Arbitrage in Equity Markets

Robert Fernholz

INTECH
One Palmer Square
Princeton, NJ 08542

March 26, 1998

Abstract

Suppose that an equity market is composed of stocks that do not pay dividends. If the relative variance of each of the stocks with respect to the market is bounded away from zero, then there exist well-behaved portfolios that dominate the market portfolio.

Key words: Arbitrage, equity market.
Classification code: G11, C62.

1 Introduction

The absence of arbitrage is a common hypothesis in current financial theory (see, e.g., Duffie (1992) and Karatzas (1996)). While there are examples of markets with arbitrage, these examples appear to be mathematical oddities that do not resemble "real" equity markets. In this paper we consider an equity market composed of stocks that do not pay dividends, and we assume that the relative variance of every stock with respect to the market is bounded away from zero. We show that with this nondegeneracy condition, it is possible to construct a well-behaved portfolio that dominates the market portfolio. The portfolio we construct uses neither borrowing nor short sales, the weight of each stock in the portfolio is uniformly bounded by a fixed multiple of the market weight of that stock, and the value of the portfolio is bounded from below by a fixed positive multiple of the market value.

We construct the portfolio through the use of a portfolio generating function (see Fernholz (1999b)). The logarithmic return of the generated portfolio relative to the market portfolio can be expressed as the sum of the change in the value of the logarithm of the generating function plus a monotonically increasing process. There is a positive lower bound on the value of the generating function we use, and the nondegeneracy condition ensures that the rate of increase of the monotonic process is bounded away from zero. This combination of lower bounds implies that the generated portfolio will dominate the market portfolio.

The nondegeneracy condition we impose is broadly consistent with the observed behavior of equity markets and is similar to nondegeneracy conditions that appear in the literature. In contrast, the absence of arbitrage usually depends on some technical and non-observable condition like the existence of an equivalent martingale measure (see Harrison and Kreps (1979), Harrison and Pliska (1981), and Dybvig and Huang (1988)). Hence, our construction is not merely a mathematical anomaly, but rather shows that arbitrage can be present under realistic market hypotheses.

We shall use a model of stock price processes represented by continuous semimartingales that is fairly standard in continuous-time financial theory (see, e.g., Karatzas and Shreve (1991)). We shall make certain simplifying assumptions, among them:

1. Companies do not merge or break up, and the total number of shares of a company remains constant. The list of companies in the market is fixed.
2. There are no dividend payments.
3. There are no transaction costs, taxes, or problems with the indivisibility of shares.

2 Stochastic portfolio theory

In this section we shall review the basic definitions and results needed in the later sections. Much of the material in this section can also be found in Fernholz (1999a), but since it may be unfamiliar, it is presented here also. We shall generally follow the definitions and notation used in Karatzas and Shreve (1991) and Karatzas (1996).

Let

$$W = \{W(t) = (W_1(t), \dots, W_n(t)), \mathcal{F}_t, t \in [0, \infty)\}$$

be a standard n-dimensional Brownian motion defined on a probability space $\{\Omega, \mathcal{F}, P\}$ where $\{\mathcal{F}_t\}$ is the augmentation under P of the natural filtration $\{\mathcal{F}_t^W = \sigma(W(s); 0 \le s \le t)\}$. We say that a process $\{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is *adapted* if $X(t)$ is $\mathcal{F}_t$-measurable for $t \in [0,\infty)$. If X and Y are processes defined on $\{\Omega, \mathcal{F}, P\}$, we shall use the notation $X = Y$ if

$$P\{X(t) = Y(t), \quad t \in [0,\infty)\} = 1.$$

For continuous, square-integrable martingales $\{M(t), \mathcal{F}_t, t \in [0,\infty)\}$ and $\{N(t), \mathcal{F}_t, t \in [0,\infty)\}$, we can define the *cross-variation process* $\langle M, N\rangle$. The cross-variation process is adapted, continuous, and of bounded variation, and the operation $\langle \cdot, \cdot \rangle$ is bilinear on the real vector space of continuous, square-integrable martingales. If $M = N$, we shall use the notation $\langle M\rangle = \langle M, M\rangle$; $\langle M\rangle$ is called the *quadratic variation process* of M, and has continuous, nondecreasing sample paths. The Brownian motion process defined above is a continuous, square-integrable martingale, and it is characterized by its cross-variation processes

$$\langle W_i, W_j\rangle_t = \delta_{ij} t, \quad t \in [0,\infty),$$

where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise.

A *continuous semimartingale* $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a measurable, adapted process that has the decomposition,

$$X(t) = X(0) + M_X(t) + V_X(t), \quad t \in [0,\infty), \quad \text{a.s.}, \tag{2.1}$$

where $\{M_X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a continuous, square-integrable martingale and $\{V_X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a continuous, adapted process that is locally of bounded variation. It can be shown that this decomposition is a.s. unique (see Karatzas and Shreve (1991)), so we can define the cross-variation process for continuous semimartingales X and Y by

$$\langle X, Y\rangle = \langle M_X, M_Y\rangle,$$

where M_X and M_Y are the martingale parts of X and Y, respectively.

Definition 2.1. Let X_0 be a positive number. A *stock* $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a process of the form

$$X(t) = X_0 \exp\left(\int_0^t \gamma(s)\, ds + \int_0^t \sum_{\nu=1}^n \xi_\nu(s)\, dW_\nu(s)\right), \quad t \in [0,\infty), \tag{2.2}$$

where $\gamma = \{\gamma(t), \mathcal{F}_t, t \in [0,\infty)\}$ is measurable, adapted, and satisfies $\int_0^t |\gamma(s)| ds < \infty$, for all $t \in [0,\infty)$, a.s., and for $\nu = 1, \ldots, n$, $\xi_\nu = \{\xi_\nu(t), \mathcal{F}_t, t \in [0,\infty)\}$ is measurable, adapted, and satisfies $\int_0^t \xi_\nu^2(s) ds < \infty$ for all $t \in [0,\infty)$, a.s., and such that there exists a number $\varepsilon > 0$ for which $\xi_1^2(t) + \cdots + \xi_n^2(t) > \varepsilon$, $t \in [0,\infty)$, a.s.

It follows directly from Definition 2.1 that X is adapted, that $X(t) > 0$ for all $t \in [0,\infty)$, a.s., and that X has initial value $X(0) = X_0$. We shall set the initial value X_0 to be the total capitalization of the company represented by X at time $t = 0$, and we shall assume that this total capitalization is positive. This is equivalent to assuming that there is a single share of stock outstanding, and $X(t)$ represents its price at time t. We assume that stock shares are infinitely divisible, so there is no loss of generality in assuming a single share outstanding. The process γ is called the *growth rate (process)* of X and, for each ν, the process ξ_ν represents the sensitivity of X to the ν-th source of uncertainty W_ν.

We shall find it convenient to use a logarithmic representation for stocks (see Fernholz and Shay (1982)). Equation (2.2) is equivalent to

$$\log X(t) = \log X_0 + \int_0^t \gamma(s)\, ds + \int_0^t \sum_{\nu=1}^n \xi_\nu(s)\, dW_\nu(s),$$

or, in differential form,

$$d\log X(t) = \gamma(t)\,dt + \sum_{\nu=1}^{n} \xi_\nu(t)\,dW_\nu(t). \tag{2.3}$$

From this it is clear that $\log X(t)$ is a continuous semimartingale.

Suppose that we have a family of stocks $X_i, i = 1, \ldots, n$,

$$X_i(t) = X_0^i \exp\biggl(\int_0^t \gamma_i(s)\,ds + \int_0^t \sum_{\nu=1}^{n} \xi_{i\nu}(s)\,dW_\nu(s)\biggr), \quad t \in [0,\infty). \tag{2.4}$$

Consider the matrix valued process ξ defined by $\xi(t) = (\xi_{i\nu}(t))_{1\le i,\nu\le n}$ and define the *covariance process* σ where $\sigma(t) = \xi(t)\xi^T(t)$. The cross-variation processes for $\log X_i$ and $\log X_j$ are related to σ by

$$\langle \log X_i, \log X_j\rangle_t = \int_0^t \sigma_{ij}(s)\,ds, \quad t \in [0,\infty), \quad \text{a.s.} \tag{2.5}$$

Since the processes $\xi_{i\nu}$ are assumed to be square integrable in Definition 2.1, it follows that for all i and j,

$$\int_0^t \sigma_{ij}(s)\,ds < \infty, \quad t \in [0,\infty), \quad \text{a.s.}$$

Definition 2.2. A *market* is a family $\mathcal{M} = \{X_i, \ldots, X_n\}$ of stocks, defined as in (2.4). A *portfolio* in $\mathcal{M}$ is a measurable, adapted process $\pi = \{\pi(t) = (\pi_1(t), \ldots, \pi_n(t)), \mathcal{F}_t, t \in [0,\infty)\}$ such that $\pi(t)$ is bounded on $[0,\infty) \times \Omega$ and

$$\pi_1(t) + \cdots + \pi_n(t) = 1, \quad t \in [0,\infty), \quad \text{a.s.}$$

The processes π_i represent the respective proportions, or weights, of each stock in the portfolio. A negative value for $\pi_i(t)$ indicates a short sale. Suppose $Z_\pi(t)$ represents the value of an investment in π at time t. Then the amount invested in the i-th stock X_i will be

$$\pi_i(t)Z_\pi(t),$$

so if the price of X_i changes by $dX_i(t)$, the induced change in the portfolio value will be

$$\pi_i(t)Z_\pi(t)\frac{dX_i(t)}{X_i(t)}.$$

Hence the total change in the portfolio value at time t will be

$$dZ_\pi(t) = \sum_{i=1}^{n} \pi_i(t)Z_\pi(t)\frac{dX_i(t)}{X_i(t)},$$

or, equivalently,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^{n} \pi_i(t)\frac{dX_i(t)}{X_i(t)}. \tag{2.6}$$

Since we are interested in the behavior of portfolios, we are interested in solutions to (2.6). The following proposition and corollary are proved in Fernholz (1999a).

Proposition 2.1. *Let π be a portfolio and let*

$$\gamma_\pi(t) = \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t), \tag{2.7}$$

where

$$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)\Big). \tag{2.8}$$

Then, for any positive initial value Z_0^π, the process Z_π defined by

$$Z_\pi(t) = Z_0^\pi \exp\Big(\int_0^t \gamma_\pi(s)\,ds + \int_0^t \sum_{i,\nu=1}^n \pi_i(s)\xi_{i\nu}(s)\,dW_\nu(s)\Big), \quad t \in [0,\infty), \tag{2.9}$$

is a strong solution of (2.6)*.*

Corollary 2.1. *Let π be a portfolio and Z_π be its value process. Then for $t \in [0,\infty)$,*

$$d\log Z_\pi(t) = \sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \gamma_\pi^*(t)\,dt. \tag{2.10}$$

The process γ_π in (2.7) is called the *portfolio growth rate (process)* of the portfolio π, and γ_π^* in (2.8) is called the *excess growth rate (process)*. It was proved in Fernholz (1999a) that for portfolios with non-negative weights, the excess growth rate is non-negative.

For any stock X_i and portfolio π we can consider the quotient process X_i/Z_π defined by

$$\log\big(X_i(t)/Z_\pi(t)\big) = \log X_i(t) - \log Z_\pi(t). \tag{2.11}$$

This process is a continuous semimartingale with

$$\langle \log(X_i/Z_\pi), \log(X_j/Z_\pi)\rangle_t = \langle \log X_i, \log X_j\rangle_t - \langle \log X_i, \log Z_\pi\rangle_t - \langle \log X_j, \log Z_\pi\rangle_t + \langle \log Z_\pi\rangle_t. \tag{2.12}$$

If we define the process $\sigma_{i\pi}$ by

$$\sigma_{i\pi}(t) = \sum_{j=1}^n \pi_j(t)\sigma_{ij}(t),$$

for $i = 1, \ldots, n$, then

$$\langle \log X_i, \log Z_\pi\rangle_t = \int_0^t \sigma_{i\pi}(s)\,ds.$$

Define the *relative covariance (process)* τ^π to be the matrix valued process

$$\tau^\pi(t) = (\tau_{ij}^\pi(t))_{1\le i,j\le n},$$

where

$$\tau_{ij}^\pi(t) = \sigma_{ij}(t) - \sigma_{i\pi}(t) - \sigma_{j\pi}(t) + \sigma_{\pi\pi}(t), \tag{2.13}$$

for $i, j = 1, \ldots, n$. Then for all i and j,

$$\langle \log(X_i/Z_\pi), \log(X_j/Z_\pi)\rangle_t = \int_0^t \tau_{ij}^\pi(s)\,ds. \tag{2.14}$$

In the case that $i = j$, we know that $\langle \log(X_i/Z_\pi)\rangle_t$ is non-decreasing, so

$$\tau_{ii}^\pi(t) \ge 0, \quad t \in [0,\infty), \quad \text{a.s.}$$

3 Admissible portfolios in nondegenerate markets

In this section we impose a nondegeneracy condition on the market and consider the behavior of certain well-behaved portfolios under this condition. Let us assume from now on that the market is $\mathcal{M} = \{X_1, \ldots, X_n\}$, with $n > 1$.

Definition 3.1. The portfolio

$$\mu = \{\mu(t) = (\mu_1(t), \ldots, \mu_n(t)), \mathcal{F}_t, t \in [0, \infty)\},$$

where

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \tag{3.1}$$

for $i = 1, \ldots, n$, is called the *market portfolio (process).*

It is clear that the μ_i defined by (3.1) satisfy the requirements of Definition 2.2. If we let

$$Z(t) = X_1(t) + \cdots + X_n(t), \tag{3.2}$$

then $Z(t)$ satisfies (2.6) with proportions $\mu_i(t)$ given by (3.1). Hence, the value of the market portfolio represents the combined capitalization of all the stocks in the market. From this point on, we shall let μ exclusively represent the market portfolio and $Z(t)$ represent its value. We shall use the notation $\tau_{ij}(t) = \tau^{\mu}_{ij}(t)$ to represent relative covariances of the stocks with respect to the market as in (2.13).

Definition 3.2. The market M is *nondegenerate* if there exists an $\varepsilon > 0$ such that for $i = 1, \ldots, n$,

$$\tau_{ii}(t) \geq \varepsilon, \quad t \in (0, \infty), \quad \text{a.s.}$$

This definition is similar to the *uniform ellipticity* condition which states that there exists a number $\varepsilon > 0$ such that

$$x\sigma(t)x^T \geq \varepsilon \, \|x\|^2, \quad x \in \mathbb{R}^n, t \in [0, \infty), \quad \text{a.s.} \tag{3.3}$$

Condition (3.3) is fairly common in the literature, and can be found, for example, in Karatzas and Shreve (1991) and Karatzas and Kou (1996), and Duffie (1992). Under certain circumstances, condition (3.3) makes it possible to invoke Novikov's theorem (see Karatzas and Shreve (1991)) to prove the existence of an equivalent martingale measure (see Harrison and Kreps (1979), Harrison and Pliska (1981), and Dybvig and Huang (1988), as well as Duffie (1992) and Karatzas (1996)). For example, if (3.3) holds and γ_i and $\xi_{i\nu}$ in (2.4) are a.s. bounded on $[0, \infty)$ for $i, \nu = 1, \ldots, n$, then there exists an equivalent martingale measure (see, e.g., Karatzas (1996)). It is well known that the existence of an equivalent martingale measure implies that the market is arbitrage-free (see, e.g., Harrison and Kreps (1979), Duffie (1992), or Karatzas (1996)).

Definition 3.2 is compatible with condition (3.3), but neither one implies the other. The values of the relative variances in Definition 3.2 should be fairly simple to estimate in actual markets, while condition (3.3) could be more difficult to verify because it depends on the values of the eigenvalues of the matrix $\sigma(t)$. Both of these conditions are probably consistent with the behavior of actual equity markets. There is one significant difference, however: condition (3.3) is helpful in proving the absence of arbitrage, while Definition 3.2 implies its presence, as we shall show below.

Definition 3.3. A portfolio π is *admissible* if *i)* For $i = 1, \ldots, n$, $\pi_i(t) \geq 0, \quad t \in [0, \infty)$;

ii) There exists a constant M such that for $i = 1, \ldots, n$,

$$\pi_i(t)/\mu_i(t) \leq M, \quad t \in [0, \infty), \quad \text{a.s.};$$

iii) There exists a constant $c > 0$ such that

$$Z_\pi(t)/Z_\pi(0) \geq cZ(t)/Z(0), \quad t \in [0, \infty), \quad \text{a.s.}$$

The conditions imposed in this definition prevent the use of "doubling" strategies that will permit unlimited returns at unlimited risk (see Karatzas (1996)). Conditions of this nature are not uniform in the literature, and a portfolio that satisfies Definition 3.3 may not be "admissible" in other settings. Condition *ii* prevents arbitrarily high overweighting of any particular stock, and condition *iii* implies limited risk with respect to the market as numeraire. The market is a natural numeraire for equity managers whose performance is measured versus the market as benchmark.

Definition 3.4. Let η and ξ be portfolios. Then η *strictly dominates* ξ if there is a number $t > 0$ such that

$$P\{Z_\eta(t)/Z_\eta(0) > Z_\xi(t)/Z_\xi(0)\} = 1. \tag{3.4}$$

This definition is stronger than than the usual definition of "dominates" in which (3.4) is replaced by $P\{Z_\eta(t)/Z_\eta(0) \geq Z_\xi(t)/Z_\xi(0)\} = 1$ and $P\{Z_\eta(t)/Z_\eta(0) > Z_\xi(t)/Z_\xi(0)\} > 0$. The existence of a portfolio which strictly dominates another implies the existence of an arbitrage opportunity because we can buy a dollar's worth of the dominating portfolio and pay for it by selling a dollar's worth of the dominated portfolio short.

Theorem 3.1. *If the market $\mathcal{M}$ is nondegenerate then there exists an admissible portfolio that strictly dominates the market portfolio.*

Proof. Consider the function $\mathbf{S} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathbf{S}(x) = 1 - \frac{1}{2}\sum_{i=1}^n x_i^2. \tag{3.5}$$

By Theorem A.1 in the Appendix, there exists a portfolio π with weights

$$\pi_i(t) = \left(\frac{2 - \mu_i(t)}{\mathbf{S}(\mu(t))} - 1\right)\mu_i(t),$$

for $i = 1, \ldots, n$, which satisfies

$$\log\bigl(Z_\pi(T)/Z_\pi(0)\bigr) - \log\bigl(Z(T)/Z(0)\bigr) = \\ \log\bigl(\mathbf{S}(\mu(T))/\mathbf{S}(\mu(0))\bigr) + \int_0^T \frac{1}{2\,\mathbf{S}(\mu(t))}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \quad T \in [0, \infty), \quad \text{a.s.} \tag{3.6}$$

Let us first show that π is admissible. From (3.5) it is clear that

$$\frac{1}{2} < \mathbf{S}(\mu(t)) < 1, \quad t \in [0, \infty), \quad \text{a.s.} \tag{3.7}$$

Hence, for $i = 1, \dots, n$,

$$0 < \pi_i(t) < 3\mu_i(t), \quad t \in [0, \infty), \quad \text{a.s.},$$

so i and ii of Definition 3.3 are satisfied. Since for all i, $\tau_{ii}(t) \geq 0$, the integral in (3.6) is non-negative, and therefore

$$\log\bigl(Z_\pi(T)/Z_\pi(0)\bigr) - \log\bigl(Z(T)/Z(0)\bigr) \geq \log\bigl(\mathbf{S}(\mu(T))/\mathbf{S}(\mu(0))\bigr), \quad T \in [0, \infty), \quad \text{a.s.}$$

From (3.7) it follows that $\mathbf{S}(\mu(T))/\mathbf{S}(\mu(0)) \geq 1/2$, so

$$Z_\pi(T)/Z_\pi(0) \geq \frac{1}{2} Z(T)/Z(0), \quad T \in [0, \infty), \quad \text{a.s.},$$

and hence π is admissible.

Now we must show that π strictly dominates μ. Since $\mathcal{M}$ is nondegenerate, there is an $\varepsilon > 0$ such that for $i = 1, \dots, n$, $\tau_{ii}(t) \geq \varepsilon$, for all $t \in [0, \infty)$, a.s. Since $\sum_{i=1}^n \mu_i^2(t) \geq 1/n$,

$$\int_0^T \frac{1}{2\,\mathbf{S}(\mu(t))} \sum_{i=1}^n \mu_i^2(t) \tau_{ii}(t)\, dt \geq \frac{\varepsilon}{2n} T, \quad T \in [0, \infty), \quad \text{a.s.}$$

It follows that if $T > 2n \log 2/\varepsilon$, then $P\{Z_\pi(T)/Z_\pi(0) > Z(T)/Z(0)\} = 1$. □

4 Conclusion

In a nondegenerate equity market in which the stocks pay no dividends, it is possible to construct an admissible portfolio that strictly dominates the market portfolio.

A Appendix: Portfolio generating functions

In this appendix we shall show that certain real-valued functions of the market weights can be used to generate dynamic portfolios. The function $\mathbf{S}$ in (3.5) is such a function, and it generates the portfolio π used to prove Theorem 3.1. A general discussion of portfolio generating functions, including examples, can be found in Fernholz (1999b). Let μ be the market portfolio and $\tau = (\tau_{ij}^\mu)$ be the relative covariance process for μ, as in Section 3.

Lemma A.1. *$\mu(t)$ is in the null space of $\tau(t)$.*

Proof. It follows from (2.13) that for any portfolio π, a.s. for $t \in [0, \infty)$,

$$\pi(t)\tau(t)\pi^T(t) = \bigl(\pi(t) - \mu(t)\bigr)\sigma(t)\bigl(\pi(t) - \mu(t)\bigr)^T,$$

and the lemma follows. □

The following lemma expresses the excess growth in terms of the relative covariance process.

Lemma A.2. *Let π be a portfolio. Then for $t \in [0, \infty)$,*

$$\gamma_\pi^*(t) = \frac{1}{2}\Bigl(\sum_{i=1}^n \pi_i(t)\tau_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t)\Bigr).$$

Proof. The proof is a direct calculation using (2.13). □

We shall consider real-valued functions defined on the open simplex

$$\Delta^n = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 1, \quad 0 < x_i < 1, \quad i = 1, \ldots, n\}.$$

It will be convenient to use the standard coordinate system in $\mathbb{R}^n$, even though it is not a coordinate system on Δ^n. For this reason we shall consider functions that are defined in an open neighborhood $U \subset \mathbb{R}^n$ of Δ^n. A real-valued function defined on a subset of $\mathbb{R}^n$ is C^2 if it is twice continuously differentiable in all variables. We shall use the notation D_i for the partial derivative with respect to the i-th variable, and D_{ij} for the second partial derivative with respect to the i-th and j-th variables.

Definition A.1. Let U be an open neighborhood of Δ^n and $\mathbf{S}$ be a positive C^2 function defined in U. Then $\mathbf{S}$ is the *generating function* of the portfolio π if there exists a measurable, adapted process $\Theta = \{\Theta(t), \mathcal{F}_t, t \in [0, \infty)\}$ such that

$$d\log\big(Z_\pi(t)/Z(t)\big) = d\log \mathbf{S}(\mu(t)) + \Theta(t)dt, \quad t \in [0,\infty), \quad \text{a.s.} \tag{A.1}$$

Θ is called the *drift process* corresponding to $\mathbf{S}$.

We shall say that the function $\mathbf{S}$ *generates* π. What follows is the main theorem on portfolio generating functions.

Theorem A.1. *Let $\mathbf{S}$ be a positive C^2 function defined on a neighborhood U of Δ^n such that for $i = 1, \ldots, n$, $x_i D_i \log \mathbf{S}(x)$ is bounded on Δ^n. Then $\mathbf{S}$ generates the portfolio π with weights*

$$\pi_i(t) = \Big(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}(\mu(t))\Big)\mu_i(t), \quad t \in [0,\infty), \quad \text{a.s.,} \tag{A.2}$$

for $i = 1, \ldots, n$, and drift process

$$\Theta(t) = \frac{-1}{2\,\mathbf{S}(\mu(t))} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t), \quad t \in [0,\infty), \quad \text{a.s.} \tag{A.3}$$

Proof. The weight process μ_i is a quotient process with $\mu_i(t) = X_i(t)/Z(t)$ for all t. By (2.14) it follows that

$$d\langle \log \mu_i, \log \mu_j \rangle_t = \tau_{ij}(t)\, dt, \quad t \in [0,\infty), \quad \text{a.s.,}$$

so by Itô's Lemma,

$$d\mu_i(t) = \mu_i(t)\, d\log \mu_i(t) + \frac{1}{2}\mu_i(t)\tau_{ii}(t)\, dt, \quad t \in [0,\infty), \quad \text{a.s.,}$$

and

$$d\langle \mu_i, \mu_j \rangle_t = \mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt, \quad t \in [0,\infty), \quad \text{a.s.} \tag{A.4}$$

Itô's lemma, along with (A.4), implies that a.s. for all $t \in [0,\infty)$,

$$d\log \mathbf{S}(\mu(t)) = \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} D_{ij} \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt.$$

Now,

$$D_{ij}\log \mathbf{S}(\mu(t)) = \frac{D_{ij}\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(t))} - D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t)),$$

so, a.s., for all $t \in [0,\infty)$,

$$\begin{aligned} d\log \mathbf{S}(\mu(t)) = \sum_{i=1}^n D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t) + \frac{1}{2\,\mathbf{S}(\mu(t))} \sum_{i,j=1}^n D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt \\ - \frac{1}{2}\sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt. \end{aligned} \tag{A.5}$$

In order for (A.1) to hold, the martingale parts of $\log \mathbf{S}(\mu(t))$ and $\log\big(Z_\pi(t)/Z(t)\big)$ must be equal. Corollary 2.1 implies that for the portfolio π, a.s. for all $t \in [0,\infty)$,

$$\begin{aligned} d\log\big(Z_\pi(t)/Z(t)\big) &= \sum_{i=1}^n \pi_i(t)\, d\log\big(X_i(t)/Z(t)\big) + \gamma_\pi^*(t)\, dt \\ &= \sum_{i=1}^n \pi_i(t)\, d\log \mu_i(t) + \gamma_\pi^*(t)\, dt \\ &= \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)}\, d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t)\, dt \end{aligned}$$

by Lemma A.2. Suppose that

$$\pi_i(t) = \big(D_i \log \mathbf{S}(\mu(t)) + \varphi(t)\big)\mu_i(t), \tag{A.6}$$

where $\varphi(t)$ is chosen such that $\sum_{i=1}^n \pi_i(t) = 1$. Then, a.s. for all $t \in [0,\infty)$,

$$\begin{aligned} \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)}\, d\mu_i(t) &= \sum_{i=1}^n D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t) + \varphi(t)\sum_{i=1}^n d\mu_i(t) \\ &= \sum_{i=1}^n D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t), \end{aligned}$$

since $\sum_{i=1}^n d\mu_i(t) = 0$. Hence, the martingale parts of $\log \mathbf{S}(\mu(t))$ and $\log\big(Z_\pi(t)/Z(t)\big)$ are equal. If

$$\varphi(t) = 1 - \sum_{j=1}^n \mu_j(t) D_j \log \mathbf{S}(\mu(t)),$$

then $\sum_{i=1}^n \pi_i(t) = 1$, and (A.2) is proved.

If $\pi_i(t)$ satisfies (A.6), then a.s. for all $t \in [0,\infty)$,

$$\begin{aligned} \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t) &= \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &\quad + 2\varphi(t)\sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t) + \varphi^2(t)\sum_{i,j=1}^n \mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &= \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t), \end{aligned}$$

since $\mu(t)$ is in the null space of $\tau(t)$ by Lemma A.1. Hence, a.s. for all $t \in [0, \infty)$,

$$d \log\big(Z_\pi(t)/Z(t)\big) = \sum_{i=1}^n D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t) - \frac{1}{2} \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t)) \mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt.$$

This equation and (A.5) imply that a.s. for all $t \in [0, \infty)$,

$$d \log\big(Z_\pi(t)/Z(t)\big) = d \log \mathbf{S}(\mu(t)) - \frac{1}{2\,\mathbf{S}(\mu(t))} \sum_{i,j=1}^n D_{ij} \mathbf{S}(\mu(t)) \mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt,$$

so (A.3) is proved. The process Θ defined by (A.3) is clearly measurable and adapted. □

References

Duffie, D. (1992). *Dynamic Asset Pricing Theory*. Princeton, NJ: Princeton University Press.

Dybvig, P. and C. F. Huang (1988). Nonnegative wealth, absence of arbitrage, and feasible consumption plans. *Review of Financial Studies 1*, 377–401.

Fernholz, R. (1999a). On the diversity of equity markets. *Journal of Mathematical Economics 31*(3), 393–417.

Fernholz, R. (1999b). Portfolio generating functions. In M. Avellaneda (Ed.), *Quantitative Analysis in Financial Markets*, River Edge, NJ. World Scientific.

Fernholz, R. and B. Shay (1982). Stochastic portfolio theory and stock market equilibrium. *Journal of Finance 37*, 615–624.

Harrison, J. M. and D. Kreps (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory 20*, 381–408.

Harrison, J. M. and S. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Their Applications 11*, 215–260.

Karatzas, I. (1996). *Lectures on the Mathematics of Finance*. Providence, RI: American Mathematical Society.

Karatzas, I. and S. G. Kou (1996). On the pricing of contingent claims under constraints. *The Annals of Applied Probability 6*, 321–369.

Karatzas, I. and S. Shreve (1991). *Brownian Motion and Stochastic Calculus*. New York: Springer-Verlag.