quodque folum, certa nitri figna præbere, fed plura concurrere debere, ut de vero nitro producto dubium non relinquatur.

LII. *An Essay towards solving a Problem in the Doctrine of Chances.* By the late Rev. Mr. Bayes, *F. R. S.* communicated by Mr. Price, *in a Letter to* John Canton, *A. M. F. R. S.*

Dear Sir,

I Now fend you an effay which I have found among the papers of our deceafed friend Mr. Bayes, and which, in my opinion, has great merit, and well deferves to be preferved. Experimental philofophy, you will find, is nearly interefted in the fubject of it; and on this account there feems to be particular reafon for thinking that a communication of it to the Royal Society cannot be improper.

He had, you know, the honour of being a member of that illuftrious Society, and was much efteemed by many in it as a very able mathematician. In an introduction which he has writ to this Effay, he fays, that his defign at firft in thinking on the fubject of it was, to find out a method by which we might judge concerning the probability that an event has to happen, in given circumftances, upon fuppofition that we know nothing concerning it but that, under the fame

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circumſtances, it has happened a certain number of times, and failed a certain other number of times. He adds, that he ſoon perceived that it would not be very difficult to do this, provided ſome rule could be found according to which we ought to eſtimate the chance that the probability for the happening of an event perfectly unknown, ſhould lie between any two named degrees of probability, antecedently to any experiments made about it ; and that it appeared to him that the rule muſt be to ſuppoſe the chance the ſame that it ſhould lie between any two equidifferent degrees ; which, if it were allowed, all the reſt might be eaſily calculated in the common method of proceeding in the doctrine of chances. Accordingly, I find among his papers a very ingenious ſolution of this problem in this way. But he afterwards conſidered, that the *poſtulate* on which he had argued might not perhaps be looked upon by all as reaſonable; and therefore he choſe to lay down in another form the propoſition in which he thought the ſolution of the problem is contained, and in a *ſcholium* to ſubjoin the reaſons why he thought ſo, rather than to take into his mathematical reaſoning any thing that might admit diſpute. This, you will obſerve, is the method which he has purſued in this eſſay.

Every judicious perſon will be ſenſible that the problem now mentioned is by no means merely a curious ſpeculation in the doctrine of chances, but neceſſary to be ſolved in order to a ſure foundation for all our reaſonings concerning paſt facts, and what is likely to be hereafter. Common ſenſe is indeed ſufficient to ſhew us that, from the obſervation of what has in former inſtances been the conſequence of a certain

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caufe or action, one may make a judgment what is likely to be the confequence of it another time, and that the larger number of experiments we have to fupport a conclufion, fo much the more reafon we have to take it for granted. But it is certain that we cannot determine, at leaft not to any nicety, in what degree repeated experiments confirm a conclufion, without the particular difcuffion of the beforementioned problem; which, therefore, is neceffary to be confidered by any one who would give a clear account of the ftrength of *analogical* or *inductive reafoning*; concerning, which at prefent, we feem to know little more than that it does fometimes in fact convince us, and at other times not; and that, as it is the means of cquainting us with many truths, of which otherwife we muft have been ignorant; fo it is, in all probability, the fource of many errors, which perhaps might in fome meafure be avoided, if the force that this fort of reafoning ought to have with us were more diftinctly and clearly underftood.

Thefe obfervations prove that the problem enquired after in this effay is no lefs important than it is curious. It may be fafely added, I fancy, that it is alfo a problem that has never before been folved. Mr. De Moivre, indeed, the great improver of this part of mathematics, has in his *Laws of chance* *, after Bernoulli, and to a greater degree of exactnefs, given rules to find the probability there is, that if a very great number of trials be made concerning any event,

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* See Mr. De Moivre's *Doctrine of Chances*, p. 243, &c. He has omitted the demonftrations of his rules, but thefe have been fince fupplied by Mr. Simpfon at the conclufion of his treatife on *The Nature and Laws of Chance*.

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the proportion of the number of times it will happen, to the number of times it will fail in thofe trials, fhould differ lefs than by fmall affigned limits from the proportion of the probability of its happening to the probability of its failing in one fingle trial. But I know of no perfon who has fhewn how to deduce the folution of the converfe problem to this; namely, " the number of times an unknown event " has happened and failed being given, to find the " chance that the probability of its happening fhould " lie fomewhere between any two named degrees of " probability." What Mr. De Moivre has done therefore cannot be thought fufficient to make the confideration of this point unneceffary : efpecially, as the rules he has given are not pretended to be rigoroufly exact, except on fuppofition that the number of trials made are infinite ; from whence it is not obvious how large the number of trials muft be in order to make them exact enough to be depended on in practice.

Mr. De Moivre calls the problem he has thus folved, the hardeft that can be propofed on the fubject of chance. His folution he has applied to a very important purpofe, and thereby fhewn that thofe a remuch miftaken who have infinuated that the Doctrine of Chances in mathematics is of trivial confequence, and cannot have a place in any ferious enquiry *. The purpofe I mean is, to fhew what reafon we have for believing that there are in the conftitution of things fixt laws according to which events happen, and that, therefore, the frame of the world muft be

* See his Doctrine of Chances, p. 252, &c.

the effect of the wisdom and power of an intelligent cause; and thus to confirm the argument taken from final causes for the existence of the Deity. It will be easy to see that the converse problem solved in this essay is more directly applicable to this purpose; for it shews us, with distinctness and precision, in every case of any particular order or recurrency of events, what reason there is to think that such recurrency or order is derived from stable causes or regulations in nature, and not from any of the irregularities of chance.

The two last rules in this essay are given without the deductions of them. I have chosen to do this because these deductions, taking up a good deal of room, would swell the essay too much; and also because these rules, though of considerable use, do not answer the purpose for which they are given as perfectly as could be wished. They are however ready to be produced, if a communication of them should be thought proper. I have in some places writ short notes, and to the whole I have added an application of the rules in the essay to some particular cases, in order to convey a clearer idea of the nature of the problem, and to shew how far the solution of it has been carried.

I am sensible that your time is so much taken up that I cannot reasonably expect that you should minutely examine every part of what I now send you. Some of the calculations, particularly in the Appendix, no one can make without a good deal of labour. I have taken so much care about them, that I believe there can be no material error in any of them; but should there be any such errors, I am the only person who ought to be considered as answerable for them.

<div style="text-align:right">Mr.</div>

Mr. Bayes has thought fit to begin his work with a brief demonſtration of the general laws of chance. His reaſon for doing this, as he ſays in his introduction, was not merely that his reader might not have the trouble of ſearching elſewhere for the principles on which he has argued, but becauſe he did not know whither to refer him for a clear demonſtration of them. He has alſo made an apology for the peculiar definition he has given of the word *chance* or *probability*. His deſign herein was to cut off all diſpute about the meaning of the word, which in common language is uſed in different ſenſes by perſons of different opinions, and according as it is applied to *paſt* or *future* facts. But whatever different ſenſes it may have, all (he obſerves) will allow that an expectation depending on the truth of any *paſt* fact, or the happening of any *future* event, ought to be eſtimated ſo much the more valuable as the fact is more likely to be true, or the event more likely to happen. Inſtead therefore, of the proper ſenſe of the word *probability*, he has given that which all will allow to be its proper meaſure in every caſe where the word is uſed. But it is time to conclude this letter. Experimental philoſophy is indebted to you for ſeveral diſcoveries and improvements; and, therefore, I cannot help thinking that there is a peculiar propriety in directing to you the following eſſay and appendix. That your enquiries may be rewarded with many further ſucceſſes, and that you may enjoy every every valuable bleſſing, is the ſincere wiſh of, Sir,

your very humble ſervant,

Newington-Green,
Nov. 10, 1763.

Richard Price.

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PROBLEM.

Given the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of its happening in a fingle trial lies fomewhere between any two degrees of probability that can be named.

SECTION I.

DEFINITION 1. Several events are *inconfiftent*, when if one of them happens, none of the reft can.

2. Two events are *contrary* when one, or other of them muft; and both together cannot happen.

3. An event is faid to *fail*, when it cannot happen; or, which comes to the fame thing, when its contrary has happened.

4. An event is faid to be determined when it has either happened or failed.

5. The *probability of any event* is the ratio between the value at which an expectation depending on the happening of the event ought to be computed, and the value of the thing expected upon it's happening.

6. By *chance* I mean the fame as probability.

7. Events are independent when the happening of any one of them does neither increafe nor abate the probability of the reft.

PROP. 1.

When feveral events are inconfiftent the probability of the happening of one or other of them is the fum of the probabilities of each of them.

Suppofe

Suppose there be three such events, and which ever of them happens I am to receive N, and that the probability of the 1ft, 2d, and 3d are respectively $\frac{a}{N}$, $\frac{b}{N}$, $\frac{c}{N}$. Then (by the definition of probability) the value of my expectation from the 1ft will be a, from the 2d b, and from the 3d c. Wherefore the value of my expectations from all three will be $a + b + c$. But the sum of my expectations from all three is in this case an expectation of receiving N upon the happening of one or other of them. Wherefore (by definition 5) the probability of one or other of them is $\frac{a+b+c}{N}$ or $\frac{a}{N} + \frac{b}{N} + \frac{c}{N}$. The sum of the probabilities of each of them.

Corollary. If it be certain that one or other of the three events must happen, then $a + b + c = N$. For in this case all the expectations together amounting to a certain expectation of receiving N, their values together must be equal to N. And from hence it is plain that the probability of an event added to the probability of its failure (or of its contrary) is the ratio of equality. For these are two inconsistent events, one of which necessarily happens. Wherefore if the probability of an event is $\frac{P}{N}$ that of it's failure will be $\frac{N-P}{N}$.

PROP. 2.

If a person has an expectation depending on the happening of an event, the probability of the event is to the probability of its failure as his loss if it fails to his gain if it happens.

Suppose a person has an expectation of receiving N, depending on an event the probability of which

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is $\frac{P}{N}$. Then (by definition 5) the value of his expectation is P, and therefore if the event fail, he lofes that which in value is P ; and if it happens he receives N, but his expectation ceafes. His gain therefore is N—P. Likewife fince the probability of the event is $\frac{P}{N}$, that of its failure (by corollary prop. 1) is $\frac{N—P}{N}$. But $\frac{P}{N}$ is to $\frac{N—P}{N}$ as P is to N—P, i. e. the probability of the event is to the probability of it's failure, as his lofs if it fails to his gain if it happens.

P R O P. 3.

The probability that two fubfequent events will both happen is a ratio compounded of the probability of the 1ft, and the probability of the 2d on fuppofition the 1ft happens.

Suppofe that, if both events happen, I am to receive N, that the probability both will happen is $\frac{P}{N}$, that the 1ft will is $\frac{a}{N}$ (and confequently that the 1ft will not is $\frac{N—a}{N}$) and that the 2d will happen upon fuppofition the 1ft does is $\frac{b}{N}$. Then (by definition 5) P will be the value of my expectation, which will become b if the 1ft happens. Confequently if the 1ft happens, my gain by it is b—P, and if it fails my lofs is P. Wherefore, by the foregoing propofition, $\frac{a}{N}$ is to $\frac{N—a}{N}$, i. e. a is to N—a as P is to b—P. Wherefore (componendo inverfè) a is to N as P is to b. But the ratio of P to N is compounded of the ratio of P to b, and that of b to N. Wherefore the

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fame ratio of P to N is compounded of the ratio of a to N and that of b to N, i. e. the probability that the two fubfequent events will both happen is compounded of the probability of the 1ft and the probability of the 2d on fuppofition the 1ft happens.

Corollary. Hence if of two fubfequent events the probability of the 1ft be $\frac{a}{N}$, and the probability of both together be $\frac{P}{N}$, then the probability of the 2d on fuppofition the 1ft happens is $\frac{P}{a}$.

P R O P. 4.

If there be two fubfequent events to be determined every day, and each day the probability of the 2d is $\frac{b}{N}$ and the probability of both $\frac{P}{N}$, and I am to receive N if both the events happen the 1ft day on which the 2d does; I fay, according to thefe conditions, the probability of my obtaining N is $\frac{P}{b}$. For if not, let the probability of my obtaining N be $\frac{x}{N}$ and let y be to x as N—b to N. Then fince $\frac{x}{N}$ is the probability of my obtaining N (by definition 1) x is the value of my expectation. And again, becaufe according to the foregoing conditions the 1ft day I have an expectation of obtaining N depending on the happening of both the events together, the probability of which is $\frac{P}{N}$, the value of this expectation is P. Likewife, if this coincident fhould not happen I have an expectation of being reinftated in my former circumftances, i. e. of receiving that which in value is x depending

pending on the failure of the 2d event the probability of which (by cor. prop. 1) is $\frac{N-b}{N}$ or $\frac{y}{x}$, becaufe y is to x as $N-b$ to N. Wherefore fince x is the thing expected and $\frac{y}{x}$ the probability of obtaining it, the value of this expectation is y. But thefe two laft expectations together are evidently the fame with my original expectation, the value of which is x, and therefore $P + y = x$. But y is to x as $N-b$ is to N. Wherefore x is to P as N is to b, and $\frac{x}{N}$ (the probability of my obtaining N) is $\frac{P}{b}$.

Cor. Suppofe after the expectation given me in the foregoing propofition, and before it is at all known whether the 1ft event has happened or not, I fhould find that the 2d event has happened; from hence I can only infer that the event is determined on which my expectation depended, and have no reafon to efteem the value of my expectation either greater or lefs than it was before. For if I have reafon to think it lefs, it would be reafonable for me to give fomething to be reinftated in my former circumftances, and this over and over again as often as I fhould be informed that the 2d event had happened, which is evidently abfurd. And the like abfurdity plainly follows if you fay I ought to fet a greater value on my expectation than before, for then it would be reafonable for me to refufe fomething if offered me upon condition I would relinquifh it, and be reinftated in my former circumftances; and this likewife over and over again as often as (nothing being known concerning the 1ft event) it fhould appear that the 2d had happened. Notwithftanding therefore this difcovery that the 2d

event

event has happened, my expectation ought to be esteemed the same in value as before, i. e. x, and consequently the probability of my obtaining N is (by definition 5) still $\frac{x}{N}$ or $\frac{P}{b}$ *. But after this discovery the probability of my obtaining N is the probability that the 1st of two subsequent events has happened upon the supposition that the 2d has, whose probabilities were as before specified. But the probability that an event has happened is the same as the probability I have to guess right if I guess it has happened. Wherefore the following proposition is evident.

P R O P. 5.

If there be two subsequent events, the probability of the 2d $\frac{b}{N}$ and the probability of both together $\frac{P}{N}$, and it being 1st discovered that the 2d event has happened, from hence I guess that the 1st event has also happened, the probability I am in the right is $\frac{P}{b}$ †.

P R O P.

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* What is here said may perhaps be a little illustrated by considering that all that can be lost by the happening of the 2d event is the chance I should have had of being reinstated in my former circumstances, if the event on which my expectation depended had been determined in the manner expressed in the proposition. But this chance is always as much *against* me as it is *for* me. If the 1st event happens, it is *against* me, and equal to the chance for the 2d event's failing. If the 1st event does not happen, it is *for* me, and equal also to the chance for the 2d event's failing. The loss of it, therefore, can be no disadvantage.

† What is proved by Mr. Bayes in this and the preceding proposition is the same with the answer to the following question. What is the probability that a certain event, when it happens, will

PROP. 6.

The probability that feveral independent events fhall all happen is a ratio compounded of the probabilities of each.

For from the nature of independent events, the probability that any one happens is not altered by the happening or failing of any of the reft, and confequently the probability that the 2d event happens on fuppofition the 1ft does is the fame with its original probability; but the probability that any two events happen is a ratio compounded of the probability of the 1ft event, and the probability of the 2d on fuppofition the 1ft happens by prop. 3. Wherefore the probability that any two independent events both happen is a ratio compounded of the probability of the 1ft and the probability of the 2d. And in like manner confidering the 1ft and 2d event together as one event; the probability that three independent events all happen is a ratio compounded of the probability that the two 1ft both happen and the probability of the 3d. And thus you

be accompanied with another to be determined at the fame time ? In this cafe, as one of the events is given, nothing can be due for the expectation of it; and, confequently, the value of an expectation depending on the happening of both events muft be the fame with the value of an expectation depending on the happening of one of them. In other words; the probability that, when one of two events happens, the other will, is the fame with the probability of this other. Call x then the probability of this other, and if $\frac{b}{N}$ be the probability of the given event, and $\frac{p}{N}$ the probability of both, becaufe $\frac{p}{N} = \frac{b}{N} \times x$, $x = \frac{p}{b} =$ the probability mentioned in thefe propofitions.

may

may proceed if there be ever fo many fuch events; from whence the propofition is manifeft.

Cor. 1. If there be feveral independent events, the probability that the 1ft happens the 2d fails, the 3d fails and the 4th happens, &c. is a ratio compounded of the probability of the 1ft, and the probability of the failure of the 2d, and the probability of the failure of the 3d, and the probability of the 4th, &c. For the failure of an event may always be confidered as the happening of its contrary.

Cor. 2. If there be feveral independent events, and the probability of each one be a, and that of its failing be b, the probability that the 1ft happens and the 2d fails, and the 3d fails and the 4th happens, &c. will be $abba$, &c. For, according to the algebraic way of notation, if a denote any ratio and b another, $abba$ denotes the ratio compounded of the ratios a, b, b, a. This corollary therefore is only a particular cafe of the foregoing.

Definition. If in confequence of certain data there arifes a probability that a certain event fhould happen, its happening or failing, in confequence of thefe data, I call it's happening or failing in the 1ft trial. And if the fame data be again re-repeated, the happening or failing of the event in confequence of them I call its happening or failing in the 2d trial; and fo on as often as the fame data are repeated. And hence it is manifeft that the happening or failing of the fame event in fo many diffe-trials, is in reality the happening or failing of fo many diftinct independent events exactly fimilar to each other.

P R O P.

PROP. 7.

If the probability of an event be a, and that of its failure be b in each fingle trial, the probability of its happening p times, and failing q times in $p+q$ trials is E $a^p b^q$ if E be the coefficient of the term in which occurs $a^p b^q$ when the binomial $\overline{a+b}\big|^{b+q}$ is expanded.

For the happening or failing of an event in different trials are fo many independent events. Wherefore (by cor. 2. prop. 6.) the probability that the event happens the 1ft trial, fails the 2d and 3d, and happens the 4th, fails the 5th, &c. (thus happening and failing till the number of times it happens be p and the number it fails be q) is $abbab$ &c. till the number of a's be p and the number of b's be q, that is; 'tis $a^p b^q$. In like manner if you confider the event as happening p times and failing q times in any other particular order, the probability for it is $a^p b^q$; but the number of different orders according to which an event may happen or fail, fo as in all to happen p times and fail q, in $p+q$ trials is equal to the number of permutations that $aaaa\ bbb$ admit of when the number of a's is p, and the number of b's is q. And this number is equal to E, the coefficient of the term in which occurs $a^p b^q$ when $\overline{a+b}\big|^{p+q}$ is expanded. The event therefore may happen p times and fail q in $p+q$ trials E different ways and no more, and its happening and failing thefe feveral different ways are fo many inconfiftent events, the probability for each of which is $a^p b^q$, and therefore by

prop.

prop. 1. the probability that some way or other it happens p times and fails q times in $p + q$ trials is E $a^p\, b^q$.

SECTION II.

Postulate. 1. I Suppose the square table or plane A B C D to be so made and levelled, that if either of the balls o or W be thrown upon it, there shall be the same probability that it rests upon any one equal part of the plane as another, and that it must necessarily rest somewhere upon it.

2. I suppose that the ball W shall be 1st thrown, and through the point where it rests a line os shall be drawn parallel to A D, and meeting C D and A B in s and o; and that afterwards the ball O shall be thrown $p + q$ or n times, and that its resting between A D and os after a single throw be called the happening of the event M in a single trial. These things supposed,

Lem. 1. The probability that the point o will fall between any two points in the line A B is the ratio of the distance between the two points to the whole line A B.

Let any two points be named, as f and b in the line A B, and through them parallel to A D draw f F, b L meeting C D in F and L. Then if the rectangles C f, F b, L A are
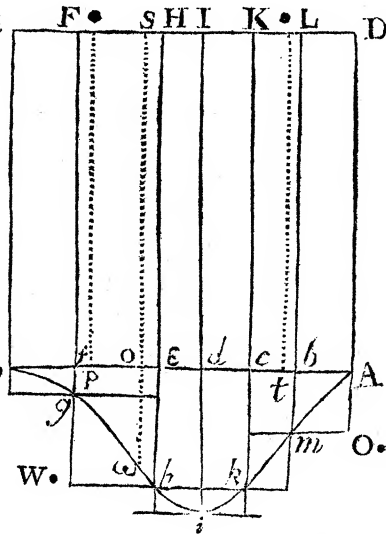
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commenfurable to each other, they may each be divided into the fame equal parts, which being done, and the ball W thrown, the probability it will reft fomewhere upon any number of thefe equal parts will be the fum of the probabilities it has to reft upon each one of them, becaufe its refting upon any different parts of the plane A C are fo many inconfiftent events ; and this fum, becaufe the probability it fhould reft upon any one equal part as another is the fame, is the probability it fhould reft upon any one equal part multiplied by the number of parts. Confequently, the probability there is that the ball W fhould reft fomewhere upon F b is the probability it has to reft upon one equal part multiplied by the number of equal parts in Fb; and the probability it refts fomewhere upon Cf or LA, i.e. that it dont reft upon Fb (becaufe it muft reft fomewhere upon A C) is the probability it refts upon one equal part multiplied by the number of equal parts in Cf, LA taken together. Wherefore, the probability it refts upon F b is to the probability it dont as the number of equal parts in F b is to the number of equal parts in Cf, LA together, or as F b to Cf, LA together, or as fb to Bf, Ab together. Wherefore the probability it reft upon F b is to the probability it dont as fb to Bf, Ab together. And *(componendo inverfe)* the probability it refts upon F b is to the probability it refts upon F b added to the probability it dont, as fb to A B, or as the ratio of fb to A B to the ratio of A B to A B. But the probability of any event added to the probability of its failure is the ratio of equality ; wherefore, the probability it reft upon F b is to the ratio of equality as the ratio of fb to A B to the ratio of A B to A B, or the ratio of equality ; and therefore the probability it reft upon

F b

F b is the ratio of fb to A B. But *ex hypothefi* according as the ball W falls upon F b or not the point o will lie between f and b or not, and therefore the probability the point o will lie between f and b is the ratio of fb to A B.

Again; if the rectangles Cf, Fb, L A are not commenfurable, yet the laft mentioned probability can be neither greater nor lefs than the ratio of fb to A B; for, if it be lefs, let it be the ratio of fc to AB, and upon the line fb take the points p and t, fo that pt fhall be greater than fc, and the three lines Bp, pt, t A commenfurable (which it is evident may be always done by dividing A B into equal parts lefs than half cb, and taking p and t the neareft points of divifion to f and c that lie upon fb). Then becaufe Bp, pt, t A are commenfurable, fo are the rectangles Cp, Dt, and that upon pt compleating the fquare AB. Wherefore, by what has been faid, the probability that the point o will lie between p and t is the ratio of pt to A B. But if it lies between p and t it muft lie between f and b. Wherefore, the probability it fhould lie between f and b cannot be lefs than the ratio of pt to A B, and therefore muft be greater than the ratio of fc to A B (fince pt is greater than fc). And after the fame manner you may prove that the forementioned probability cannot be greater than the ratio of fb to A B, it muft therefore be the fame.

Lem. 2. The ball W having been thrown, and the line os drawn, the probability of the event M in a fingle trial is the ratio of Ao to A B.

For, in the fame manner as in the foregoing lemma, the probability that the ball o being thrown fhall

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reſt ſomewhere upon Do or between A D and so is
is the ratio of A o to A B. But the reſting of the
ball o between A D and so after a ſingle throw is
the happening of the event M in a ſingle trial.
Wherefore the lemma is manifeſt.

P R O P. 8.

If upon B A you erect the figure B$ghikm$A
whoſe property is this, that (the baſe B A being di-
vided into any two parts, as Ab, and Bb and at the
point of diviſion b a perpendicular being erected and
terminated by the figure in m; and y, x, r repre-
ſenting reſpectively the ratio of bm, Ab, and Bb to
A B, and E being the the coefficient of the term in
which occurs $a^p b^q$ when the binomial $\overline{a+b}|^{p+q}$ is
expanded) $y = E x^p r^q$. I ſay that before the ball W
is thrown, the probability the point o ſhould fall be-
tween f and b, any two points named in the line
A B, and withall that the event M ſhould happen p
times and fail q in $p+q$ trials, is the ratio of
$fghikmb$, the part of the figure B$ghikm$A in-
tercepted between the perpendiculars fg, bm raiſed
upon the line A B, to C A the ſquare upon A B.

D E M O N S T R A T I O N.

For if not; 1ſt let it be the ratio of D a figure
greater than $fghikmb$ to C A, and through the
points e, d, c draw perpendiculars to fb meeting the
curve AmigB in h, i, k; the point d being ſo
placed that di ſhall be the longeſt of the perpendi-
5   culars

culars terminated by the line fb, and the curve A m i g B ; and the points e, d, c being so many and so placed that the rectangles, bk, ci, ei, fb taken together shall differ less from $fghikmb$ than D does ; all which may be easily done by the help of the equation of the curve, and the difference between D and the figure $fghikmb$ given. Then since di is the longest of the perpendicular ordinates that insist upon fb, the rest will gradually decrease as they are farther and farther from it on each side, as appears from the construction of the figure, and consequently eh is greater than gf or any other ordinate that insists upon ef.

Now if A o were equal to A e, then by lem. 2. the probability of the event M in a single trial would be the ratio of A e to A B, and consequently by cor. Prop. 1. the probability of it's failure would be the ratio of B e to A B. Wherefore, if x and r be the two forementioned ratios respectively, by Prop. 7. the probability of the event M happening p times and failing q in $p + q$ trials would be $\mathrm{E}\,x^p\,r^q$. But x and r being respectively the ratios of A e to A B and B e to A B, if y is the ratio of eh to A B, then, by construction of the figure A i B, $y = \mathrm{E}\,x^p\,r^q$. Wherefore, if A o were equal to A e the probability of the event M happening p times and failing q in $p + q$ trials would be y, or the ratio of eh to A B. And if A o were equal to Af, or were any mean between A e and Af, the last mentioned probability for the same reasons would be the ratio of fg or some other of the ordinates insisting upon ef, to A B. But eh is the greatest of all the ordinates that insist upon ef. Wherefore, upon supposition the point should lie

any where between f and e, the probability that the event M happens p times and fails q in $p + q$ trials can't be greater than the ratio of $e h$ to A B. There then being thefe two fubfequent events, the 1ft that the point o will lie between e and f, the 2d that the event M will happen p times and fail q in $p + q$ trials, and the probability of the 1ft (by lemma 1ft) is the ratio of $e f$ to A B, and upon fuppofition the 1ft happens, by what has been now proved, the probability of the 2d cannot be greater than the ratio of $e h$ to A B, it evidently follows (from Prop. 3.) that the probability both together will happen cannot be greater than the ratio compounded of that of $e f$ to A B and that of $e h$ to A B, which compound ratio is the ratio of $f h$ to C A   Wherefore, the probability that the point o will lie between f and e, and the event M happen p times and fail q, is not greater than the ratio of $f h$ to C A. And in like, manner the probability the point o will lie between e and d, and the event M happen and fail as before, cannot be greater than the ratio of $e i$ to C A. And again, the probability the point o will lie between d and c, and the event M happen and fail as before, cannot be greater than the ratio of $c i$ to C A. And laftly, the probability that the point o will lie between c and b, and the event M happen and fail as before, cannot be greater than the ratio of $b k$ to C A. Add now all thefe feveral probabilities together, and their fum (by Prop. 1.) will be the probability that the point will lie fomewhere between f and b, and the event M happen p times and fail q in $p + q$ trials. Add likewife the correfpondent ratios together, and their fum will be the ratio of the fum of the antecedents to

to their common confequent, i. e. the ratio of fh, ei, ci, bk together to C A; which ratio is lefs than that of D to C A, becaufe D is greater than fh, ei, ci, bk together. And therefore, the probability that the point o will lie between f and b, and withal that the event M will happen p times and fail q in $p+q$ trials, is *lefs* than the ratio of D to C A; but it was fuppofed the fame which is abfurd. And in like manner, by infcribing rectangles within the figure, as eg, dh, dk, cm, you may prove that the laft mentioned probability is *greater* than the ratio of any figure lefs than $fghikmb$ to C A.

Wherefore, that probability muft be the ratio of $fghikmb$ to C A.

Cor. Before the ball W is thrown the probability that the point o will lie fomewhere between A and B, or fomewhere upon the line A B, and withal that the event M will happen p times, and fail q in $p+q$ trials is the ratio of the whole figure A i B to C A. But it is certain that the point o will lie fomewhere upon A B. Wherefore, before the ball W is thrown the probability the event M will happen p times and fail q in $p+q$ trials is the ratio of A i B to C A.

P R O P. 9.

If before any thing is difcovered concerning the place of the point o, it fhould appear that the event M had happened p times and failed q in $p+q$ trials, and from hence I guefs that the point o lies between any two points in the line A B, as f and b, and confequently that the probability of the event M in a fingle trial was fomewhere between the ratio of A b to A B and that of A f to A B: the probability I am in

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the right is the ratio of that part of the figure A *i* B defcribed as before which is intercepted between perpendiculars erected upon A B at the points *f* and *b*, to the whole figure A *i* B.

For, there being thefe two fubfequent events, the firft that the point *o* will lie between *f* and *b*; the fecond that the event M fhould happen *p* times and fail *q* in *p* + *q* trials; and (by cor. prop. 8.) the original probability of the fecond is the ratio of A *i* B to C A, and (by prop. 8.) the probability of both is the ratio of *f g b i m b* to C A; wherefore (by prop. 5) it being firft difcovered that the fecond has happened, and from hence I guefs that the firft has happened alfo, the probability I am in the right is the ratio of *f g b i m b* to A *i* B, the point which was to be proved.

Cor. The fame things fuppofed, if I guefs that the probability of the event M lies fomewhere between *o* and the ratio of A *b* to A B, my chance to be in the right is the ratio of A *b m* to A *i* B.

SCHOLIUM.

From the preceding propofition it is plain, that in the cafe of fuch an event as I there call M, from the number of times it happens and fails in a certain number of trials, without knowing any thing more concerning it, one may give a guefs whereabouts it's probability is, and, by the ufual methods computing the magnitudes of the areas there mentioned, fee the chance that the guefs is right. And that the fame rule is the proper one to be ufed in the cafe of an event concerning the probability of which

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we abfolutely know nothing antecedently to any trials made concerning it, feems to appear from the following confideration; viz. that concerning fuch an event I have no reafon to think that, in a certain number of trials, it fhould rather happen any one poffible number of times than another. For, on this account, I may juftly reafon concerning it as if its probability had been at firft unfixed, and then determined in fuch a manner as to give me no reafon to think that, in a certain number of trials, it fhould rather happen any one poffible number of times than another. But this is exactly the cafe of the event M. For before the ball W is thrown, which determines it's probability in a fingle trial, (by cor. prop. 8.) the probability it has to happen p times and fail q in $p + q$ or n trials is the ratio of A i B to C A, which ratio is the fame when $p + q$ or n is given, whatever number p is; as will appear by computing the magnitude of A i B by the method * of fluxions. And confequently before the place of the point o is difcovered or the number of times the event M has happened in n trials, I can have no reafon to think it fhould rather happen one pof-fible number of times than another.

In what follows therefore I fhall take for granted that the rule given concerning the event M in prop. 9. is alfo the rule to be ufed in relation to any event concerning the probability of which nothing

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* It will be proved prefently in art. 4. by computing in the method here mentioned that A i B contracted in the ratio of E to 1 is to C A as 1 to $\overline{n + 1} \times E$: from whence it plainly follows that, antecedently to this contraction, A i B muft be to C A in the ratio of 1 to $n + 1$, which is a conftant ratio when n is given, whatever p is.

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at all is known antecedently to any trials made or ob-
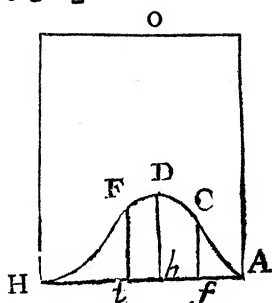ferved concerning it. And fuch an event I fhall call
an unknown event.

Cor. Hence, by fuppofing the ordinates in the fi-
gure A *i* B to be contracted in the ratio of E to one,
which makes no alteration in the proportion of the
parts of the figure intercepted between them, and
applying what is faid of the event M to an unknown
event, we have the following propofition, which gives
the rules for finding the probability of an event from
the number of times it actually happens and fails.

P R O P. 10.

If a figure be defcribed upon any bafe A H (Vid.
Fig.) having for it's equation $y = x^p r^q$; where y,
x, r are refpectively the ratios of an ordinate of the
figure infifting on the bafe at right angles, of the
fegment of the bafe intercepted between the ordinate
and A the beginning of the bafe, and of the other
fegment of the bafe lying between the ordinate and
the point H, to the bafe as their common confequent.
I fay then that if an unknown event has happened
p times and failed q in $p + q$ trials, and in the bafe
A H taking any two points as f and t you erect the
ordinates fc, t F at right angles with it, the chance
that the probability of the event lies fomewhere be-
tween the ratio of Af to A H and that of A t to
A H, is the ratio of t F C f, that part of the before-
defcribed figure which is intercepted between the two
ordinates, to A C F H the whole figure infifting on
the bafe A H.

This is evident from prop. 9. and the remarks made
in the foregoing fcholium and corollary.

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Now

Now, in order to reduce the forego-ing rule to practice, we muſt find the value of the area of the figure de-ſcribed and the ſe-veral parts of it ſe-parated, by ordi-nates perpendicu-lar to its baſe. For which purpoſe, ſuppoſe $AH = 1$ and HO the ſquare upon AH likewiſe $= 1$, and Cf will be $= y$, and $Af = x$, and $Hf = r$, becauſe y, x and r denote the ratios of Cf, Af, and Hf reſpectively to AH. And by the equation of the curve $y = x^p r^q$ and (be-cauſe $Af + fH = AH$) $r + x = 1$. Wherefore

$$y = x^p \times \overline{1-x}|^q = x^p - q x^{p+1} + q \times \frac{q-1}{2} \times x^{p+2} - q$$

$$\times \frac{q-1}{2} \times \frac{q-2}{3} \times x^{p+3} + \&c.$$ Now the abſciſſe being x and the ordinate x^p the correſpondent area is $\frac{x^{p+1}}{p+1}$ (by prop. 10. caſ. 1. Quadrat. Newt.) * and the ordi-nate being $q x^{p+1}$ the area is $\frac{q x^{p+2}}{p+2}$; and in like man-

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* Tis very evident here, without having recourſe to Sir Iſaac Newton, that the fluxion of the area ACf being $y\dot{x} = x^p \dot{x} - q x^{p+1} \dot{x} + q \times \frac{q-1}{2} x^{p+2} \dot{x}$ &c. the fluent or area itſelf is $\frac{x^{p+1}}{p+1}$ $- q \times \frac{x^{p+2}}{p+2} + q \times \frac{q-1}{2} \times \frac{x^{p+3}}{p+3}$ &c.

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ner of the reſt. Wherefore, the abſciſſe being x and the ordinate y or $x^p - q\,x^{p+1} +$ &c. the correſpondent area is $\dfrac{x^{p+1}}{p+1} - q \times \dfrac{x^{p+2}}{p+2} + q \times \dfrac{q-1}{2} \times \dfrac{x^{p+3}}{p+3} - q \times \dfrac{q-1}{2} \times$

$\dfrac{q-2}{3} \times \dfrac{x^{p+4}}{p+4} +$ &c. Wherefore, if $x = Af = \dfrac{Af}{AH}$,

and $y = Cf = \dfrac{Cf}{AH}$, then $ACf = \dfrac{ACf}{HO} = \dfrac{x^{p+1}}{p+1}$

$- \dfrac{q}{p+2} \times x^{p+2} + q \times \dfrac{q-1}{2} \times \dfrac{x^{p+3}}{p+3} -$ &c.

From which equation, if q be a ſmall number, it is eaſy to find the value of the ratio of ACf to HO. and in like manner as that was found out, it will appear that the ratio of HCf to HO is $\dfrac{r^{q+1}}{q+1} - p \times$

$\dfrac{r^{q+2}}{q+2} + p \times \dfrac{p-1}{2} \times \dfrac{r^{q+3}}{q+3} - p \times \dfrac{p-1}{2} \times \dfrac{p-2}{3} \times \dfrac{r^{q+4}}{q+4}$ &c.

which ſeries will conſiſt of few terms and therefore is to be uſed when p is ſmall.

2. The ſame things ſuppoſed as before, the ratio of ACf to HO is $\dfrac{x^{p+1}\,r^q}{p+1} + \dfrac{q}{p+1} \times \dfrac{x^{p+2}\,r^{q-1}}{p+2} + \dfrac{q}{p+1} \times$

$\dfrac{q-1}{p+2} \times \dfrac{x^{p+3}\,r^{q-2}}{p+3} + \dfrac{q}{p+1} \times \dfrac{q-1}{p+2} \times \dfrac{q-2}{p+3} \times \dfrac{x^{p+4}\,r^{q-3}}{p+4} +$

&c.

&c. $+ x^{\frac{n+1}{n+1}} \times \frac{q}{p+1} \times \frac{q-1}{p+2} \times$ &c. $\times \frac{1}{n}$ where $n =$

$p + q$. For this series is the same with $x^{\frac{p+1}{p+1}} - q \times$

$x^{\frac{p+2}{p+2}}$ &c. set down in Art. 1st. as the value of the ratio of A C f to H O ; as will easily be seen by putting in the former instead of r its value $1-x$, and expanding the terms and ordering them according to the powers of x. Or, more readily, by comparing the fluxions of the two series, and in the former instead of $\dot{r}$ substituting $-\dot{x}$ *.

* The fluxion of the first series is $x^p \, r^q \, \dot{x} + q \, x^{\frac{p+1}{}} \, r^{\frac{p-1}{p+1}} \dot{r} +$

$q \, x^{\frac{p+1}{p+1}} \, r^{q-1} \dot{x} + q \times \frac{q-1}{p+1} \times \frac{x^{\frac{p+2}{p+2}} \, r^{q-2} \, \dot{r}}{} + q \times \frac{q-1}{p+1} \times x^{\frac{p+2}{p+2}} \, r^{q-2} \dot{x}$

$+ q \times \frac{q-1}{p+1} \times \frac{q-3}{p+2} \times x^{\frac{p+3}{p+3}} \, r^{q-3}_{r}$ &c. or, substituting $- \dot{x}$ for $\dot{r}$,

$x^p \, r^q \, \dot{x} - \frac{q \, x^{\frac{p+1}{p+1}} \, r^{q-1} \, \dot{x}}{} + \frac{q \, x^{\frac{p+1}{p+1}} \, r^{q-1} \dot{x}}{} - q \times \frac{q-1}{p+1} \times$

$\frac{x^{\frac{p+2}{p+2}} \, r^{q-2} \dot{x}}{} + q \times \frac{q-1}{p+1} \times \frac{x^{\frac{p+2}{p+2}} \, r^{q-2} \dot{x}}{}$ &c. which, as all the terms after the first destroy one another, is equal to $x^p \, r^q \, \dot{x} =$
$x^p \times \overline{1-x}|^q \, \dot{x} = x^p \, \dot{x} \times \overline{1 - qx + q \times \frac{q-1}{2} x^2}$ &c. $= x^p \, \dot{x} -$
$q \, x^{\frac{p+1}{p+1}} \dot{x} + q \times \frac{q-1}{2} x^{\frac{p+2}{p+2}} \, \dot{x}$ &c. $=$ the fluxion of the latter series or of $x^{\frac{p+1}{p+1}} - q \times x^{\frac{p+2}{p+2}}$ &c. The two series therefore are the same.

3. In like manner, the ratio of HCf to HO is

$$\frac{r^{\overset{q+1}{}} x^p}{q+1} + \frac{p}{q+1} \times \frac{r^{\overset{q+2}{}} x^{p-1}}{q+2} + \frac{p}{q+1} \times \frac{p-1}{q+2} \times \frac{r^{\overset{q+3}{}} x^{p-2}}{q+3} + $$

&c.

4. If E be the coefficient of that term of the binomical $\overline{a+b}|^{p+q}$ expanded in which occurs $a^p\, b^q$, the ratio of the whole figure $ACFH$ to HO is $\frac{1}{n+1} \times \frac{1}{E}$, n being $= p+q$. For, when $Af = AH$ $x = 1$, $r = 0$. Wherefore, all the terms of the feries fet down in Art. 2. as expreffing the ratio of ACf to HO will vanifh except the laft, and that becomes $\frac{1}{n+1} \times \frac{q}{p+1} \times \frac{q-1}{p+2} \times$ &c. $\times \frac{1}{n}$. But E being the coefficient of that term in the binomial $\overline{a+b}|^n$ expanded in which occurs $a^p\, b^q$ is equal to $\frac{p+1}{q} \times \frac{p+2}{q-1} \times$ &c. $\times \frac{n}{1}$. And, becaufe Af is fuppofed to become $= AH$, $ACf = ACH$. From whence this article is plain.

5. The ratio of ACf to the whole figure $ACFH$ is (by Art. 1. and 4.) $\overline{n+1} \times E \times \frac{x^{\overset{p+1}{}}}{p+1} - q \times \frac{x^{\overset{p+2}{}}}{p+2} + q \times \frac{q-1}{2} \times \frac{x^{\overset{p+3}{}}}{p+3}$ &c. and if, as x expreffes the ratio of Af to AH, X fhould exprefs the ratio of At to AH; the ratio of AFt to $ACFH$ would be $\overline{n+1} \times E \times \frac{X^{\overset{p+1}{}}}{p+1} - q\frac{X^{\overset{p+2}{}}}{p+2} + q \times \frac{q-1}{2} \times \frac{X^{\overset{p+3}{}}}{p+3} -$ &c. and confequently the ratio of $tFCf$ to $ACFH$ is $\overline{n+1} \times E \times x^d$ into the difference between

between the two series. Compare this with prop. 10. and we shall have the following practical rule.

RULE 1.

If nothing is known concerning an event but that it has happened p times and failed q in $p + q$ or n trials, and from hence I guess that the probability of its happening in a single trial lies somewhere between any two degrees of probability as X and x, the chance I am in the right in my guess is $\overline{n+1}$ $\times E \times^{d}$ into the difference between the series $X^{p+1} \over p+1$

$$ - q \frac{X^{p+2}}{p+2} + q \times \frac{q-1}{2} \times \frac{X^{p+3}}{p+3} - \&c. \text{ and the} $$

series $\dfrac{x^{p+1}}{p+1} - q\dfrac{x^{p+2}}{p+2} + q \times \dfrac{q-1}{2} \times \dfrac{x^{p+3}}{p+3} - \&c.$ E

being the coefficient of $a^p\, b^q$ when $\overline{a + b}\,|^n$ is expanded.

This is the proper rule to be used when q is a small number; but if q is large and p small, change every where in the series here set down p into q and q into p and x into r or $1-x$, and X into R $=$ $1-$X; which will not make any alteration in the difference between the two seriefes.

Thus far Mr. Bayes's essay.

With respect to the rule here given, it is further to be observed, that when both p and q are very large numbers, it will not be possible to apply it to practice on account of the multitude of terms which the se-riefes in it will contain. Mr. Bayes, therefore, by

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an inveſtigation which it would be too tedious to give here, has deduced from this rule another, which is as follows.

R U L E 2.

If nothing is known concerning an event but that it has happened p times and failed q in $p + q$ or n trials, and from hence I gueſs that the probability of its happening in a ſingle trial lies between $\frac{p}{n} + z$ and $\frac{p}{n} - z$; if $m^2 = \frac{n^3}{pq}$ $a = \frac{p}{n}$, $b = \frac{q}{n}$, E the coefficient of the term in which occurs $a^p b^q$ when $\overline{a + b}|^n$ is expanded, and $\Sigma = \frac{n+1}{n} \times \frac{\sqrt{2pq}}{\sqrt{n}} \times E\ a^p b^q \times$ by the ſeries $mz - \frac{m^3 z^3}{3} + \frac{n-2}{2n} \times \frac{m^5 z^5}{5} - \frac{\overline{n-2} \times \overline{n-4}}{2n \times 3n}$

$\times \frac{m^7 z^7}{7} + \frac{n-2}{2n} \times \frac{n-4}{3n} \times \frac{n-6}{4n} \times \frac{m^9 z^9}{9}$ &c. my chance to be in the right is greater than

$$\overline{1 + 2\ E\ a^p b^q} + \overline{\dfrac{2\ E\ a^p b^q}{n}} \quad * \quad \text{and leſs than}$$

$$\overline{1-2\ E\ a_p b^q} - \overline{\dfrac{2\ E\ a^p b^q}{n}}.\ \text{And if } p = q \text{ my chance}$$

is $2\ \Sigma$ exactly.

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* In Mr. Bayes's manuſcript this chance is made to be gieater than $\dfrac{2\ \Sigma}{1+2\ E\ a^p b^q}$ and leſs than $\dfrac{2\ \Sigma}{1-2\ E\ a^p b^q}$. The third term in the two diviſors, as I have given them, being omitted. But this being evidently owing to a ſmall overſight in the deduction of this rule, which I have reaſon to think Mr. Bayes had himſelf diſcovered, I have ventured to correct his copy, and to give the rule as I am ſatisfied it ought to be given.

In order to render this rule fit for ufe in all cafes it is only neceffary to know how to find within fufficient nearnefs the value of E $a^p b^q$ and alfo of the feries $m z - \dfrac{m^3 z^3}{3}$ &c*. With refpect to the former Mr. Bayes has proved that, fuppofing K to fignify the ratio of the quadrantal arc to it's radius, E $a^p b^q$ will be equal to $\dfrac{\sqrt{n}}{2\sqrt{K p q}}$ × by the *ratio* whofe *hyperbolic* logarithm is

$$\tfrac{1}{12} \times \overline{\tfrac{1}{n} - \tfrac{1}{p} - \tfrac{1}{q}} - \tfrac{1}{360} \times \overline{\tfrac{1}{n^3} - \tfrac{1}{p^3} - \tfrac{1}{q^3}} + \tfrac{1}{1260} \times \overline{\tfrac{1}{n^5} - \tfrac{1}{p^5} - \tfrac{1}{q^5}} - \tfrac{1}{1680} \times \overline{\tfrac{1}{n^7} - \tfrac{1}{p^7} - \tfrac{1}{q^7}} + \tfrac{1}{1188} \times \overline{\tfrac{1}{n^9} - \tfrac{1}{p^9} - \tfrac{1}{q^9}}$$

&c. where the numeral coefficients may be found in the following manner. Call them A, B, C, D, E, &c. Then

$$A = \frac{1}{2.\,2.\,3} = \frac{1}{3.\,4}. \quad B = \frac{1}{2.\,4.\,5} - \frac{A}{3}. \quad C = \frac{1}{2.\,6.\,7} - \frac{10\,B + A}{5}.$$

$$D = \frac{1}{2.\,8.\,9} - \frac{35\,C + 21\,B + A}{7}. \quad E = \frac{1}{2.\,10.\,11} - \frac{126\,C + 84\,D + 36\,B + A}{9}. \quad F = \frac{1}{2.\,12.\,13} -$$

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* A very few terms of this feries will generally give the hyperbolic logarithm to a fufficient degree of exactnefs. A fimilar feries has been given by Mr. De Moivre, Mr. Simpfon and other eminent mathematicians in an expreffion for the fum of the logarithms of the numbers 1, 2, 3, 4, 5 to x, which fum they have afferted to be equal to $\frac{1}{2}$ log. $c + \overline{x + \frac{1}{2}} \times$ log. $x - x + \frac{1}{12 x} - \frac{1}{360} x^3 + \frac{1}{1260} x^5$ &c. c denoting the circumference of a circle whofe radius is unity. But Mr. Bayes, in a preceding paper in this volume, has demonftrated that, though this expreffion will very nearly approach to the value of this fum when only a proper number of the firft terms is taken, the whole feries cannot exprefs any quantity at all, becaufe, let x be what it will, there will be always a part of the feries where it will begin to diverge. This obfervation, though it does not much affect the ufe of this feries, feems well worth the noticeof mathematicians.

$$\frac{462\,D + 330\,C + 165\,E + 55\,B + A}{11}$$

&c. where the co-efficients of B, C, D, E, F, &c. in the values of D, E, F, &c. are the 2, 3, 4, &c. higheſt coefficients in $\overline{a+b}|^7$, $\overline{a+b}|^9$, $\overline{a+b}|^{11}$, &c. expanded; affixing in every particular value the leaſt of theſe coefficents to B, the next in magnitude to the furtheſt letter from B, the next to C, the next to the furtheſt but one, the next to D, the next to the furtheſt but two, and ſo on *.

With reſpect to the value of the ſeries $m\,z -$ $\frac{m^3\,z^3}{3} + \frac{n-2}{2n} \times \frac{m^5\,z^5}{5}$ &c. he has obſerved that it may be calculated directly when $m\,z$ is leſs than 1, or even not greater than $\sqrt{3}$: but when $m\,z$ is much larger it becomes impracticable to do this; in which caſe he ſhews a way of eaſily finding two values of it very nearly equal between which it's true value muſt lie.

The theorem he gives for this purpoſe is as follows.

Let K, as before, ſtand for the ratio of the quadrantal arc to its radius, and H for the ratio whoſe hyperbolic logarithm is $\frac{2^2-1}{2n} - \frac{2^4-1}{360\,n^3} + \frac{2^6-1}{1260\,n^5} - \frac{2^8-1}{1680\,n^7}$ &c. Then the ſeries $m\,z - \frac{m^3\,z^3}{3}$ &c. will be greater or leſs than the ſeries $\frac{H\,n}{n+1} \times \frac{\sqrt{K}}{\sqrt{2}} - \frac{n}{n+2} \times$

$$\frac{\overline{1 - 2\,m^2\,z^2}\,|^{\frac{n}{2}} + 1}{2\,m\,z} + \frac{n^2}{n+2} \times \frac{\overline{2\,m^2\,z^2}\,|^{\frac{n}{2}} + 2}{1 - \dfrac{n}{n+4} \times 4\,m^3\,z^3} +$$

* This method of finding theſe coefficients I have deduced from the demonſtration of the third lemma at the end of Mr. Simpſon's Treatiſe on the Nature and Laws of Chance.

$$\frac{3\,n^3}{n+2} \times \frac{\overline{1-\dfrac{2\,m^2\,z^2}{n}}\Big]^{\frac{n}{2}}+3}{n+4\times n+6 \times 8\,m^5 z^5} + \frac{3\times 5 \times n^4}{n+2}\times\frac{\overline{1-\dfrac{2\,m^2\,z^2}{n}}\Big]^{\frac{n}{2}}+4}{n+4\times n+6\times n+8\times 16 z^7 m^7}$$

— &c. continued to any number of terms, according as the laſt term has a poſitive or a negative ſign before it.

From ſubſtituting theſe values of E $a^p\,b^q$ and $m\,z$ — $\dfrac{m^3\,z^3}{3} + \dfrac{n-2}{2\,n} \times \dfrac{m^5\,z^5}{5}$ &c. in the 2d rule ariſes a 3d rule, which is the rule to be uſed when $m\,z$ is of ſome conſiderable magnitude.

R U L E 3.

If nothing is known of an event but that it has happened p times and failed q in $p+q$ or n trials, and from hence I judge that the probability of it's happening in a ſingle trial lies between $\dfrac{p}{n} + z$ and $\dfrac{p}{n} - z$ my chance to be right is *greater* than

$$\frac{\sqrt{K\,p\,q}\times b}{2\sqrt{K\,p\,q}+b\,n^{\frac{1}{2}}+b\,n^{-\frac{1}{2}}} \times 2H - \frac{\sqrt{2}}{\sqrt{K}}\times\frac{n+1}{n+2}\times\frac{1}{m\,z}$$

$$\times \overline{1-\frac{2\,m^2\,z^2}{n}}\Big]^{\frac{n}{2}}+1$$ and *leſs* than $\dfrac{\sqrt{K\,p\,q}\times b}{2\sqrt{K\,p\,q}-b\,n^{\frac{1}{2}}-b\,n^{-\frac{1}{2}}}$

multiplied by the 3 terms $2H - \dfrac{\sqrt{2}}{\sqrt{K}} \times \dfrac{n+1}{n+2}$

$$\times\frac{1}{m\,z}\times\overline{1-\frac{2\,m^2\,z^2}{n}}\Big]^{\frac{n}{2}}+1 + \frac{\sqrt{2}}{\sqrt{K}}\times\frac{n}{n+2}\times$$

$$\frac{n+1}{n+4}\times\frac{1}{2m^3\,z^3}\times\overline{1-\frac{2\,m^2\,z^2}{n}}\Big]^{\frac{n}{2}}+2$$ where m^2, K, b and H ſtand for the quantities already explained.

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An APPENDIX,

CONTAINING

An Application of the foregoing Rules to some particular Cases.

THE first rule gives a direct and perfect solution in all cases; and the two following rules are only particular methods of approximating to the solution given in the first rule, when the labour of applying it becomes too great.

The first rule may be used in all cases where either p or q are nothing or not large. The second rule may be used in all cases where $m z$ is less than $\sqrt{3}$; and the 3d in all cases where $m^2 z^2$ is greater than 1 and less than $\frac{n}{2}$, if n is an even number and very large. If n is not large this last rule cannot be much wanted, because, m decreasing continually as n is diminished, the value of z may in this case be taken large, (and therefore a considerable interval had between $\frac{p}{n} - z$ and $\frac{p}{n} + z$,) and yet the operation be carried on by the 2d rule; or $m z$ not exceed $\sqrt{3}$.

But in order to shew distinctly and fully the nature of the present problem, and how far Mr. Bayes has carried the solution of it; I shall give the result of this solution in a few cases, beginning with the lowest and most simple.

Let

Let us then firſt ſuppoſe, of ſuch an event as that called M in the eſſay, or an event about the probability of which, antecedently to trials, we know nothing, that it has happened *once*, and that it is enquired what concluſion we may draw from hence with reſpect to the probability of it's happening on a *ſecond* trial.

The anſwer is that there would be an odds of three to one for ſomewhat more than an even chance that it would happen on a ſecond trial.

For in this caſe, and in all others where q is nothing, the expreſſion $\overline{n+1} \times \dfrac{\overline{X^{p+1}} - \overline{x^{p+1}}}{p+1}$ or $X^{p+1} - x^{p+1}$ gives the ſolution, as will appear from conſidering the firſt rule. Put therefore in this expreſſion $\overline{p+1} = 2$, $X = 1$ and $x = \frac{1}{2}$ and it will be $1 - \overline{\tfrac{1}{2}}\rvert^2$ or $\frac{3}{4}$; which ſhews the chance there is that the probability of an event that has happened once lies ſomewhere between 1 and $\frac{1}{2}$; or (which is the ſame) the odds that it is ſomewhat more than an even chance that it will happen on a ſecond trial *.

In the ſame manner it will appear that if the event has happened twice, the odds now mentioned will be ſeven to one; if thrice, fifteen to one; and in general, if the event has happened p times, there will be an odds of $2^{p+1} - 1$ to one, for *more* than an equal chance that it will happen on further trials.

Again, ſuppoſe all I know of an event to be that it has happened ten times without failing, and the

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* There can, I ſuppoſe, be no reaſon for obſerving that on this ſubject unity is always made to ſtand for certainty, and $\frac{1}{2}$ for an even chance.

G g g enquiry

enquiry to be what reason we shall have to think we are right if we guess that the probability of it's happening in a single trial lies somewhere between $\frac{16}{17}$ and $\frac{2}{3}$, or that the ratio of the causes of it's happening to those of it's failure is some ratio between that of sixteen to one and two to one.

Here $p + 1 = 11$, $X = \frac{16}{17}$ and $x = \frac{2}{3}$ and $X^{p+1} - x^{p+1} = \overline{\frac{16}{17}}\,|^{11} - \overline{\frac{2}{3}}\,|^{11} = .5013$ &c. The answer therefore is, that we shall have very nearly an equal chance for being right.

In this manner we may determine in any case what conclusion we ought to draw from a given number of experiments which are unopposed by contrary experiments. Every one sees in general that there is reason to expect an event with more or less confidence according to the greater or less number of times in which, under given circumstances, it has happened without failing; but we here see exactly what this reason is, on what principles it is founded, and how we ought to regulate our expectations.

But it will be proper to dwell longer on this head.

Suppose a solid or die of whose number of sides and constitution we know nothing; and that we are to judge of these from experiments made in throwing it.

In this case, it should be observed, that it would be in the highest degree improbable that the solid should, in the first trial, turn any one side which could be assigned before hand; because it would be known that some side it must turn, and that there was an infinity of other sides, or sides otherwise marked, which it was equally likely that it should turn. The first

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throw

throw only ſhews that *it has* the ſide then thrown, without giving any reaſon to think that it has it any one number of times rather than any other. It will appear, therefore, that *after* the firſt throw and not before, we ſhould be in the circumſtances required by the conditions of the preſent problem, and that the whole effect of this throw would be to bring us into theſe circumſtances. That is: the turning the ſide firſt thrown in any ſubſequent ſingle trial would be an event about the probability or improbability of which we could form no judgment, and of which we ſhould know no more than that it lay ſomewhere between nothing and certainty. With the ſecond trial then our calculations muſt begin; and if in that trial the ſuppoſed ſolid turns again the ſame ſide, there will ariſe the probability of three to one that it has more of that ſort of ſides than of *all* others; or (which comes to the ſame) that there is ſomewhat in its conſtitution diſpoſing it to turn that ſide ofteneſt: And this probability will increaſe, in the manner already explained, with the number of times in which that ſide has been thrown without failing. It ſhould not, however, be imagined that any number of ſuch experiments can give ſufficient reaſon for thinking that it would *never* turn any other ſide. For, ſuppoſe it has turned the ſame ſide in every trial a million of times. In theſe circumſtances there would be an improbability that it had *leſs* than 1.400,000 more of theſe ſides than all others; but there would alſo be an improbability that it had *above* 1.600,000 times more. The chance for the latter is expreſſed by $\frac{1600000}{1600001}$ raiſed to the millioneth power ſubſtracted from unity, which is equal to .4647 &c. and

the

the chance for the former is equal to $\frac{1400000}{1400001}$ raifed to the fame power, or to .4895; which, being both lefs than an equal chance, proves what I have faid. But though it would be thus improbable that it had *above* 1.600,000 times more or *lefs* than 1.400,000 times *more* of thefe fides than of all others, it by no means follows that we have any reafon for judging that the true proportion in this cafe lies fomewhere between that of 1.600,000 to one and 1.400,000 to one. For he that will take the pains to make the calculation will find that there is nearly the probability exprefled by .527, or but little more than an equal chance, that it lies fomewhere between that of 600,000 to one and three millions to one. It may deferve to be added, that it is more probable that this proportion lies fomewhere between that of 900,000 to 1 and 1.900,000 to 1 than between any other two proportions whofe antecedents are to one another as 900,000 to 1.900,000, and confequents unity.

I have made thefe obfervations chiefly becaufe they are all ftrictly applicable to the events and appearances of nature. Antecedently to all experience, it would be improbable as infinite to one, that any particular event, before-hand imagined, fhould follow the application of any one natural object to another; becaufe there would be an equal chance for any one of an infinity of other events. But if we had once feen any particular effects, as the burning of wood on putting it into fire, or the falling of a ftone on detaching it from all contiguous objects, then the conclufions to be drawn from any number of fubfequent events of the fame kind would be to be determined in the fame manner with the conclufions juft mentioned relating to the conftitution of the folid I have

fuppofed. ——— In other words. The firſt experiment ſuppoſed to be ever made on any natural object would only inform us of one event that may follow a particular change in the circumſtances of thoſe objects; but it would not ſuggeſt to us any ideas of uniformity in nature, or give us the leaſt reaſon to apprehend that it was, in that inſtance or in any other, regular rather than irregular in its operations. But if the ſame event has followed without interruption in any one or more ſubſequent experiments, then ſome degree of uniformity will be obſerved; reaſon will be given to expect the ſame ſucceſs in further experiments, and the calculations directed by the ſolution of this problem may be made.

One example here it will not be amiſs to give.

Let us imagine to ourſelves the caſe of a perſon juſt brought forth into this, world and left to collect from his obſervation of the order and courſe of events what powers and cauſes take place in it. The Sun would, probably, be the firſt object that would engage his attention; but after loſing it the firſt night he would be entirely ignorant whether he ſhould ever ſee it again. He would therefore be in the condtion of a perſon making a firſt experiment about an event entirely unknown to him. But let him ſee a ſecond appearance or one *return* of the Sun, and an expectation would be raiſed in him of a ſecond return, and he might know that there was an odds of 3 to 1 for *ſome* probability of this. This odds would increaſe, as before repreſented, with the number of returns to which he was witneſs. But no finite number of returns would be ſufficient to produce abſolute or phyſical certainty. For let it be ſuppoſed that he has ſeen it return at regular and ſtated intervals a million of times. The concluſions

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this would warrant would be fuch as follow ——————
There would be the odds of the millioneth power
of 2, to one, that it was likely that it would return again
at the end of the ufual interval. There would be the
probability expreffed by .5352, that the odds for this
was not *greater* than 1.600,000 to 1 ; And the pro-
bability expreffed by .5105, that it was not *lefs* than
1.400,000 to 1.

It fhould be carefully remembered that thefe de-
ductions fuppofe a previous total ignorance of nature.
After having obferved for fome time the courfe of
events it would be found that the operations of nature
are in general regular, and that the powers and laws
which prevail in it are ftable and parmanent. The
confideration of this will caufe one or a few experi-
ments often to produce a much ftronger expectation of
fuccefs in further experiments than would otherwife
have been reafonable ; juft as the frequent obfervation
that things of a fort are difpofed together in any place
would lead us to conclude, upon difcovering there
any object of a particular fort, that there are laid up
with it many others of the fame fort. It is obvious
that this, fo far from contradicting the foregoing de-
ductions, is only one particular cafe to which they are
to be applied.

What has been faid feems fufficient to fhew us
what conclufions to draw from *uniform* experience.
It demonftrates, particularly, that inftead of proving
that events will *always* happen agreeably to it, there
will be always reafon againft this conclufion. In other
words, where the courfe of nature has been the moft
conftant, we can have only reafon to reckon upon a
recurrency of events proportioned to the degree of
this

this conſtancy; but we can have no reaſon for thin k-ing that there are no cauſes in nature which will *ever* inrerfere with the operations of the cauſes from which this conſtancy is derived, or no circumſtances of the world in which it will fail. And if this is true, ſup-poſing our only *data* derived from experience, we ſhall find additional reaſon for thinking thus if we ap-ply other principles, or have recourſe to ſuch confi-derations as reaſon, independently of experience, can ſuggeſt.

But I have gone further than I intended here ; and it is time to turn our thoughts to another branch of this ſubject: I mean, to caſes where an experiment has ſometimes ſucceeded and ſometimes failed.

Here, again, in order to be as plain and explicit as poſſible, it will be proper to put the following caſe, which is the eaſieſt and ſimpleſt I can think of.

Let us then imagine a perſon preſent at the drawing of a lottery, who knows nothing of its ſcheme or of the proportion of *Blanks* to *Prizes* in it. Let it further be ſuppoſed, that he is obliged to infer this from the number of *blanks* he hears drawn compared with the number of *prizes*; and that it is enquired what con-cluſions in theſe circumſtances he may reaſonably make.

Let him firſt hear *ten* blanks drawn and *one* prize, and let it be enquired what chance he will have for be-ing right if he gueſſes that the proportion of *blanks* to *prizes* in the lottery lies ſomewhere between the pro-portions of 9 to 1 and 11 to 1.

Here taking $X = \frac{11}{12}$, $x = \frac{9}{10}$, $p = 10$, $q = 1$, $n = 11$, $E = 11$, the required chance, according to the firſt rule,

rule, is $\overline{n+1} \times$ E into the difference between

$$\overset{p+1}{\mathrm{X}} - q \overset{p+2}{\mathrm{X}} \quad \text{and} \quad \overset{p+1}{x} - q \overset{p+2}{x} = 12 \times 11$$
$$\overline{p+1} \quad \overline{p+2} \quad\quad \overline{p+1} \quad \overline{p+2}$$

$$\times \frac{\overline{\frac{11}{12}}\Big|^{11} - \overline{\frac{11}{12}}\Big|^{12}}{11} - \frac{\overline{\frac{9}{10}}\Big|^{11} - \overline{\frac{9}{10}}\Big|^{12}}{12} = .07699$$

&c. There would therefore be an odds of about 923 to 76, or nearly 12 to 1 *against* his being right. Had he gueſſed only in general that there were leſs than 9 blanks to a prize, there would have been a probability of his being right equal to .6589, or the odds of 65 to 34.

Again, ſuppoſe that he has heard 20 *blanks* drawn and 2 *prizes*; what chance will he have for being right if he makes the ſame gueſs?

Here X and x being the ſame, we have $n = 22$, $p = 20$, $q = 2$, E $= 231$, and the required chance

$$\text{equal to } \overline{n+1} \times \text{E} \times \frac{\overset{p+1}{\mathrm{X}} - q\overset{p+2}{\mathrm{X}} + q \times \frac{q-1}{2} \times \overset{p+3}{\mathrm{X}}}{\underset{p+1}{} \quad \overline{p+2} \quad\quad \overline{p+3}}$$

$$- \frac{\overset{p+1}{x}}{p+1} - q\frac{\overset{p+2}{x}}{p+2} + q \times \frac{q-1}{2} \times \frac{\overset{p+3}{x}}{p+3} = .10843 \text{ &c.}$$

He will, therefore, have a better chance for being right than in the former inſtance, the odds againſt him now being 892 to 108 or about 9 to 1. But ſhould he only gueſs in general, as before, that there were leſs than 9 blanks to a prize, his chance for being right will be worſe; for inſtead of .6589 or an odds of near two to one, it will be .584, or an odds of 584 to 415.

Suppoſe,

Suppofe, further, that he has heard 40 *blanks* drawn and 4 *prizes*; what will the before-mentioned chances be?

The anfwer here is .1525, for the former of thefe chances; and .527, for the latter. There will, therefore, now be an odds of only $5\frac{1}{2}$ to 1 againft the proportion of blanks to prizes lying between 9 to 1 and 11 to 1; and but little more than an equal chance that it is lefs than 9 to 1.

Once more. Suppofe he has heard 100 *blanks* drawn and 10 *prizes*.

The anfwer here may ftill be found by the firft rule; and the chance for a proportion of blanks to prizes *lefs* than 9 to 1 will be .44109, and for a proportion *greater* than 11 to 1 .3082. It would therefore be likely that there were not *fewer* than 9 or *more* than 11 blanks to a prize. But at the fame time it will remain unlikely * that the true proportion fhould lie between 9 to 1 and 11 to 1, the chance for this being .2506 &c. There will therefore be ftill an odds of near 3 to 1 againft this.

From thefe calculations it appears that, in the circumftances I have fuppofed, the chance for being right in guefling the proportion of *blanks* to *prizes* to be nearly the fame with that of the number of *blanks*

* I fuppofe no attentive perfon will find any difficulty in this. It is only faying that, fuppofing the interval between nothing and certainty divided into a hundred equal chances, there will be 44 of them for a lefs proportion of blanks to prizes than 9 to 1, 31 for a greater than 11 to 1, and 25 for fome proportion between 9 to 1 and 11 to 1; in which it is obvious that, though one of thefe fuppofitions muft be true, yet, having each of them more chances againft them than for them, they are all feparately unlikely.

drawn in a given time to the number of prizes drawn, is continually increasing as these numbers increase; and that therefore, when they are considerably large, this conclusion may be looked upon as morally certain. By parity of reason, it follows universally, with respect to every event about which a great number of experiments has been made, that the causes of its happening bear the same proportion to the causes of its failing, with the number of happenings to the number of failures; and that, if an event whose causes are supposed to be known, happens oftener or seldomer than is agreeable to this conclusion, there will be reason to believe that there are some unknown causes which disturb the operations of the known ones. With respect, therefore, particularly to the course of events in nature, it appears, that there is demonstrative evidence to prove that they are derived from permanent causes, or laws originally established in the constitution of nature in order to produce that order of events which we observe, and not from any of the powers of chance *. This is just as evident as it would be, in the case I have insisted on, that the reason of drawing 10 times more *blanks* than *prizes* in millions of trials, was, that there were in the wheel about so many more *blanks* than *prizes*.

But to proceed a little further in the demonstration of this point.

We have seen that supposing a person, ignorant of the whole scheme of a lottery, should be led to conjecture, from hearing 100 *blanks* and 10 prizes drawn,

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* See Mr. De Moivre's Doctrine of Chances, pag. 250.

that the proportion of *blanks* to *prizes* in the lottery was fomewhere between 9 to 1 and 11 to 1, the chance for his being right would be .2506 &c. Let now enquire what this chance would be in fome higher cafes.

Let it be fuppofed that *blanks* have been drawn 1000 times, and prizes 100 times in 1100 trials.

In this cafe the powers of X and x rife fo high, and the number of terms in the two feriefes $\overline{X^{p+1}} - \frac{q}{p+2} X^{p+1}$ &c. and $\overline{x^{p+1}} - \frac{q}{p+2} x^{p+2}$ &c. become fo numerous that it would require immenfe labour to obtain the anfwer by the firft rule. 'Tis neceffary, therefore, to have recourfe to the fecond rule. But in order to make ufe of it, the interval between X and x muft be a little altered. $\frac{10}{11} - \frac{9}{10}$ is $\frac{1}{110}$, and therefore the interval between $\frac{10}{11} - \frac{1}{110}$ and $\frac{10}{11} + \frac{1}{110}$ will be nearly the fame with the interval between $\frac{9}{10}$ and $\frac{11}{12}$, only fomewhat larger. If then we make the queftion to be; what chance there would be (fuppofing no more known than that blanks have been drawn 1000 times and prizes 100 times in 1100 trials) that the probability of drawing a blank in a fingle trial would lie fomewhere between $\frac{10}{11} - \frac{1}{110}$ and $\frac{10}{11} + \frac{1}{110}$ we fhall have a queftion of the fame kind with the preceding queftions, and deviate but little from the limits affigned in them.

The anfwer, according to the fecond rule, is that this chance is greater than $\dfrac{2 \Sigma}{1 - 2 \overline{\mathrm{E} \, a^p \, b^q} + 2 \overline{\mathrm{E} \, a^p \, b^q}}{n}$

and

and lefs than $\overline{\dfrac{2\,\Sigma}{1 - 2\,\mathrm{E}\,a^p\,b^q} - 2\,\mathrm{E}\,\overline{a^p\,b^q}}$, E being $\dfrac{n+1}{n}$

$\times\,\dfrac{\sqrt{2\,p\,q}}{\sqrt{n}}\times \mathrm{E}\,a^p\,p^q\times m\,z - \dfrac{m^3\,z^3}{3} + \dfrac{n-2}{2\,n}\times \dfrac{m^5\,z^5}{5}$ &c.

By making here $1000 = p$ $100 = q$ $1100 = n$

$\frac{1}{11\,q} = z,\; m = \dfrac{\sqrt{n^3}}{\sqrt{p\,q}} = 1.048808,\, \mathrm{E}\,a^p\,b^q = \dfrac{b}{2}\times \dfrac{\sqrt{n}}{\sqrt{K\,p\,q}}$, b

being the ratio whofe hyperbolic logarithm is $\frac{1}{12}$ $\times$

$\overline{\dfrac{1}{n} - \dfrac{1}{p} - \dfrac{1}{q} - \dfrac{1}{360}}\times \overline{\dfrac{1}{n^3} - \dfrac{1}{p^3} - \dfrac{1}{q^3}} + \dfrac{1}{1260}\times \overline{\dfrac{1}{n^5} - \dfrac{1}{p^5} - \dfrac{1}{q^5}}$ &c.

and K the ratio of the quadrantal arc to radius; the former of thefe expreffions will be found to be .7953, and the latter .9405 &c. The chance enquired after, therefore, is greater than .7953, and lefs than .9405. That is; there will be an odds for being right in gueff-ing that the proportion of blanks to prizes lies *nearly* between 9 to 1 and 11 to 1, (or *exactly* between 9 to 1 and 1111 to 99) which is greater than 4 to 1, and lefs than 16 to 1.

Suppofe, again, that no more is known than that *blanks* have been drawn 10,000 times and *prizes* 1000 times in 11000 trials; what will the chance now mentioned be?

Here the fecond as well as the firft rule becomes ufelefs, the value of $m\,z$ being fo great as to render it fcarcely poffible to calculate directly the feries $\overline{m\,z -}$ $\dfrac{m^3\,z^3}{3} + \dfrac{n-2}{2\,n}\times \dfrac{m^5\,z}{5}$ — &c. The third rule, therefore, muft be ufed; and the information it gives us is, that the required chance is greater than .97421, or more than an odds of 40 to 1.

By

By calculations fimilar to thefe may be determined univerfally, what expectations are warranted by any experiments, according to the different number of times in which they have fucceeded and failed; or what fhould be thought of the probability that any particular caufe in nature, with which we have any acquaintance, will or will not, in any fingle trial, produce an effect that has been conjoined with it.

Moft perfons, probably, might expect that the chances in the fpecimen I have given would have been greater than I have found them. But this only fhews how liable we are to error when we judge on this fubject independently of calculation. One thing, however, fhould be remembered here; and that is, the narrownefs of the interval between $\frac{9}{10}$ and $\frac{11}{12}$, or between $\frac{10}{11} + \frac{1}{110}$ and $\frac{10}{11} - \frac{1}{110}$. Had this interval been taken a little larger, there would have been a confiderable difference in the refults of the calculations. Thus had it been taken double, or $z = \frac{1}{55}$, it would have been found in the fourth inftance that inftead of odds againft there were odds for being right in judging that the probability of drawing a blank in a fingle trial lies between $\frac{10}{11} + \frac{1}{55}$ and $\frac{10}{11} - \frac{1}{55}$.

The foregoing calculations further fhew us the ufes and defects of the rules laid down in the effay. 'Tis evident that the two laft rules do not give us the required chances within fuch narrow limits as could be wifhed. But here again it fhould be confidered, that thefe limits become narrower and narrower as q is taken larger in refpect of p; and when p and q are equal, the exact folution is given in all cafes by the fecond rule. Thefe two rules therefore afford

<div align="right">a direction</div>

a direction to our judgment that may be of confiderable use till fome perfon fhall difcover a better approximation to the value of the two feries's in the firft rule.

But what moft of all recommends the folution in this *Effay* is, that it is compleat in thofe cafes where information is moft wanted, and where Mr. De Moivre's folution of the inverfe problem can give little or no direction; I mean, in all cafes where either p or q are of no confiderable magnitude. In other cafes, or when both p and q are very confiderable, it is not difficult to perceive the truth of what has been here demonftrated, or that there is reafon to believe in general that the chances for the happening of an event are to the chances for its failure in the fame *ratio* with that of p to q. But we fhall be greatly deceived if we judge in this manner when either p or q are fmall. And tho' in fuch cafes the *Data* are not fufficient to difcover the exact probability of an event, yet it is very agreeable to be able to find the limits between which it is reafonable to think it muft lie, and alfo to be able to determine the precife degree of affent which is due to any conclufions or affertions relating to them.

† Since this was written I have found out a method of confiderably improving the approximation in the 2d and 3d rules by demonftrating that the expreffion $\dfrac{2\,\Sigma}{1+2\,\mathrm{E}\,a^p\,b^q+2\,\mathrm{E}\,a^p\,b^q}$ comes almoft as near to the true value wanted as there is reafon to defire, only always fomewhat lefs. It feems neceffary to hint this here; though the proof of it cannot be given.

LIII. *An*