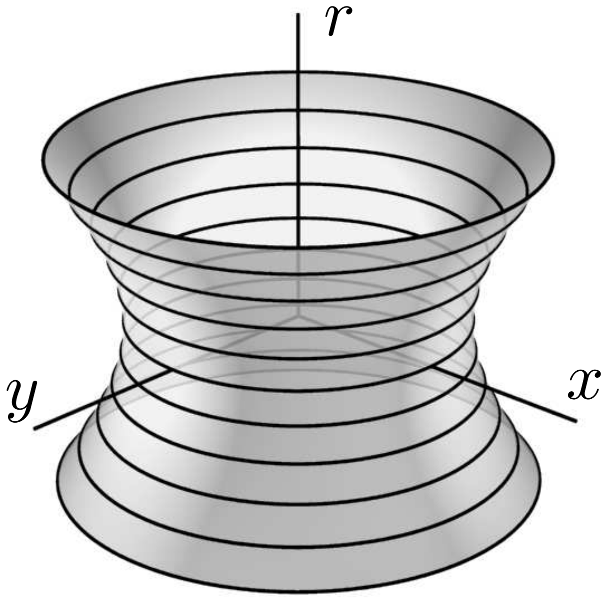
  - $\mathbf{u}$ is a unit vector iff $||\mathbf{u}||^2 = \pm 1$
  - $\forall\, \mathbf{u}$ s.t. $||\mathbf{u}|| = 0 \Rightarrow$ *light cone*

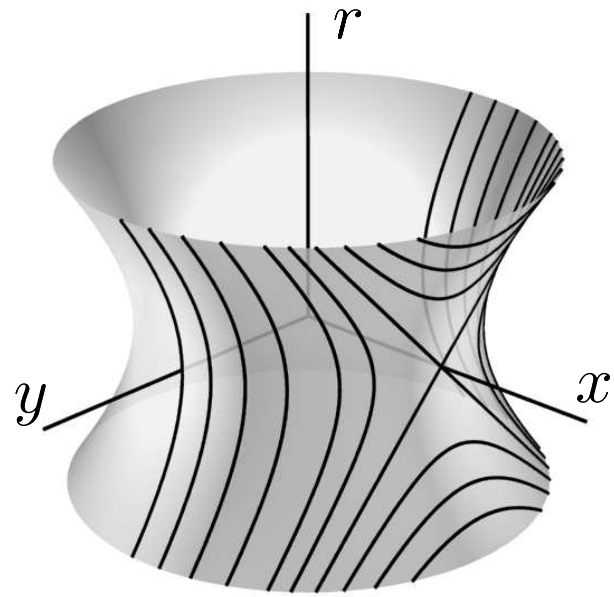


Lorentz Transforms

- $L \in SO_+(2,1) \Rightarrow L = R(\alpha_1)H(\beta)R(\alpha_2)$, where

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & \sinh\beta \\ 0 & \sinh\beta & \cosh\beta \end{pmatrix}$$





Minkowski Pythagorean Hodograph Curves

- A polynomial curve $\mathbf{c} = (x, y, r)^\top \in \mathbb{R}^{2,1}$ is called an MPH curve if there exists a polynomial σ such that $x'^2 + y'^2 - r'^2 = \sigma^2$

- If the MAT is an MPH curve, then the coordinate functions of the domain boundaries are (piecewise) rational

The envelope formula:

$$\mathbf{b}^{(\pm)}(t) = \begin{pmatrix} x \\ y \end{pmatrix} - \frac{r}{x'^2 + y'^2} \left[r' \begin{pmatrix} x' \\ y' \end{pmatrix} \pm \sqrt{x'^2 + y'^2 - r'^2} \begin{pmatrix} -y' \\ x' \end{pmatrix} \right]$$

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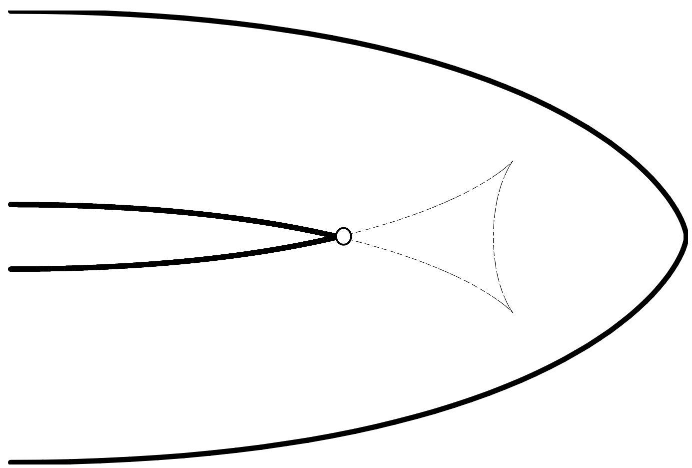
  The envelope formula for δ offsets:

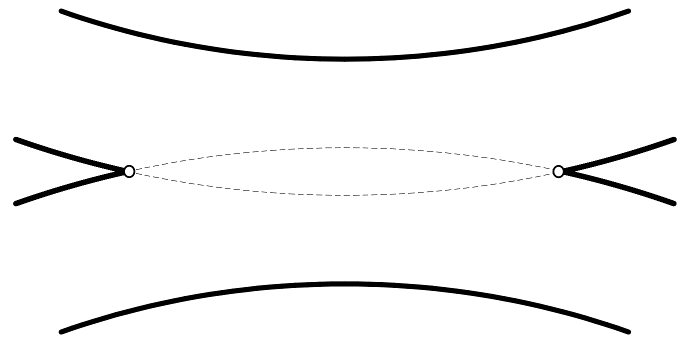
$$\mathbf{b}^{(\pm)}(t) = \begin{pmatrix} x \\ y \end{pmatrix} - \frac{r \pm \delta}{x'^2 + y'^2} \left[r' \begin{pmatrix} x' \\ y' \end{pmatrix} \pm |\sigma| \begin{pmatrix} -y' \\ x' \end{pmatrix} \right]$$

MPH

Self-intersections of Offsets

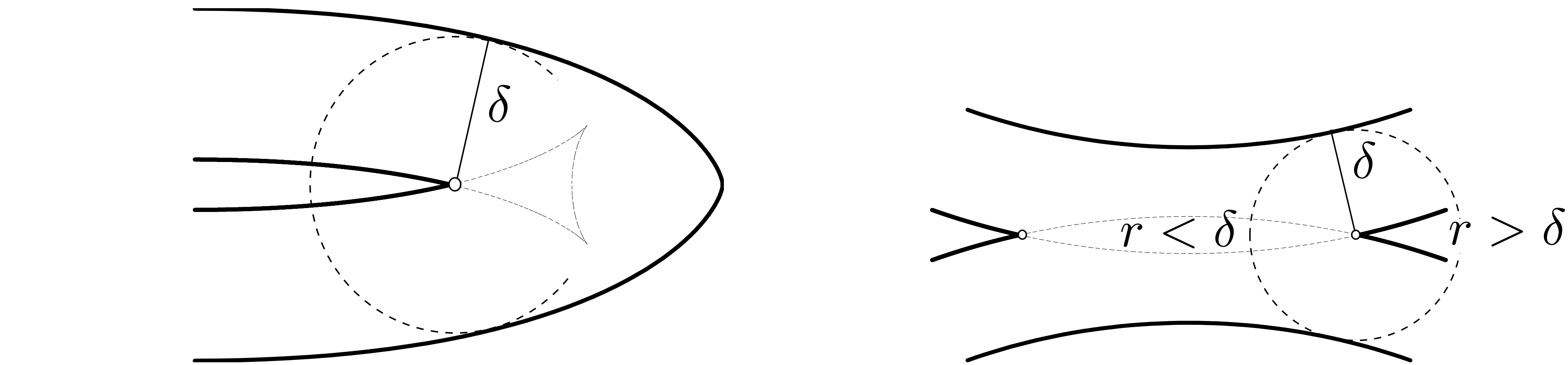
- Local and global self-intersections:





Offset Trimming with MAT

- Trimming procedure for inner offsets:



- The parts of the MAT where the corresponding circle radius r is less than δ (cutting tool radius) are trimmed

(Minkowski) Pythagorean Hodograph Curves

Planar PH: Kubota (1972)

$$\mathbf{c}(t) = (x(t), y(t))^\top \in \mathbb{R}^2$$

$$x'^2 + y'^2 = \sigma^2$$

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$$x'^2 + y'^2 = \sigma^2$$

$$\boxed{\begin{aligned} x' &= w(u^2 - v^2) \\ y' &= w(2uv) \\ \sigma &= w(u^2 + v^2) \end{aligned}}$$

(Minkowski) Pythagorean Hodograph Curves

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Planar MPH: Kubota (1972)

$$\mathbf{c}(t) = (x(t), y(t))^\top \in \mathbb{R}^{1,1}$$

$$x'^2 - y'^2 = \sigma^2$$

$$\boxed{\begin{aligned} x' &= w(u^2 + v^2) \\ y' &= w(2uv) \\ \sigma &= w(u^2 - v^2) \end{aligned}}$$

Spatial PH: Dietz et al. (1993)

$$\mathbf{c}(t) = (x(t), y(t), z(t))^\top \in \mathbb{R}^3$$

$$x'^2 + y'^2 + z'^2 = \sigma^2$$

$$\boxed{\begin{aligned} x' &= u^2 + v^2 - p^2 - q^2 \\ y' &= 2(uq + vp) \\ z' &= 2(vq - up) \\ \sigma &= u^2 + v^2 + p^2 + q^2 \end{aligned}}$$

Spatial MPH: Moon (1999)

$$\mathbf{c}(t) = (x(t), y(t), r(t))^\top \in \mathbb{R}^{2,1}$$

$$x'^2 + y'^2 - r'^2 = \sigma^2$$

$$\boxed{\begin{aligned} x' &= u^2 - v^2 - p^2 + q^2 \\ y' &= -2(uv + pq) \\ r' &= 2(uq + vp) \\ \sigma &= u^2 + v^2 - p^2 - q^2 \end{aligned}}$$

A Representation of Planar PH Curves

- Starting with any polynomial curve $(u, v)^\top$ we obtain a PH curve:

$$\begin{pmatrix} u \\ v \end{pmatrix} \quad \overset{\chi}{\longrightarrow} \quad \begin{pmatrix} u^2 - v^2 \\ 2uv \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \overset{\int}{\longrightarrow} \quad \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\textit{preimage} \qquad\qquad \textit{hodograph} \qquad\qquad \textit{PH curve}$$

$$\mathbf{z} \quad \overset{\chi}{\longrightarrow} \quad \mathbf{z}^2 \quad \overset{\int}{\longrightarrow} \quad \mathbf{c}$$

- Using complex representation: $\mathbf{z}(t) = u(t) + \mathbf{i}v(t)$

Local Hermite Interpolation in Space

- General free form curves are not (M)PH curves $\longrightarrow$ approximation techniques required
- Global approximation schemes yield highly non-linear systems
- Local interpolation is solvable using linear and quadratic equations

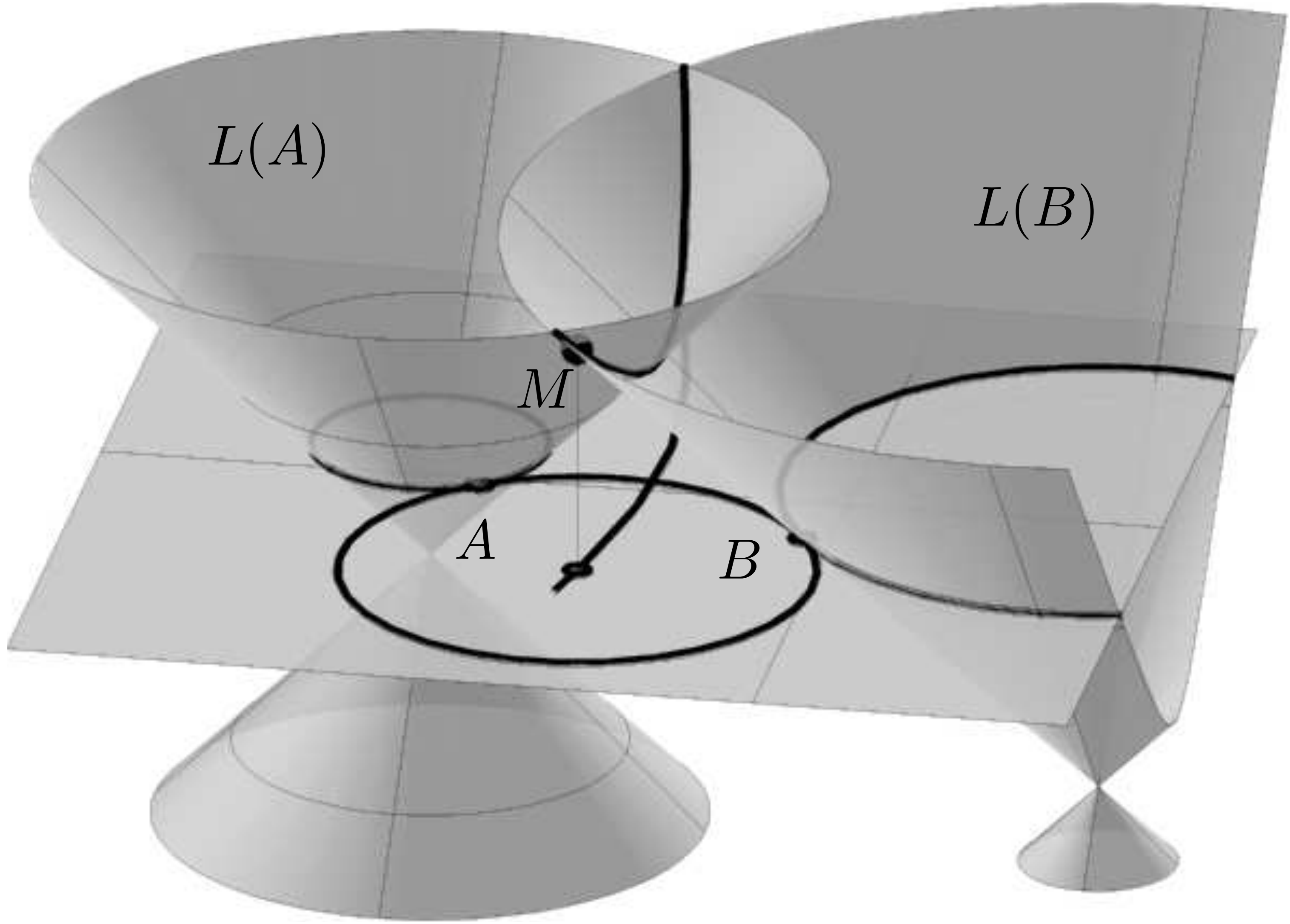
PH

- G^1 – Jüttler and Mäurer, 1999
- C^1 – Farouki et al., 2002
- C^1 – Šír and Jüttler, 2005
- C^2 – Šír and Jüttler, 2007

MPH

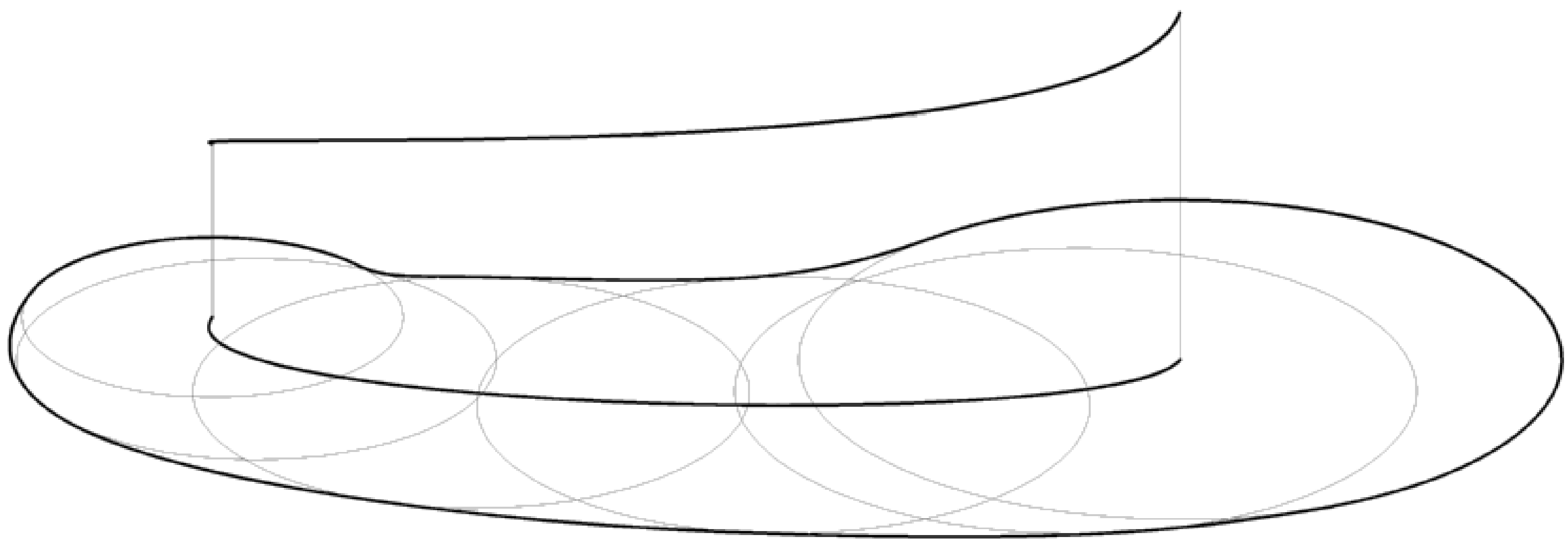
- G^1 – K. and Jüttler, 2006
- C^1 – K. and Jüttler, 2009
- $C^{1/2}$ – Kim and Ahn, 2003
- C^2 – K. and Šír, 2010

Obtaining Hermite Data



Conversion

- Converting a curve (MAT) into an MPH spline



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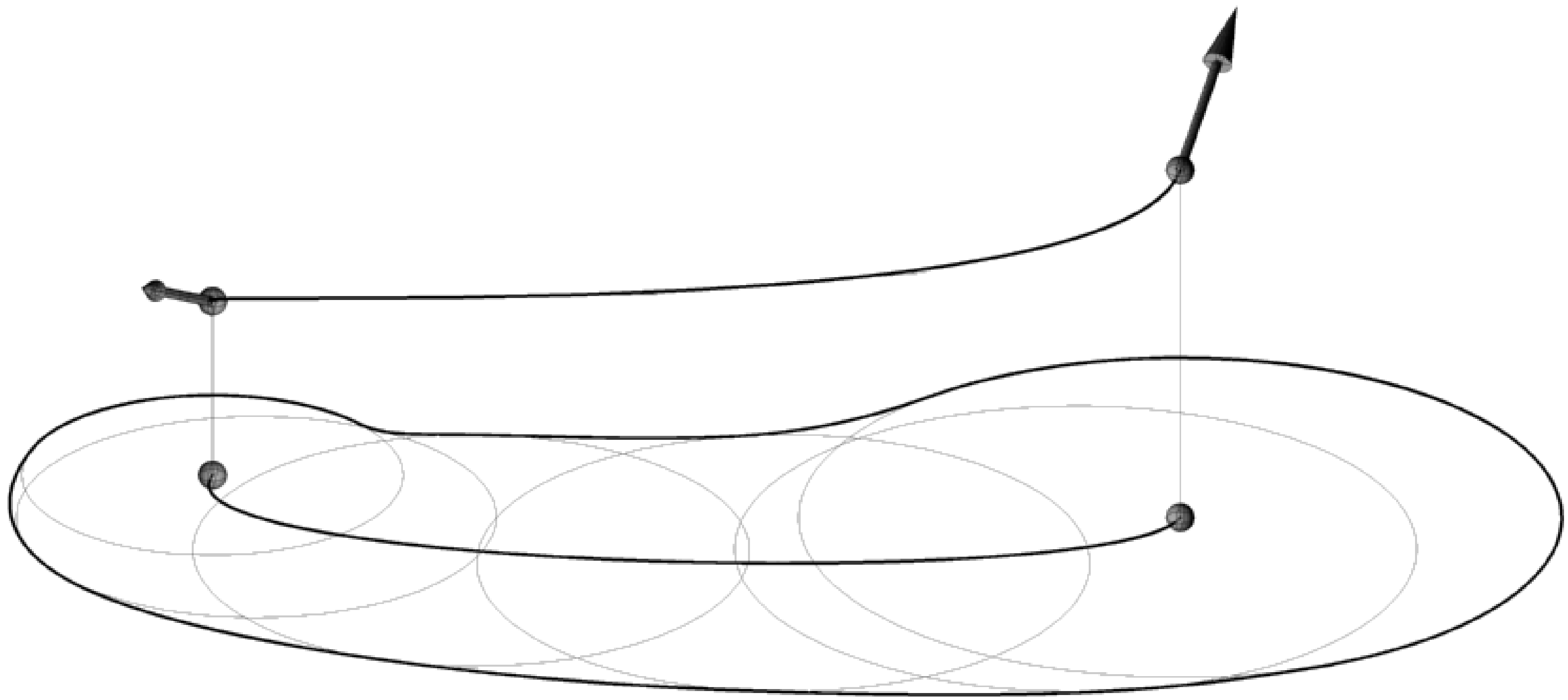

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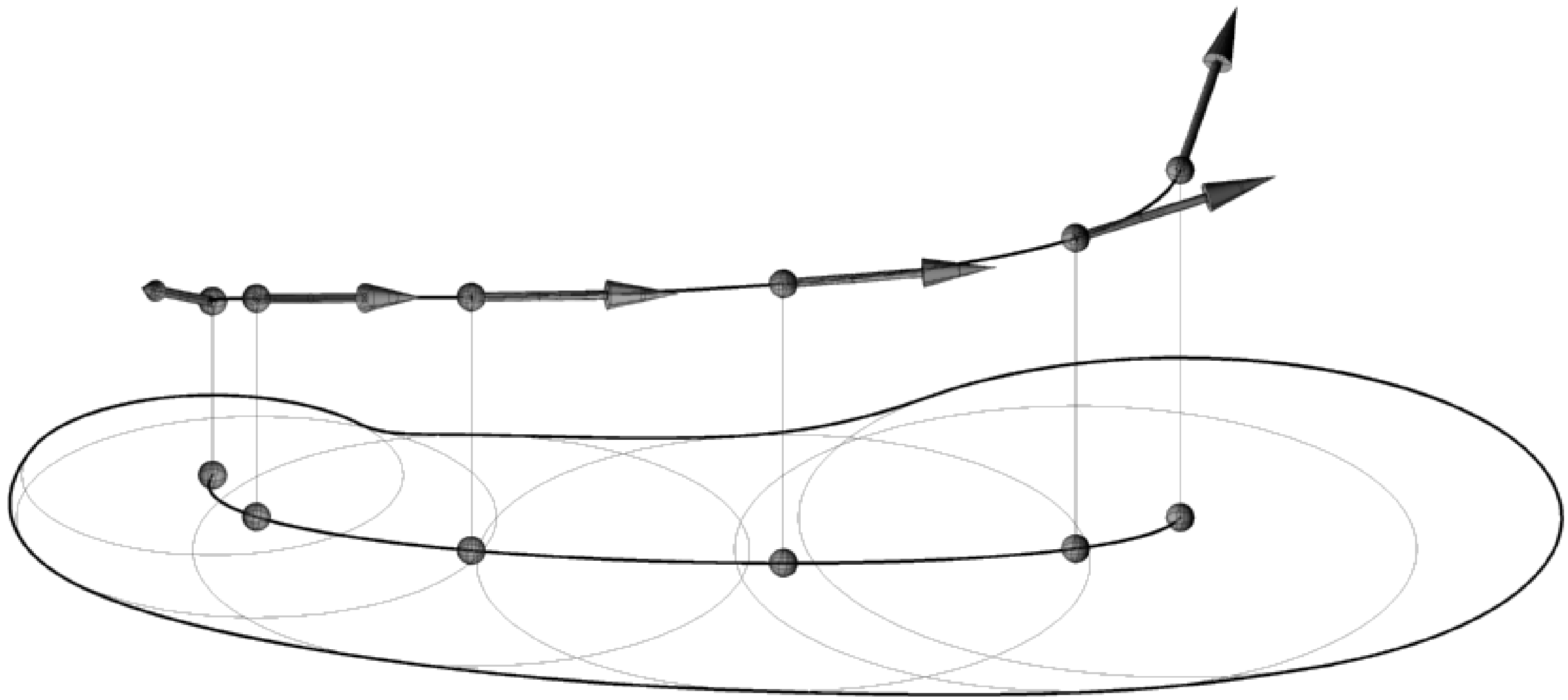

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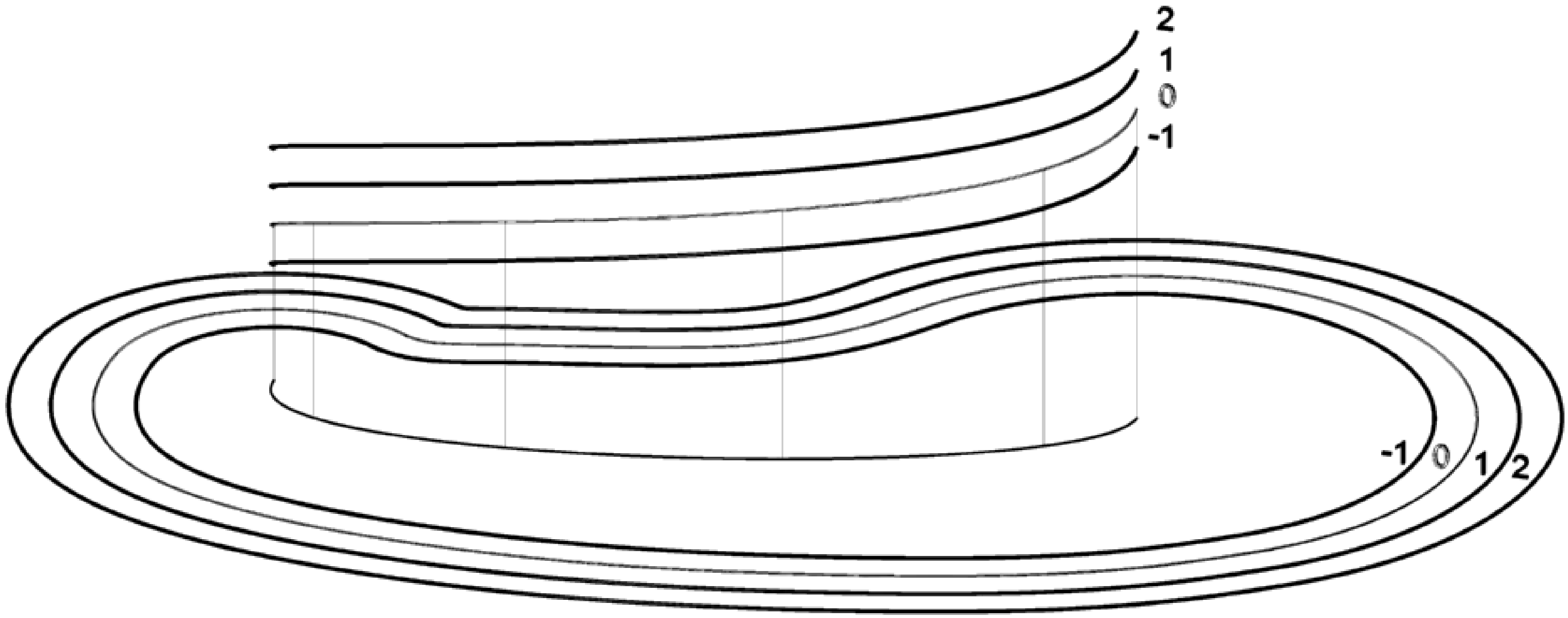
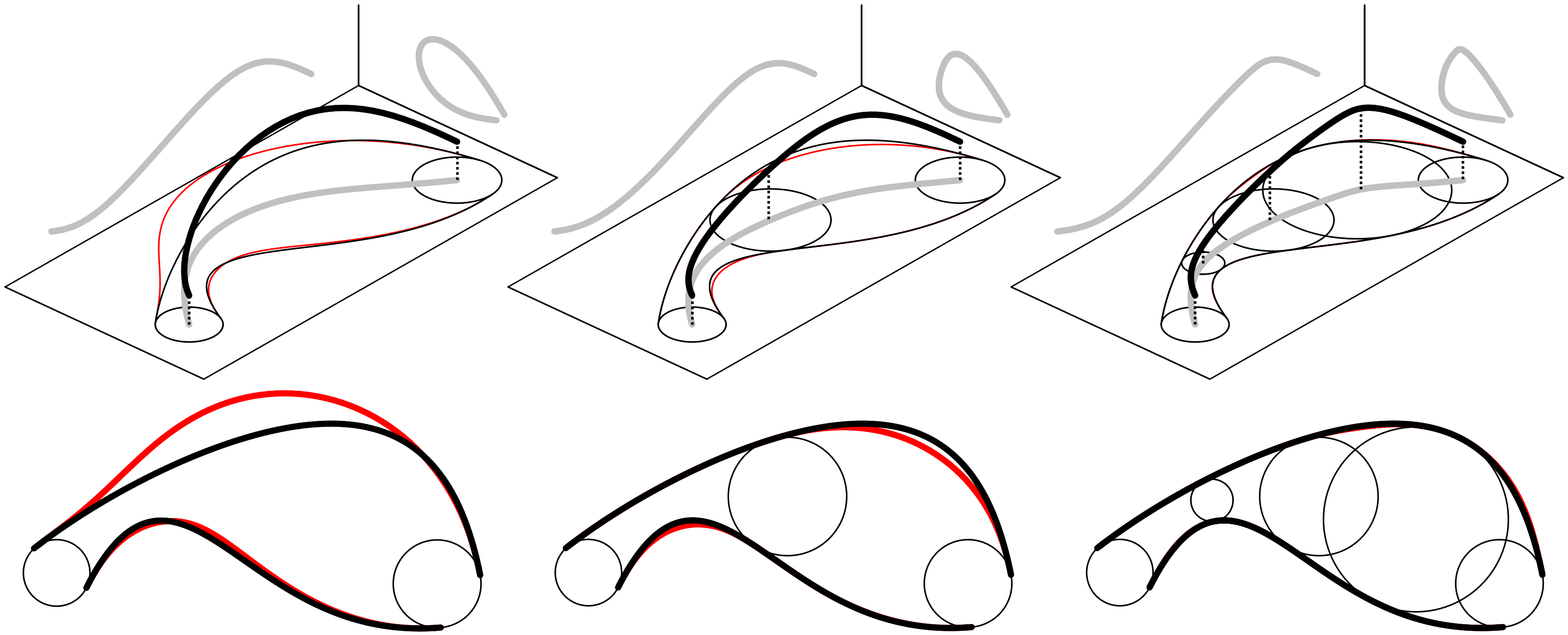


⇒ **rational** approximations of domain boundary and offsets

Example



- Three steps of subdivision applied to MAT approximation. Associated domain boundary approximations are depicted in red.

From Curves to Surfaces

PH curves

- Farouki and Sakkalis (1990)
- PH representation map
- C^k interpolation schemes
- K. and Lávička (2014)

$\longrightarrow$

PN surfaces

- Pottmann (1995)
- only dual representation
- only C^1 scheme known
- Bast, Jüttler, K., Lávička (2008)

From Curves to Surfaces

PH curves

- Farouki and Sakkalis (1990)
- PH representation map
- C^k interpolation schemes
- K. and Lávička (2014)

$\longrightarrow$

PN surfaces

- Pottmann (1995)
- only dual representation
- only C^1 scheme known
- Bast, Jüttler, K., Lávička (2008)

MPH curves

- Moon (1999)
- MPH representation map
- C^k interpolation schemes
- K. and Lávička (2014)

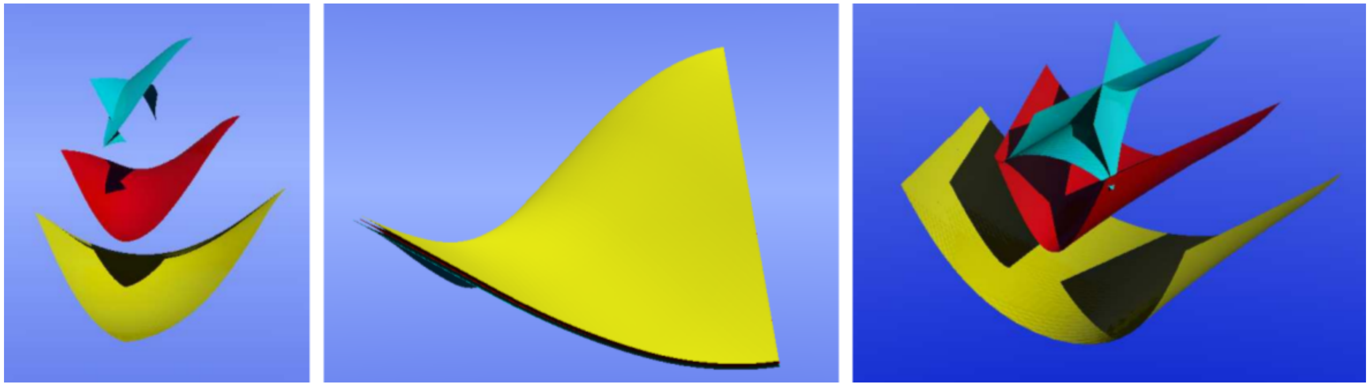
$\longrightarrow$

MOS surfaces

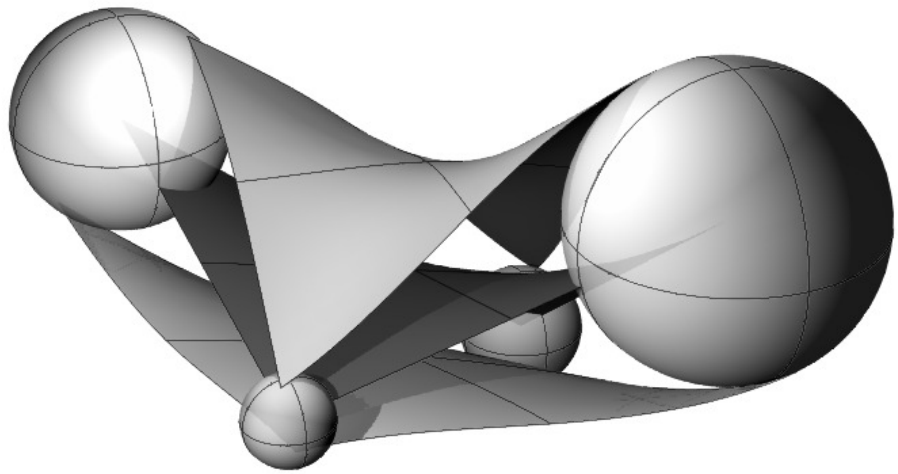
- K. and Jüttler (2007)
- only Plücker representation
- only C^1 scheme known
- Bast, Jüttler, K., Lávička (2010)

PN surfaces and Offsets

- PN surfaces: surfaces with Pythagorean normals (Pottmann, 1995)
- $\mathbf{s}(u,v)$: $||\mathbf{s}_u(u,v) \times \mathbf{s}_v(u,v)||^2 = EG - F^2 = \sigma(u,v)^2$
- PN surfaces possess rational offsets
- Quadratics are known to be PN $\rightarrow$ PN approximation algorithm (Bast, Jüttler, K., Lávička, 2008) based on C^1 quadratic splines



Medial Surface Transform

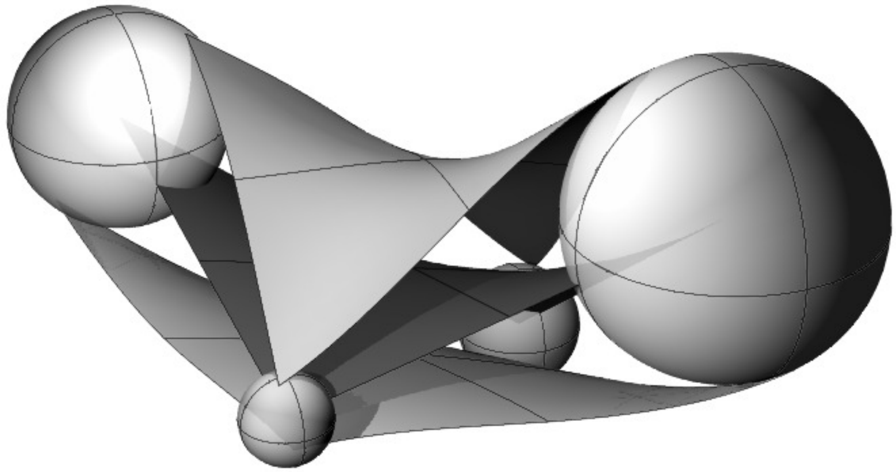


- Let $\mathbf{s}(u,v) = (x(u,v), y(u,v), z(u,v), r(u,v))^\top$, $(u,v) \in D \subseteq \mathbb{R}^2$ be a regular surface patch in $\mathbb{R}^{3,1}$ describing a sheet of the medial surface transform. Then

$$\Omega = \bigcup_{(u,v)\in D} B_{r(u,v)}(x(u,v), y(u,v), z(u,v)),$$

where $B_r(x,y,z)$ is the ball with centre $(x,y,z)^\top$ and radius r

Envelope Formula



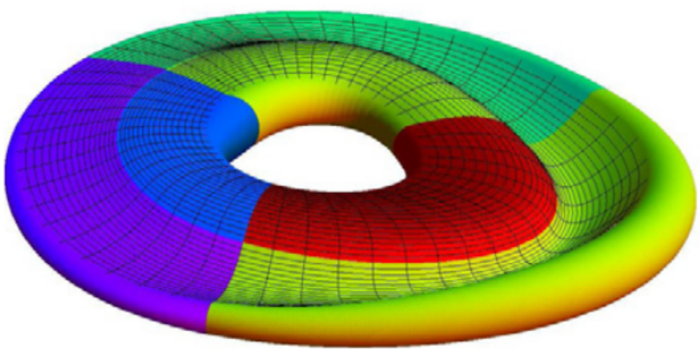
- The envelope formula reads

$$\mathbf{b}^{\pm}(u,v) = \hat{\mathbf{s}}(u,v) + \frac{r}{\hat{E}\hat{G} - \hat{F}^2}\Big(\mathbf{w} \pm \sqrt{EG - F^2} \cdot (\hat{\mathbf{s}}_u \times \hat{\mathbf{s}}_v)\Big),$$

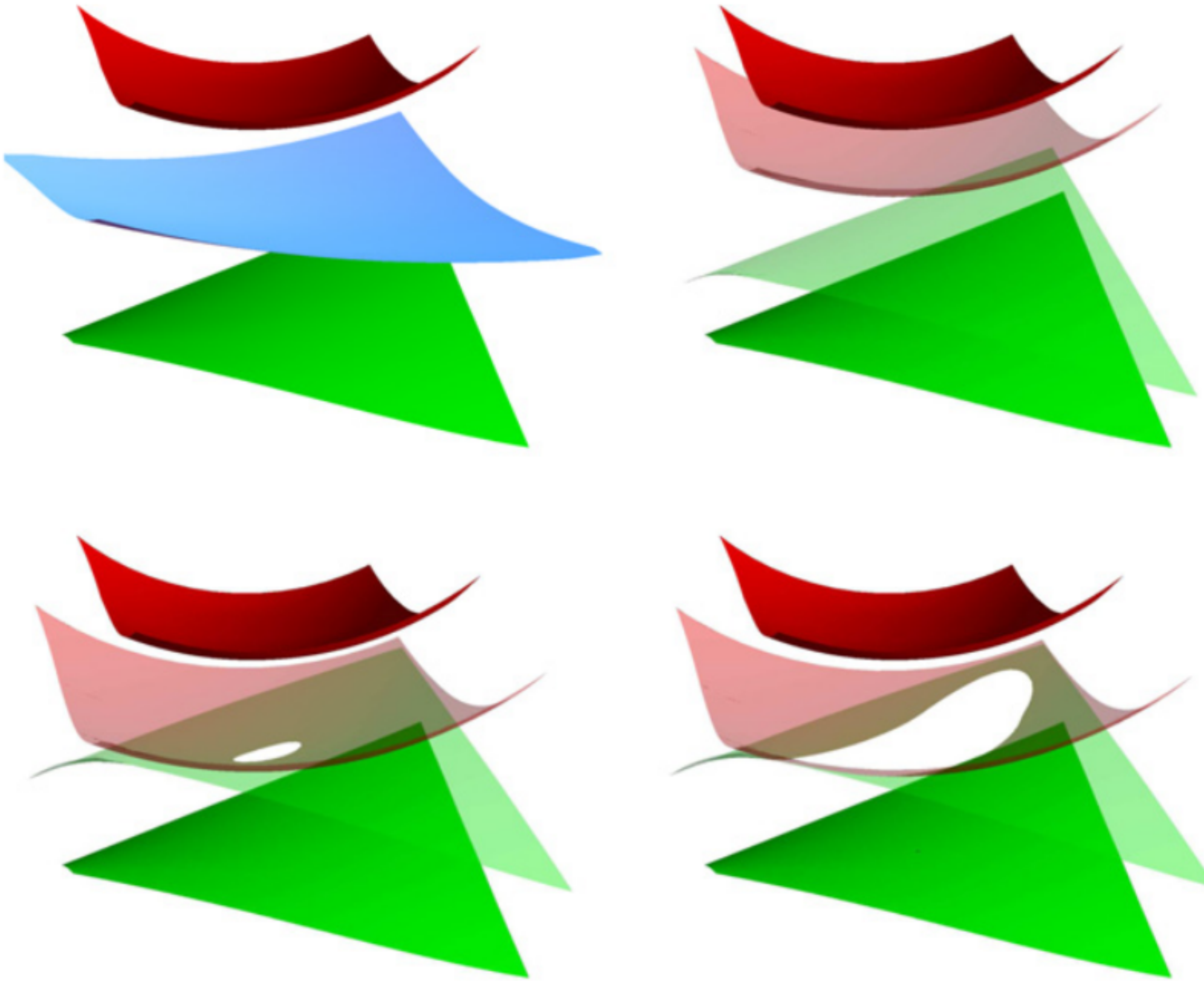
where $\hat{\mathbf{s}}(u,v) = (x(u,v), y(u,v), z(u,v))^\top$ represents the corresponding medial surface

MOS surfaces and Trimmed Offsets

- MOS surfaces: If the MAT of a domain is an MOS surface then the domain boundary and all its offsets are rational
- $\mathbf{s}(u, v)$: $||\mathbf{s}_u(u, v) \times \mathbf{s}_v(u, v)||^2 = EG - F^2 = \sigma(u, v)^2$
- Quadratics are known to be MOS $\rightarrow$ MOS approximation algorithm (Bast, Jüttler, K., Lávička, 20010) based on C^1 quadratic splines

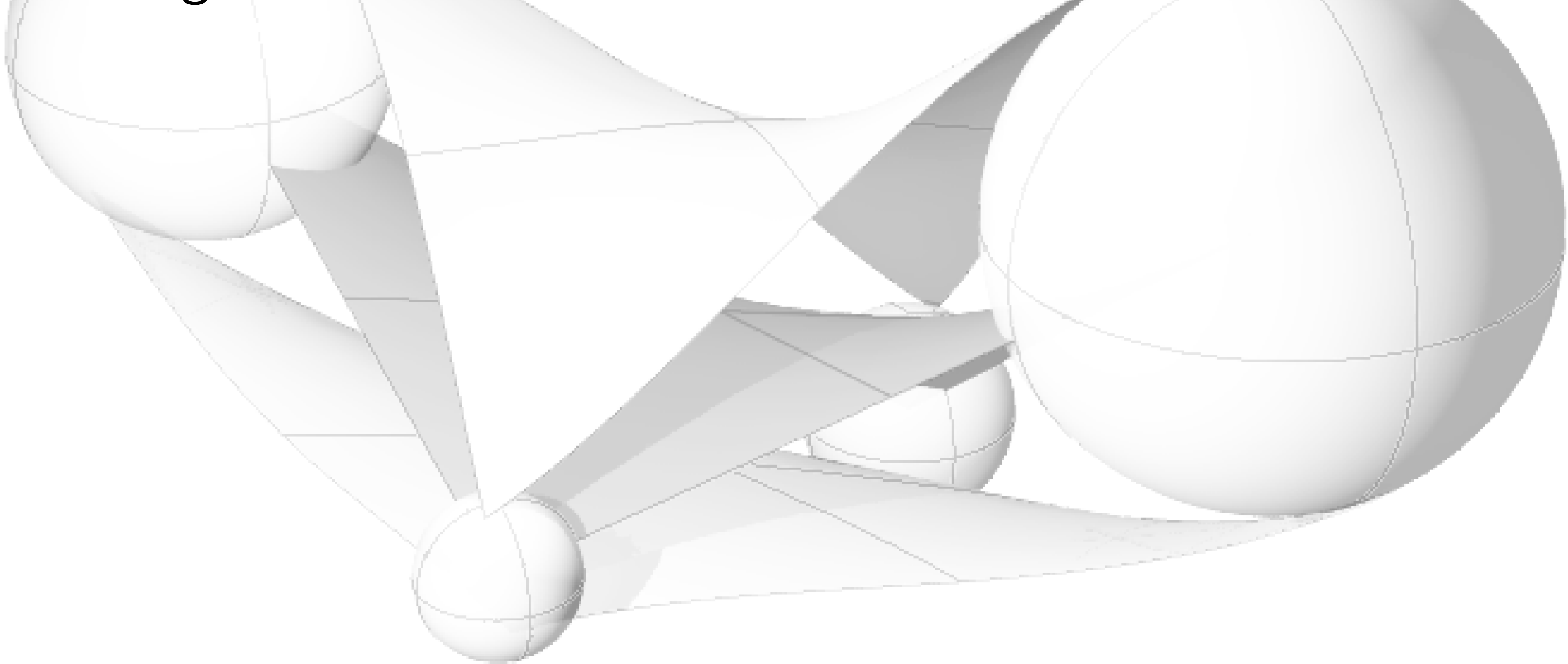


MOS surfaces and Trimmed Offsets



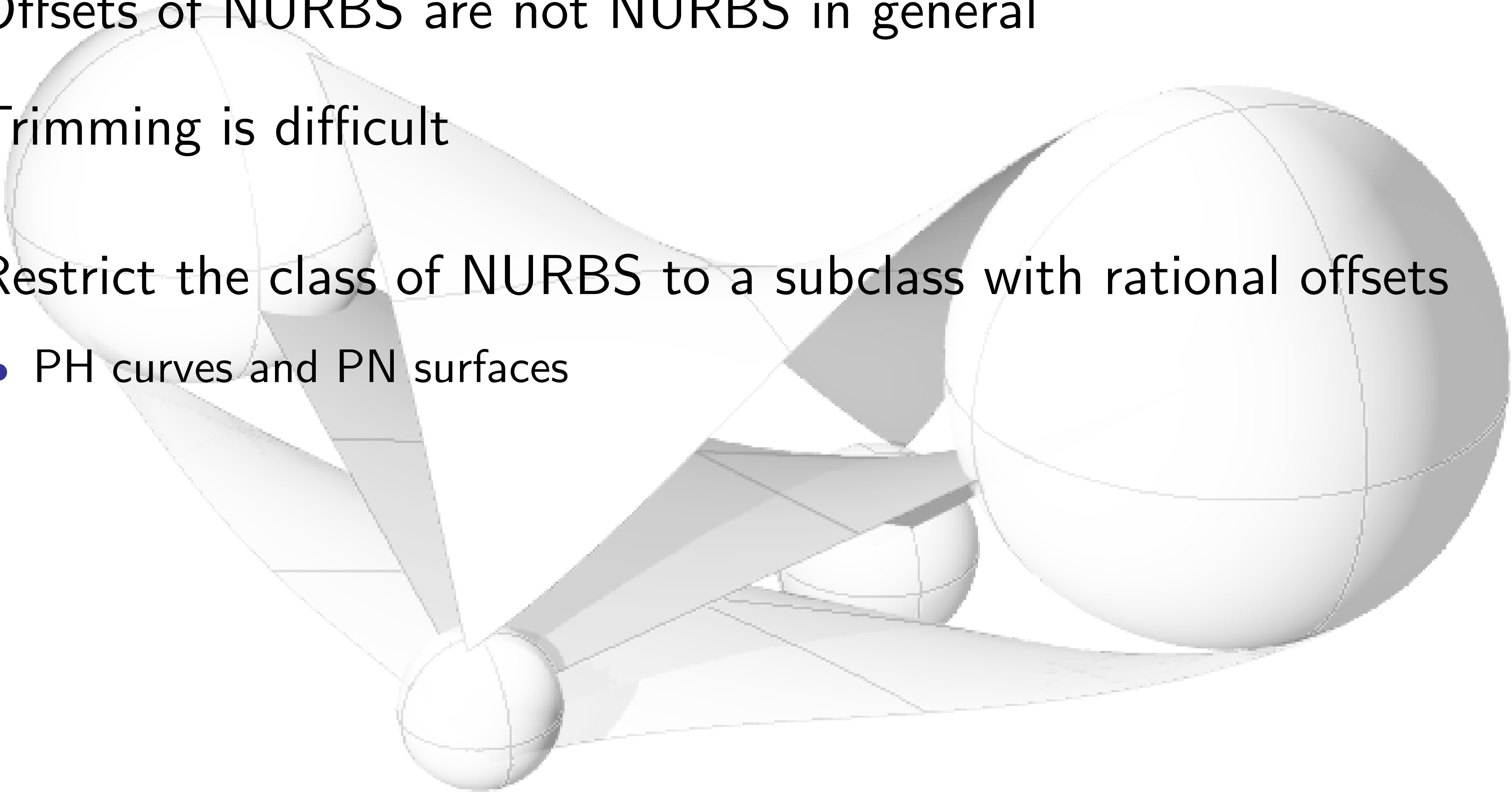
Summary

- Offsets of NURBS are not NURBS in general
- Trimming is difficult

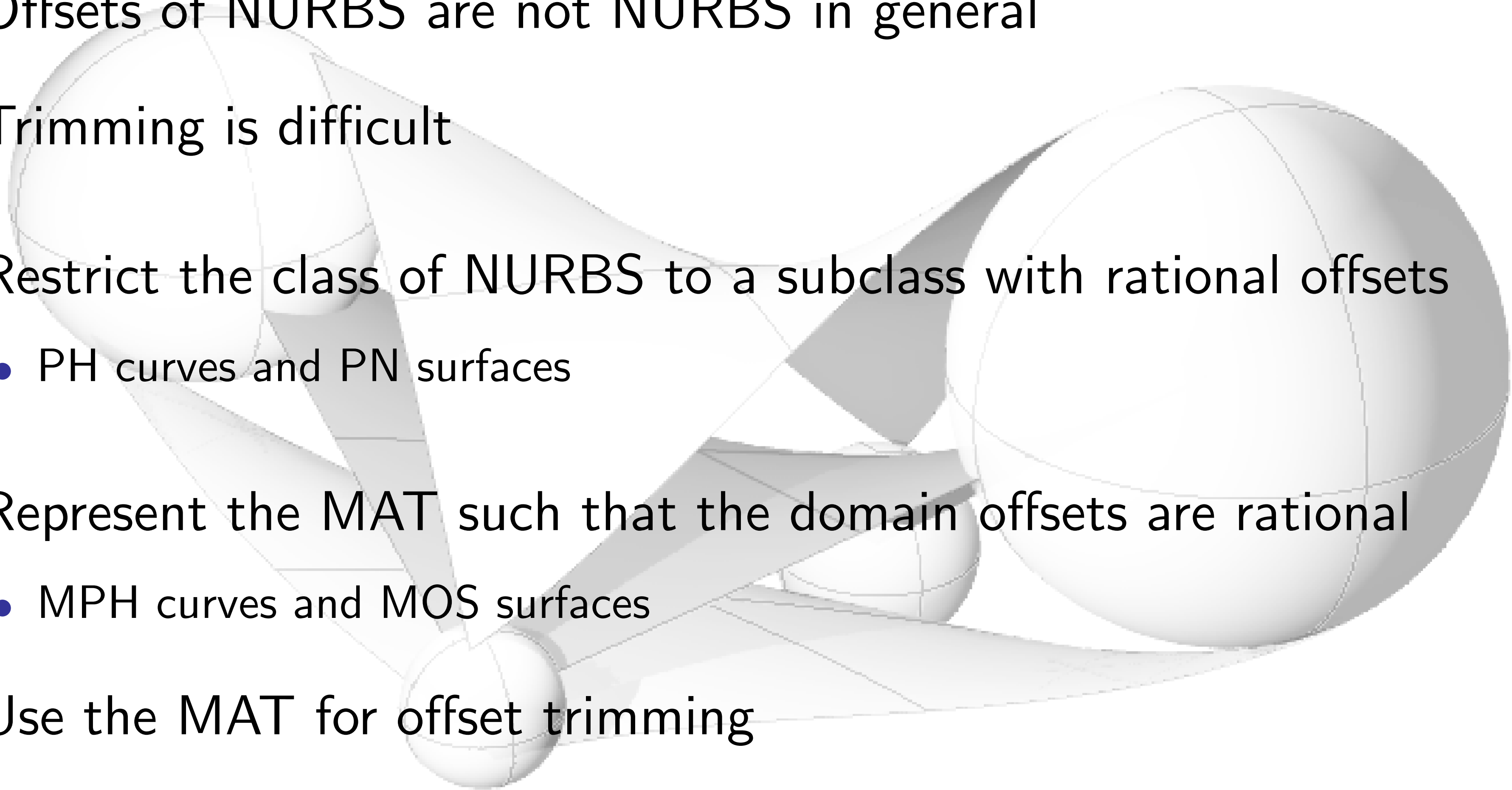


Summary

- Offsets of NURBS are not NURBS in general
- Trimming is difficult
- Restrict the class of NURBS to a subclass with rational offsets
  - PH curves and PN surfaces



Summary



- Offsets of NURBS are not NURBS in general
- Trimming is difficult
- Restrict the class of NURBS to a subclass with rational offsets
  - PH curves and PN surfaces
- Represent the MAT such that the domain offsets are rational
  - MPH curves and MOS surfaces
- Use the MAT for offset trimming

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