

An example of short-term relative arbitrage

Robert Fernholz¹

October 9, 2015

Abstract

Long-term relative arbitrage exists in markets where the excess growth rate of the market portfolio is bounded away from zero. Here it is shown that under a time-homogeneity hypothesis this condition will also imply the existence of relative arbitrage over arbitrarily short intervals.

Suppose we have a market of stocks X_1, \dots, X_n represented by positive continuous semimartingales that satisfy

$$d \log X_i(t) = \gamma_i(t) dt + \sum_{\nu=1}^d \xi_{i\nu}(t) dW_\nu(t),$$

for $i = 1, \dots, n$, where $d \geq n \geq 2$, (W_1, \dots, W_d) is a d -dimensional Brownian motion, and the processes γ_i and $\xi_{i\nu}$ are progressively measurable with respect to the underlying filtration with γ_i locally integrable and $\xi_{i\nu}$ locally square-integrable. The process X_i represents the total capitalization of the i th company, so the total capitalization of the market is $X(t) = X_1(t) + \dots + X_n(t)$ and the *market weight processes* μ_i are defined by $\mu_i(t) = X_i(t)/X(t)$, for $i = 1, \dots, n$. The ij th *covariance process* σ_{ij} is defined by

$$\sigma_{ij}(t) \triangleq \sum_{\nu=1}^d \xi_{i\nu}(t) \xi_{j\nu}(t),$$

for $i, j = 1, \dots, n$.

A *portfolio* π is defined by its *weights* π_1, \dots, π_n , which are bounded processes that are progressively measurable with respect to the Brownian filtration and add up to one. The *portfolio value process* Z_π for π satisfies

$$d \log Z_\pi(t) = \sum_{i=1}^n \pi_i(t) d \log X_i(t) + \gamma_\pi^*(t) dt, \quad \text{a.s.},$$

where the process γ_π^* defined by

$$\gamma_\pi^*(t) \triangleq \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t) \pi_j(t) \sigma_{ij}(t) \right)$$

is called the *excess growth rate process* for π . It can be shown that if $\pi_i(t) \geq 0$, for $i = 1, \dots, n$, then $\gamma_\pi^*(t) \geq 0$, a.s. The market weights μ_i define the *market portfolio* μ , and if the market portfolio value process Z_μ is initialized so that $Z_\mu(0) = X(0)$, then $Z_\mu(t) = X(t)$ for all $t \geq 0$, a.s. Since the market weights are all positive, $\gamma_\mu^*(t) \geq 0$, a.s. This introductory material can be found in Fernholz (2002).

Let \mathbf{S} be the entropy function defined by

$$\mathbf{S}(x) = - \sum_{i=1}^n x_i \log x_i,$$

for $x \in \Delta^n$, the unit simplex in \mathbb{R}^n . We see that $0 \leq \mathbf{S}(x) \leq \log n$, where the minimum value occurs only at the corners of the simplex, and the maximum value occurs only at the point where $x_i = 1/n$ for all i . For a constant $c \geq 0$, the *generalized entropy function* \mathbf{S}_c is defined by

$$\mathbf{S}_c(x) = \mathbf{S}(x) + c,$$

¹INTECH, One Palmer Square, Princeton, NJ 08542. bob@bobfernholz.com. The author thanks Christa Cuchiero, Ioannis Karatzas, Constantinos Kardaras, Johannes Ruf, and Walter Schachermayer for their invaluable comments and suggestions regarding this research.

for $x \in \Delta^n$. It can be shown that \mathbf{S}_c generates a portfolio π with weights

$$\pi_i(t) = \frac{c - \log \mu_i(t)}{\mathbf{S}_c(\mu(t))} \mu_i(t),$$

for $i = 1, \dots, n$, and the portfolio value process Z_π will satisfy

$$d \log (Z_\pi(t)/Z_\mu(t)) = d \log \mathbf{S}_c(\mu(t)) + \frac{\gamma_\mu^*(t)}{\mathbf{S}_c(\mu(t))} dt, \quad \text{a.s.} \quad (1)$$

(see Fernholz (1999), Fernholz (2002), and Fernholz and Karatzas (2005)).

Definition 1. For $T > 0$, there is *relative arbitrage* versus the market on $[0, T]$ if there exists a portfolio π such that

$$\begin{aligned} \mathbb{P}[Z_\pi(T)/Z_\mu(T) \geq Z_\pi(0)/Z_\mu(0)] &= 1, \\ \mathbb{P}[Z_\pi(T)/Z_\mu(T) > Z_\pi(0)/Z_\mu(0)] &> 0. \end{aligned}$$

If $\mathbb{P}[Z_\pi(T)/Z_\mu(T) > Z_\pi(0)/Z_\mu(0)] = 1$, then this relative arbitrage is *strong*.

Proposition 1. For $T > 0$, suppose that for the market X_1, \dots, X_n there exists a constant $\varepsilon > 0$ such that

$$\gamma_\mu^*(t) > \varepsilon, \quad \text{a.s.},$$

for all $t \in [0, T]$, and for the entropy function \mathbf{S}

$$\text{ess inf}\{\mathbf{S}(\mu(t)) : t \in [0, T/2]\} \leq \text{ess inf}\{\mathbf{S}(\mu(t)) : t \in [T/2, T]\}. \quad (2)$$

Then there is relative arbitrage versus the market on $[0, T]$.

Proof. Let

$$A = \text{ess inf}\{\mathbf{S}(\mu(t)) : t \in [0, T/2]\}. \quad (3)$$

Since $\gamma_\mu^*(t) \geq \varepsilon > 0$ on $[0, T]$, a.s., not all the μ_i can be constantly equal to $1/n$, so

$$0 \leq A < \log n.$$

Hence, we can choose $\delta > 0$ such that $A + 2\delta < \log n$ and

$$\mathbb{P}\left[\inf_{t \in [0, T/2]} \mathbf{S}(\mu(t)) < A + \delta\right] > 0,$$

so if we define the stopping time

$$\tau_1 = \inf \{t \in [0, T/2] : \mathbf{S}(\mu(t)) \leq A + \delta\} \wedge T,$$

then

$$\mathbb{P}[\tau_1 \leq T/2] > 0.$$

We can now define a second stopping time

$$\tau_2 = \inf \{t \in [\tau_1, T] : \mathbf{S}(\mu(t)) = A + 2\delta\} \wedge T,$$

and we have $\tau_1 \leq \tau_2$, a.s.

Now consider the generalized entropy function

$$\mathbf{S}_\delta(x) \triangleq \mathbf{S}(x) + \delta,$$

for the same $\delta > 0$ as we chose above, so $\mathbf{S}_\delta(x) \geq \delta$. It follows from (1) that

$$\log (Z_\pi(\tau_2)/Z_\mu(\tau_2)) - \log (Z_\pi(\tau_1)/Z_\mu(\tau_1)) = \log \mathbf{S}_\delta(\mu(\tau_2)) - \log \mathbf{S}_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma_\mu^*(t)}{\mathbf{S}_\delta(\mu(t))} dt, \quad \text{a.s.}, \quad (4)$$

for the times τ_1 and τ_2 . Suppose we are on the set where $\tau_1 \leq T/2$, so $\tau_1 < \tau_2$, a.s., and consider two cases:

1. If $\tau_2 < T$, then

$$\log \mathbf{S}_\delta(\mu(\tau_2)) - \log \mathbf{S}_\delta(\mu(\tau_1)) \geq \log(A + 3\delta) - \log(A + 2\delta) > 0, \quad \text{a.s.},$$

and since the integral in (4) is positive, a.s., we have

$$\log(Z_\pi(\tau_2)/Z_\mu(\tau_2)) - \log(Z_\pi(\tau_1)/Z_\mu(\tau_1)) > 0, \quad \text{a.s.} \quad (5)$$

2. If $\tau_2 = T$, then $A + \delta \leq \mathbf{S}_\delta(\mu(t)) < A + 3\delta$ for $t \in [\tau_1, T]$, a.s., so

$$\log \mathbf{S}_\delta(\mu(\tau_2)) - \log \mathbf{S}_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma_\mu^*(t)}{\mathbf{S}_\delta(\mu(t))} dt > \log \frac{A + \delta}{A + 2\delta} + \frac{\varepsilon T}{2(A + 3\delta)}, \quad \text{a.s.} \quad (6)$$

Again there are two cases:

(a) If $A = 0$, let

$$\delta = \frac{\varepsilon T}{6 \log 2}, \quad (7)$$

so the left-hand side of the inequality in (6) will be positive, a.s., and (4) implies that

$$\log(Z_\pi(\tau_2)/Z_\mu(\tau_2)) - \log(Z_\pi(\tau_1)/Z_\mu(\tau_1)) > 0, \quad \text{a.s.} \quad (8)$$

(b) If $A > 0$, then

$$\lim_{\delta \downarrow 0} \left[\log \frac{A + \delta}{A + 2\delta} + \frac{\varepsilon T}{2(A + 3\delta)} \right] = \frac{\varepsilon T}{2A} > 0, \quad (9)$$

so for small enough $\delta > 0$, (6) will be positive, and (8) will be valid.

Now consider the portfolio η defined by:

1. For $t \in [0, \tau_1]$, $\eta(t) = \mu(t)$, the market portfolio.
2. For $t \in [\tau_1, \tau_2]$, $\eta(t) = \pi(t)$, the portfolio generated by \mathbf{S}_δ with δ chosen according to (7) or (9), as the case may be.
3. For $t \in [\tau_2, T]$, $\eta(t) = \mu(t)$.

If $\tau_1 = T$, then $\eta(t) = \mu(t)$ for all $t \in [0, T]$, so

$$\log(Z_\eta(T)/Z_\mu(T)) = \log(Z_\eta(0)/Z_\mu(0)), \quad \text{a.s.}$$

If $\tau_1 \neq T$, then $\tau_1 \leq T/2$ and $\tau_1 < \tau_2$, a.s. By the construction of η , we have

$$\begin{aligned} \log(Z_\eta(T)/Z_\mu(T)) - \log(Z_\eta(0)/Z_\mu(0)) &= \log(Z_\pi(\tau_2)/Z_\mu(\tau_2)) - \log(Z_\pi(\tau_1)/Z_\mu(\tau_1)) \\ &> 0, \quad \text{a.s.}, \end{aligned}$$

with the inequality following from (5) or (8), as the case may be. Since $\mathbb{P}[\tau_1 \neq T] > 0$,

$$\begin{aligned} \mathbb{P}[\log(Z_\eta(T)/Z_\mu(T)) \geq \log(Z_\eta(0)/Z_\mu(0))] &= 1, \\ \mathbb{P}[\log(Z_\eta(T)/Z_\mu(T)) > \log(Z_\eta(0)/Z_\mu(0))] &> 0, \end{aligned}$$

so there is relative arbitrage versus the market on $[0, T]$. □

Let us recall that the market is *diverse* over the interval $[0, T]$ if there exists a $\delta > 0$ such that

$$\mu_i(t) < 1 - \delta, \quad \text{a.s.},$$

for $i = 1, \dots, n$ and all $t \in [0, T]$ (see, e.g., Fernholz (2002)).

Corollary 1. *Let $T > 0$ and suppose that the market is not diverse over $[0, T/2]$ and that $\gamma_\mu^*(t) > \varepsilon > 0$ for $t \in [0, T]$. Then there is relative arbitrage versus the market on $[0, T]$.*

Proof. In this case $A = 0$ in (3). □

Remark 1. Corollary 1 can be applied to *volatility-stabilized* markets, for which Banner and Fernholz (2008) have previously shown the existence of short-term strong relative arbitrage.

Remark 2. The condition (2) can be generalized to a function A defined on $[0, T]$ by

$$A(t) = \text{ess inf}\{\mathbf{S}(\mu(t))\}.$$

If A increases over any subinterval of $[0, T]$, then an argument similar to that of case 1 in Proposition 1 will establish relative arbitrage. Moreover, Johannes Ruf has pointed out that the proof of Proposition 1 can be extended to establish relative arbitrage in the case where A is slowly (enough) decreasing on $[0, T]$. By means of a remarkable construction, Karatzas and Ruf (2015) have shown that short-term relative arbitrage does not exist for arbitrary A .

References

- Banner, A. and D. Fernholz (2008). Short-term arbitrage in volatility-stabilized markets. *Annals of Finance* 4, 445–454.
- Fernholz, R. (1999). Portfolio generating functions. In M. Avellaneda (Ed.), *Quantitative Analysis in Financial Markets*, River Edge, NJ. World Scientific.
- Fernholz, R. (2002). *Stochastic Portfolio Theory*. New York: Springer-Verlag.
- Fernholz, R. and I. Karatzas (2005). Relative arbitrage in volatility-stabilized markets. *Annals of Finance* 1, 149–177.
- Karatzas, I. and J. Ruf (2015). Lyapunov functions as portfolio generators. Technical report, Columbia University and University College London.