X-Bar and S-Squared Charts

Summary
The **X-Bar and S-Squared Charts** procedure creates control charts for a single numeric variable where the data have been collected in subgroups. It creates both an X-bar chart to monitor the subgroup means and an S^2 chart to monitor the subgroup variances. Out-of-control signals are highlighted, including both points beyond the control limits and any unusual runs in the data. The charts may be constructed in either *Initial Study* (Phase 1) mode, where the current data determine the control limits, or in *Control to Standard* (Phase 2) mode, where the limits come from either a known standard or from prior data.

The output and options available in the *X-Bar and S-Squared Charts* procedure are similar to those in the *X-Bar and R Charts* procedure. This document will highlight only the differences between the two procedures. For a detailed discussion of the common options, refer to the documentation for *X-Bar and R Charts.*

Sample StatFolio: *xbarsschart.sgp*

Sample Data:
The file *pistonrings.sgd* contains measurements made on the diameter of automobile engine piston rings, taken from Montgomery (2005). The data consist of *m* = 25 samples of 5 rings each. The table below shows a partial list of the data in that file:

Sample	X1	X2	X3	X4	X5
1	74.030	74.002	74.019	73.992	74.008
2	73.995	73.992	74.001	74.011	74.004
3	73.988	74.024	74.021	74.005	74.002
4	74.002	73.996	73.993	74.015	74.009
5	73.992	74.007	74.015	73.989	74.014
6	74.009	73.994	73.997	73.985	73.993
7	73.995	74.006	73.994	74.000	74.005
8	73.985	74.003	73.993	74.015	73.988
9	74.008	73.995	74.009	74.005	74.004
10	73.998	74.000	73.99	74.007	73.995
11	73.994	73.998	73.994	73.995	73.990
12	74.004	74.000	74.007	74.000	73.996

STATGRAPHICS refers to each row of the file as a *subgroup*. The subgroup size *n* = 5.

Data Input

In entering data for this procedure, you may enter either:

1. the original measurements.
2. the subgroup means and variances.

Case #1: Entering Original Measurements

In this case, the data to be analyzed consist of the original measurements taken on a single variable.

- **Observations:** one or more numeric columns. If more than one column is entered, each row of the file is assumed to represent a subgroup with subgroup size *n* equal to the number of columns entered. If only one column is entered, then the *Date/Time/Labels or Size* field is used to form the groups.

- **Date/Time/Labels or Size**: If each set of *n* rows represents a group, enter the single value *n*. For example, entering a 5 implies that the data in rows 1-5 form the first group, rows 6-10 form the second group, and so on. If the subgroup sizes are not equal, enter the name of an additional numeric or non-numeric column containing group identifiers. The program will scan this column and place sequential rows with identical codes into the same group.

- **LSL, Nominal, USL**: optional lower specification limit, nominal (target) value, and upper specification limit.

- **Select:** subset selection.

Case #2: Entering Subgroup Statistics
In this case, the statistics for each subgroup have been computed elsewhere and entered into the datasheet, as in the table below:

Sample	Means	Variances	Sizes
1	74.0102	0.0002182	5
2	74.0006	0.0000563	5
3	74.008	0.0002175	5
4	74.003	0.0000825	5
5	74.0034	0.0001493	5
6	73.9956	0.0000758	5
7	74	0.0000305	5
8	73.9968	0.0001502	5
9	74.0042	0.0000307	5
10	73.998	0.0000395	5
11	73.9942	0.0000082	5
12	74.0014	0.0000178	5

- **Subgroup Statistics**: the names of the column containing the subgroup means, subgroup variances, and subgroup sizes.

- **Dare/Time/Labels:** optional labels for each subgroup.

- **LSL, Nominal, USL**: optional lower specification limit, nominal (target) value, and upper specification limit.

- **Select:** subset selection.

X-Bar Chart

This chart plots the subgroup means $\bar{x}_j$.
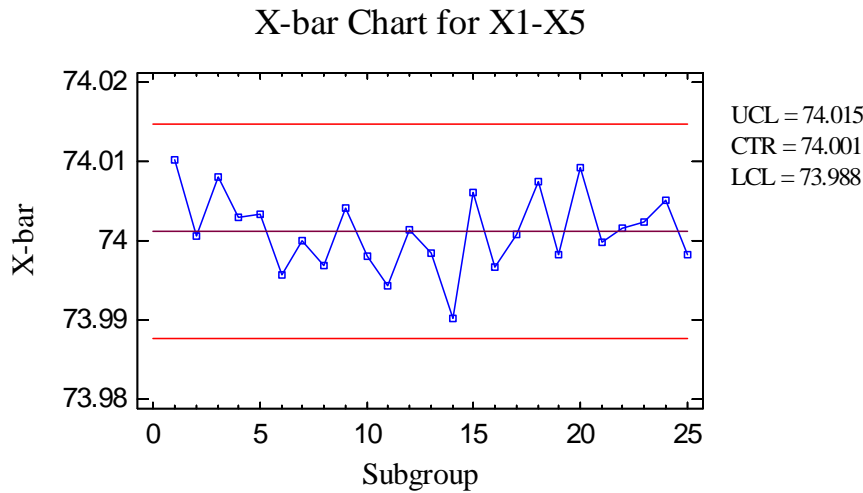
X-bar Chart for X1-X5

UCL = 74.015
CTR = 74.001
LCL = 73.988

In Phase 1 (*Initial Studies*) mode, the centerline and control limits are determined from the data. The centerline is located at the weighted average of the subgroup means:

$$\bar{\bar{x}} = \frac{\sum_{j=1}^{m} n_j \bar{x}_j}{\sum_{j=1}^{m} n_j} \qquad (1)$$

The control limits are placed above and below the centerline at:

$$\bar{\bar{x}} \pm k \frac{\hat{\sigma}}{\sqrt{n}} \qquad (2)$$

where k is the sigma multiple specified on the *Control Charts* tab of the *Preferences* dialog box ($k = 3$ except in rare cases), $\hat{\sigma}$ is the estimate of the process sigma, and n is the subgroup size. If the subgroup sizes are not equal, then depending on *Analysis Options*, n is replaced by either:

(1) $\bar{n}$, the average subgroup size. In this case, the control limits are the same for all subgroups.

(2) n_j, the individual subgroup sizes. In this case, the control limits are step functions.

The method for estimating the process sigma also depends on the settings on the *Control Charts* tab of the *Preferences* dialog box, as discussed in the *Analysis Summary* section below.

Any points beyond the control limits will be flagged using a special point symbol. Any point excluded from the analysis, usually by clicking on a point on the chart and pressing the *Exclude/Include* button, will be indicated by an X. If so indicated on the *Pane Options* dialog

box, unusual sequences of points may also be flagged. In the current chart, no unusual points or out-of-control signals are indicated.
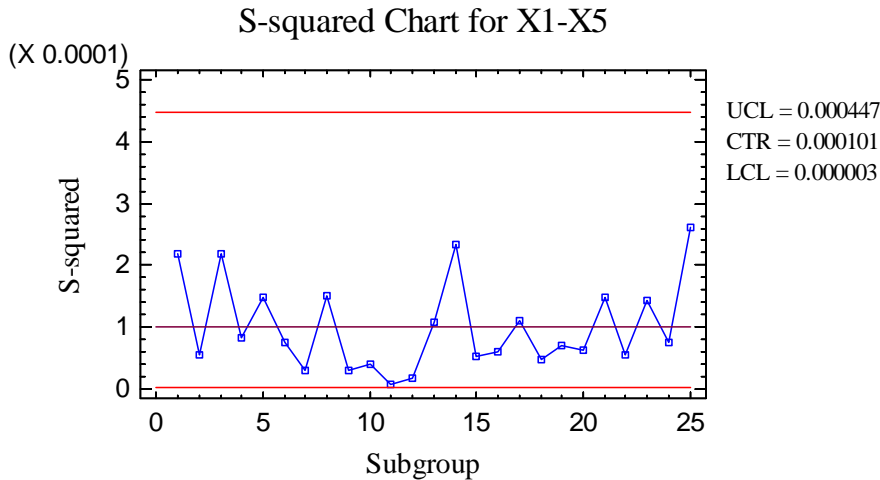
Pane Options
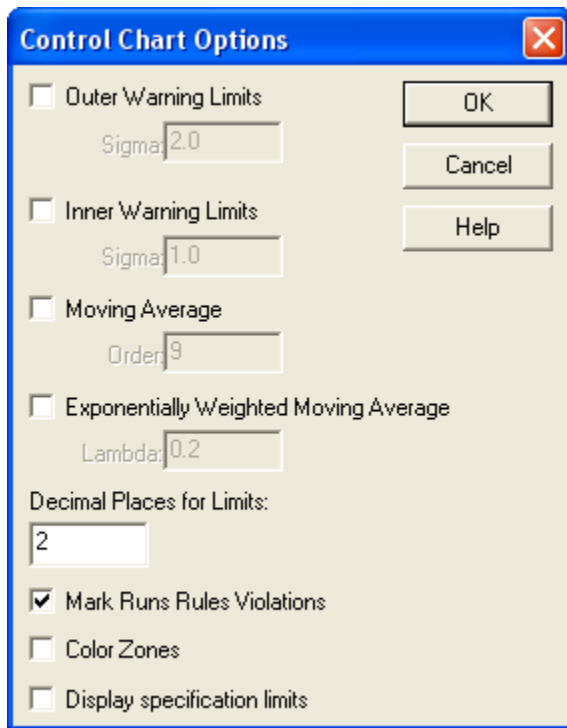
- **Outer Warning Limits**: check this box to add warning limits at the specified multiple of sigma, usually at 2 sigma.

- **Inner Warning Limits**: check this box to add warning limits at the specified multiple of sigma, usually at 1 sigma.

- **Moving Average:** check this box to add a moving average smoother to the chart. In addition to the subgroup means, the average of the most recent q points will also be displayed, where q is the *order* of the moving average. The default value $q = 9$ since the 1-sigma inner warning limits for the original subgroup means are equivalent to the 3-sigma control limits for that order moving average.

- **Exponentially Weighted Moving Average:** check this box to add an EWMA smoother to the chart. In addition to the subgroup means, an exponentially weighted moving average of the subgroup means will also be displayed, where λ is the smoothing parameter of the EWMA. The default value $\lambda = 0.2$ since the 1-sigma inner warning limits for the original subgroup means are equivalent to the 3-sigma control limits for that EWMA.

- **Decimal Places for Limits:** the number of decimal places used to display the control limits.

- **Mark Runs Rules Violations:** flags with a special point symbol any unusual sequences or runs. The runs rules applied by default are specified on the *Runs Tests* tab of the *Preferences* dialog box.

- **Color Zones:** check this box to display green, yellow and red zones.

- **Display Specification Limits**: whether to add horizontal lines to the chart displaying the location of the specification limits (if any).

S-Squared Chart

This chart plots the subgroup variances s_j^2.

S-squared Chart for X1-X5

(X 0.0001)

UCL = 0.000447
CTR = 0.000101
LCL = 0.000003

(y-axis: S-squared; x-axis: Subgroup)

In Phase 1 (*Initial Studies*) mode, the centerline and control limits are determined from the data. The centerline is located at:

$$CL = \hat{\sigma}^2 \tag{3}$$

The control limits are placed at locations determined from critical values of a chi-squared distribution:

$$UCL = \frac{\hat{\sigma}^2}{n-1} \chi^2_{\alpha/2,n-1} \tag{4}$$

$$LCL = \frac{\hat{\sigma}^2}{n-1} \chi^2_{1-\alpha/2,n-1} \tag{5}$$

where α is the false alarm probability of the chart, $\hat{\sigma}$ is the estimate of the process sigma, and n is the subgroup size. α is set to give the same false alarm probability as the sigma multiple specified on the *Control Charts* tab of the *Preferences* dialog box ($\alpha = 0.0027$ except in rare cases). If the subgroup sizes are not equal, *Analysis Options* specifies whether to use the average subgroup size or the individual subgroup sizes.

The S^2 chart for the sample data shows no unusual signals.

Pane Options
The same options exist as for the X-Bar chart.

Subgroup Reports

This pane tabulates the values plotted on the control charts:

Subgroup Reports			
All Subgroups			
X = Excluded	* = Beyond Limits		
Subgroup	*Size*	*X-bar*	*S-squared*
1	5	74.0102	0.0002182
2	5	74.0006	0.0000563
3	5	74.008	0.0002175
4	5	74.003	0.0000825
5	5	74.0034	0.0001493
6	5	73.9956	0.0000758
7	5	74.0	0.0000305
8	5	73.9968	0.0001502
9	5	74.0042	0.0000307
10	5	73.998	0.0000395
11	5	73.9942	0.0000082
12	5	74.0014	0.0000178
13	5	73.9984	0.0001093
14	5	73.9902	0.0002342
15	5	74.006	0.0000535
16	5	73.9966	0.0000608
17	5	74.0008	0.0001117
18	5	74.0074	0.0000488
19	5	73.9982	0.0000717
20	5	74.0092	0.0000637
21	5	73.9998	0.0001477
22	5	74.0016	0.0000553
23	5	74.0024	0.0001423
24	5	74.0052	0.0000757
25	5	73.9982	0.0002617

Out-of-control points are indicated by an asterisk. Points excluded from the calculations are indicated by an X.

Pane Options
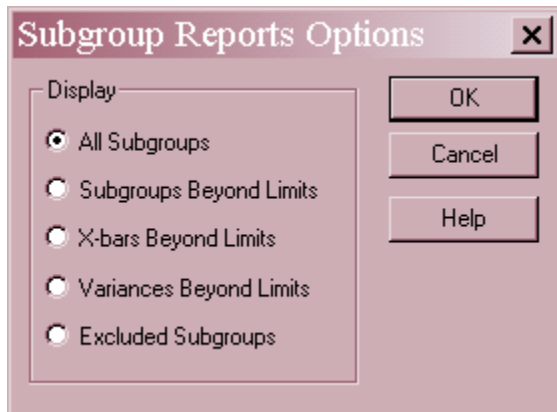
- **Display:** specify the subgroups to display in the report.

Analysis Summary

The *Analysis Summary* summarizes the data and the control charts.

X-bar and S-squared Charts - X1-X5
Number of subgroups = 25
Subgroup size = 5.0
0 subgroups excluded

Distribution: Normal
Transformation: none

X-bar Chart

Period	#1-25
UCL: +3.0 sigma	74.0147
Centerline	74.0012
LCL: -3.0 sigma	73.9877

0 beyond limits

S-squared Chart

Period	#1-25
UCL: +3.0 sigma	0.000447308
Centerline	0.000100516
LCL: -3.0 sigma	0.00000265779

0 beyond limits

Estimates

Period	#1-25
Process mean	74.0012
Process sigma	0.0100509
Pooled s	0.0100258

Sigma estimated from pooled s with bias correction

Included in the table are:

- **Subgroup Information**: the number of subgroups *m* and the average subgroup size

$$\bar{n} = \frac{\sum_{j=1}^{m} n_j}{m} \tag{6}$$

   If any subgroups have been excluded from the calculations, that number is also displayed.

- **Distribution:** the assumed distribution for the data. By default, the data are assumed to follow a normal distribution. However, one of 26 other distributions may be selected using *Analysis Options*.

- **Transformation:** any transformation that has been applied to the data. Using *Analysis Options*, you may elect to transform the data using either a common transformation such as a square root or optimize the transformation using the Box-Cox method.

- **X-bar Chart:** a summary of the centerline and control limits for the x-bar chart, which plots the subgroup means $\bar{x}_j$. Using *Analysis Options*, separate limits may be calculated for different periods (sets of subgroups).

- **S-Squared Chart:** a summary of the centerline and control limits for the S^2 chart, which plots the subgroup standard variances s_j^2.

- **Estimates:** estimates of the process mean μ and the process standard deviation σ. The process mean is estimated from the weighted average of the subgroup means:

$$\hat{\mu} = \bar{\bar{x}} \tag{7}$$

The process sigma may be estimated in any of 4 ways, depending upon the settings on the *Control Charts* tab of the *Preferences* dialog box, accessible through the *Edit* menu. There are 4 options:

  (1) *From average s with no bias correction*: the process sigma is estimated from a weighted average of the subgroup standard deviations.
  (2) *From average s with bias correction*: the process sigma is estimated from a weighted average of the subgroup standard deviations and then multiplied by a factor that makes the result an unbiased estimate of σ.
  (3) *From pooled s with no bias correction*: the process sigma is estimated from the pooled within group variance, as in a oneway ANOVA. In this case, s^2 is an unbiased estimate of σ^2 but s is a biased estimator for σ.
  (4) *From pooled s with bias correction*: the process sigma is estimated from the pooled within group variance and then multiplied by a factor that makes the result an unbiased estimate of σ.

- **Pooled s:** the pooled within subgroup standard deviation:

$$s_p = \sqrt{\frac{\sum_{j=1}^{k}(n_j - 1)s_j^2}{\sum_{j=1}^{k}(n_j - 1)}} \tag{8}$$

Analysis Options

- **Type of Study:** determines how the control limits are set. For an *Initial Study* (Phase 1) chart, the limits are estimated from the current data. For a *Control to Standard* (Phase 2) chart, the control limits are determined from the information in the *Control to Standard* section of the dialog box.

- **Normalize:** if selected, all statistics will be normalized by calculating Z-scores and the Z-scores plotted on the charts. The centerline on such a chart is always located at 0, and the control limits are always located at $\pm k$.

- **Avg. Subgroup Size:** if checked, the control limits will be horizontal lines based on the average subgroup size. If not checked, the individual subgroup sizes will be used, resulting in step function control limits if the subgroups sizes are not all equal.

- **Use Zone Format:** if checked, the statistics will be plotted using a zone chart rather than the usual format. See the example below.

- **Recalculate at:** the control limits of the chart may be recalculated at up to 4 locations along the X axis by specifying the subgroup numbers at which new estimates are to be initiated. Separate estimates of the process mean and sigma will be obtained using the data in the different sections. In such cases, the control limits will be adjusted at the start of each new period.

- **X-bar Control Limits:** specify the multiple k to use in determining the upper and lower control limits on the X-bar chart. To suppress a limit completely, enter 0.

- **S-squared Control Limits:** specify the multiple k to use in determining the upper and lower control limits on the S^2 chart. To suppress a limit completely, enter 0. The k value will be converted to an equivalent α risk for creation of the control limits.

- **Control to Standard:** to perform a Phase 2 analysis, select *Control to Standard* for the *Type of Study* and then enter either of the following:

  Specify Parameters: specify the established standard process mean and sigma (or other parameters if not assuming a normal distribution). These values will then be used to position the centerlines and control limits on the charts.

  Specify Control Limits: specify the location of the centerlines and control limits exactly where you wish them to be placed.

  Exclude button: Use this button to exclude specific subgroups from the calculations.

- **Transform Button**: Use this button to specify a transformation or non-normal distribution.

Example: Defining Multiple Sections

The time axis of the control chart can be divided into sections by requesting that the control limits be recalculated at different points in time. For example, the *Recalculate at* field on the following *Analysis Options* dialog box asks that new sections begin at subgroups 6 and 15:
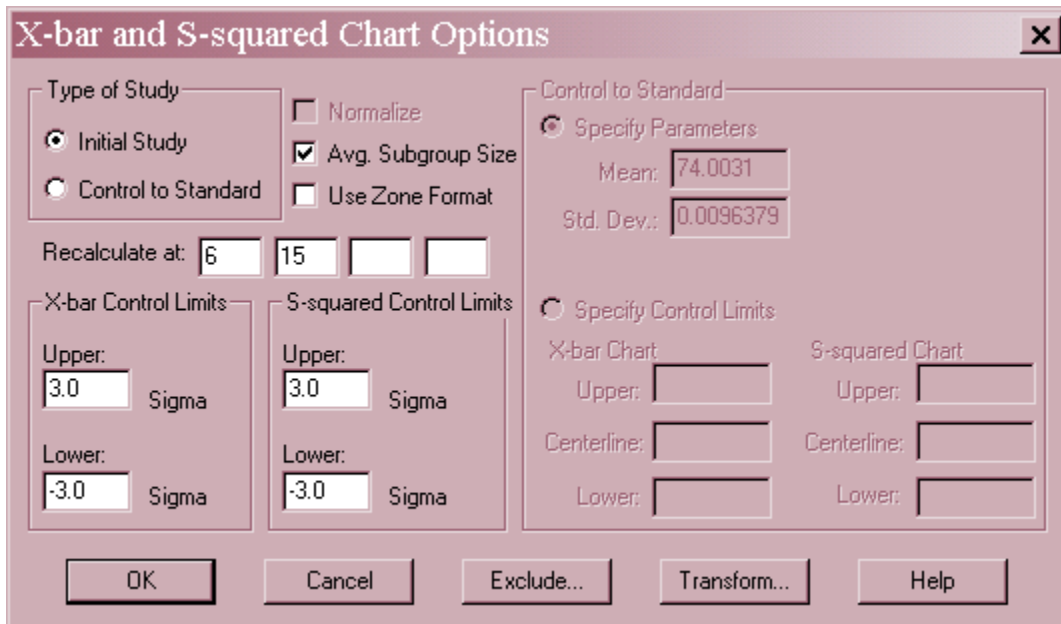
The control charts then display different limits for each section:

X-bar Chart for X1-X5

UCL = 74.0158
CTR = 74.0023
LCL = 73.9889
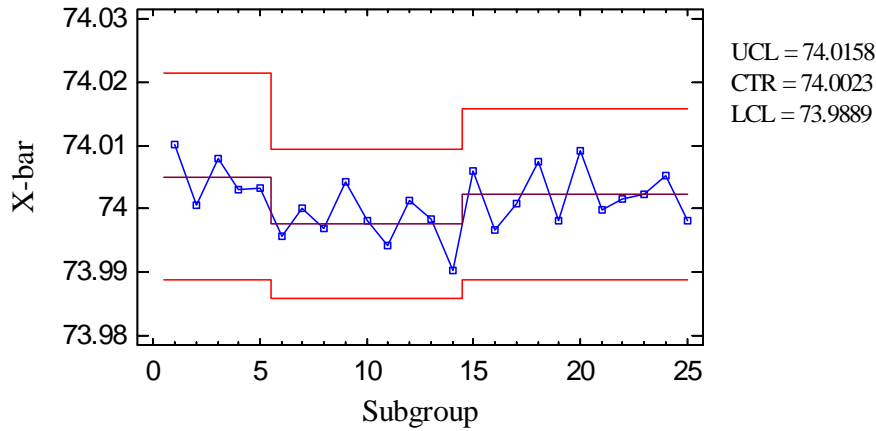
The labels along the right axis show the most recent limits.

Save Results

The following results can be saved to the datasheet:

1. *Means* – the subgroup means.
2. *Variances* – the subgroup variances.
3. *Sizes* – the subgroup sizes.
4. *Labels* – the subgroup labels.
5. *Process Mean* – the estimated process mean.
6. *Process Sigma* – the estimated process standard deviation.

<u>Calculations</u>

Estimate of Process Sigma

(1) From average s with bias no correction:

$$\hat{\sigma} = \frac{\sum\limits_{j=1}^{k} n_j s_j}{\sum\limits_{j=1}^{k} n_j} \quad (9)$$

(2) From average s with bias correction:

$$\hat{\sigma} = \frac{\sum\limits_{j=1}^{k} \left(\frac{f_j s_j}{c_4(n_j)} \right)}{\sum\limits_{j=1}^{k} h_j} \quad (10)$$

where

$$h_j = \frac{c_4^2(n_j)}{1 - c_4^2(n_j)} \quad (11)$$

(3) From pooled s with no bias correction:

$$\hat{\sigma} = \sqrt{\frac{\sum\limits_{j=1}^{k} (n_j - 1) s_j^2}{\sum\limits_{j=1}^{k} (n_j - 1)}} \quad (12)$$

(4) From pooled s with bias correction:

$$\hat{\sigma} = \frac{1}{c_4(d)} \sqrt{\frac{\sum\limits_{j=1}^{k} (n_j - 1) s_j^2}{\sum\limits_{j=1}^{m} (n_j - 1)}} \quad (13)$$

where

$$d = 1 + \sum\limits_{j=1}^{k} (n_j - 1) \quad (14)$$