DIVERSE MARKET MODELS OF COMPETING BROWNIAN PARTICLES WITH SPLITS AND MERGERS

IOANNIS KARATZAS AND ANDREY SARANTSEV

Abstract. We study models of regulatory breakup, in the spirit of Strong and Fouque (2011) but with a fluctuating number of companies. An important class of market models is based on systems of competing Brownian particles: each company has a capitalization whose logarithm behaves as a Brownian motion with drift and diffusion coefficients depending on its rank. We study such models with a fluctuating number of companies: If at some moment the share of the total market capitalization of a company reaches a fixed level, then the company is split into two parts of random size. Companies are also allowed to merge, when an exponential clock rings. We find conditions under which this system is non-explosive (that is, the number of companies remains finite at all times) and diverse, yet does not admit arbitrage opportunities.

1. Introduction

Stochastic Portfolio Theory (SPT) is a fairly recently developed area of mathematical finance. It tries to describe and understand characteristics of large, real-world equity markets using an appropriate stochastic framework, and to analyze this framework mathematically. It was introduced by E.R. Fernholz in the late 1990's, and was developed fully in his book [8]; a survey of somewhat more recent developments appeared in [12].

One real-world feature which this theory tries to capture, is diversity. A market model is called *diverse,* if at no time is a single stock allowed to dominate almost the entire market in terms of capitalization. To be a bit more precise, let us define the *market weight* of a certain company as the ratio of its capitalization (stock price, times the number of shares outstanding) to the total capitalization of all companies. If this market weight never gets above a certain threshold, a fixed number between zero and one, then this market model is called diverse.

Such models have one very important feature: with a fixed number of companies and a strictly non-degenerate covariance structure, they allow arbitrage on certain fixed, finite time-horizons: one can outperform the stock market in these models using fully invested, long-only portfolios. This was shown in [8, Chapter 3]; further examples of portfolios outperforming the market are given in [11], [13], [12, Section 11]. Some such models were constructed in [13], [23], [26] [28] and [12, Chapter 9]; see also the related articles [1], [22].

Another feature of large equity markets that SPT tries to capture, is that stocks with larger capitalizations tend to have smaller growth rates and smaller volatilities. In an attempt to model this phenomenon, the authors of [2] introduced a new model of *Competing Brownian Particles* (CBPs). Imagine a fixed finite number of particles moving on the real line; at each time, they are ranked from top to bottom, and each of them undergoes Brownian motion with drift and diffusion coefficients depending on its current rank. From these random motions, one constructs a market model with finitely many stocks: the logarithms of the companies' capitalizations evolve as a system of CBPs. Recently, these systems were studied extensively (see [24], [6], [14], [18], [10], [15], [16], [17], [27], [25], [31]) and were generalized in several directions: [32], [9], [20], [18], [30], [29]. However, these market models are *not* diverse; see [2, Section 7] and Remark 7 below.

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We would like to alter the CBP-type model a bit, in order to make it diverse. In real equity markets, diversity is a consequence of anti-monopolistic legislation and regulation: when a company becomes dominant, a governmental agency (the "regulator") forcefully splits it into smaller companies. We implement this idea in our model.

In this paper, we construct a diverse model from the above CBP-based one. We fix a certain *threshold* between 0 and 1. When a company's market weight reaches this threshold, the regulator enforces a breakup of the company into two (random) parts. We also allow for the opposite phenomenon: companies can merge at random times. The mechanism for merging companies is as follows: immediately after a split or merger, an exponential clock (with rate depending on the number of extant companies) is set. If the clock rings before any market weight has hit the threshold, the regulator picks two companies at random (according to a certain rule described right below) to be merged. If the action results in a company with market weight exceeding the threshold, then this putative merger is suppressed; otherwise, it is allowed to proceed.

We use the following rule for mergers: The company which currently occupies the highest capitalization rank is excluded from consideration, and two of the remaining $N-1$ companies are chosen randomly, according to the uniform distribution over the $\binom{N-1}{2}$ possible choices. With this rule, and with a threshold sufficiently close to 1, the merger will always be allowed to proceed. In this manner, the process of capitalizations evolves as an exponentiated system of competing Brownian particles, until either (i) one of the market weights hits the threshold, or (ii) the exponential clock rings. In case (i), the number of companies will increase by one; in case (ii), it will decrease by one.

We refer the reader to the very interesting paper [34], which considers general (that is, not just CBP-based) equity market models of regulatory breakup with a split when a market weight reaches the given threshold. Mergers in that paper obey a different rule than they do here: at the moment of any split, there is a simultaneous merger of the two smallest companies, so the total number of companies remains constant. We feel that this feature is a bit too restrictive, so in the model developed here mergers are allowed to happen independently of splits. This comes at a price, which is both "technical" and substantive: the number of companies in the model is now fluctuating randomly, in ways that need to be understood before any reasonable analysis can go through. The foundational theory for generic market models with a randomly varying number of stocks was developed by W. Strong in the important and very useful article [33].

1.1. **Preview.** The main results of this paper are as follows. First, we show that under certain conditions the *counting process* of stocks is *non-explosive:* the number of companies does not become infinite in finite time, so the model can be defined on infinite time horizons. Secondly, this model turns out to admit an equivalent martingale measure by means of a suitable Girsanov transformation: *although diverse, the model proscribes arbitrage.* This is in contrast with the models from [12], where splits/mergers are not allowed. Indeed, it was observed in [33] that in the presence of splits/mergers, diversity might not lead to arbitrage; in [34], W. Strong & J.P. Fouque established this for their models with a fixed number of companies. We establish the same result for our model, which allows the number of extant companies to fluctuate randomly.

The paper is organized as follows. Section 2 provides an informal yet somewhat detailed description of this model, and states the main results. Section 3 lays out the formal construction of the model. Section 4 is devoted to the proofs of our results. The Appendix develops a technical result.

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2. Informal Construction and Main Results

2.1. **Description of The Model.** Consider a stock market with a variable number of companies

$$X(\cdot) = \{X(t),\ 0 \le t < \infty\}, \qquad X(t) = \big(X_1(t), \ldots, X_{\mathcal{N}(t)}(t)\big)',$$

where $X_i(t) > 0$ is the capitalization of the company i at time $t \ge 0$, and $\mathcal{N}(t)$ is the number of companies in the market at that time. The integer-valued random function $t \mapsto \mathcal{N}(t)$ will be piecewise constant; we shall call it the *counting process* of our model, as it records the number of companies that are extant at any given time.

At each interval of constancy of this process, say on the set $\{t \ge 0 : \mathcal{N}(t) = N\}$, the logarithms $Y_i(\cdot) = \log X_i(\cdot),\ i = 1, \ldots, N$ behave like a system of Competing Brownian Particles (CBPs) with rank-dependent drifts and variances. More precisely, the kth largest among the N real-valued processes $Y_1(\cdot), \cdots, Y_N(\cdot)$ behaves like Brownian motion with local drift g_{Nk} and local variance σ^2_{Nk}, for $k = 1, \ldots, N$. The quantities g_{Nk} and $\sigma_{Nk} > 0$ with $N \ge 2$, $1 \le k \le N$ are given real constants. When the *market weight*

$$\mu_i(t) = \frac{X_i(t)}{\mathcal{C}(t)}, \qquad \mathcal{C}(t) := X_1(t) + \ldots + X_{\mathcal{N}(t)}(t) \tag{1}$$

of some company $i = 1, \cdots, \mathcal{N}(t)$ reaches a given, fixed lthreshold $1-\delta$, a governmental regulatory agency splits this company into two new companies; one with capitalization $\xi X_i(t)$, and the other with capitalization $(1-\xi)X_i(t)$. Here the random variable ξ is independent of everything that has happened in the past, and has a given probability distribution F on $[1/2, 1)$; whereas $\delta \in (0, 1/2)$ is a given constant.

In addition, for every integer $N \ge 2$ there is an exponential clock with rate $\lambda_N \ge 0$ (a rate of zero means that the clock never rings); we take formally $\lambda_2 = 0$, cf. Remark 1 below. When this clock rings, two companies are chosen at random, to be merged and form one company. The choice is made according to a certain probability distribution $\mathcal{P}_N(X(t))$ on the family of subsets of $\{1, \ldots, N\}$ which contain exactly two elements, and this distribution depends on the current state $X(t)$ of the system. (One example of such dependence is given below, in Assumption 4; additional clarification is provided in Subsection 3.1.) If the so-amalgamated company has market weight larger than or equal to $1-\delta$, the merger is suppressed; otherwise, the merger is allowed to proceed.

Within the framework of the model thus described in an informal way, and more formally in Section 3 below, we raise and answer the following questions:

(i) Are there explosions in this model (i.e., can the number of companies become infinite in finite time) with positive probability? Can this model be defined on an infinite time-horizon?
(ii) What is the concept of a portfolio in this model? Does the model admit (relative) arbitrage?

The answers are described in Theorems 2.1 and 2.2 below.

Remark 1. We note that this model is free of *implosions,* by its construction: when there are only two companies, their putative merger would result in a company with market weight equal to 1 and would thus be suppressed. This is the reason we took at the outset $\lambda_2 = 0$, meaning that with only two companies present the merger clock never rings. As a result, at any given moment there are at least two companies in the equity market model under consideration. □

2.2. **Portfolios and Wealth Processes.** In the context of the above model, a *portfolio* is a process

$$\pi(\cdot) = \{\pi(t),\ \ 0 \le t < \infty\}, \qquad \pi(t) = \big(\pi_1(t), \cdots, \pi_{\mathcal{N}(t)}(t)\big)'$$

for which there exists some real constant $K_\pi \ge 0$ such that $|\pi_i(t)| \le K_\pi$ holds for all $0 \le t < \infty$ and $i = 1, \cdots, \mathcal{N}(t)$. The quantity $\pi_i(t)$ is called the *portfolio weight* assigned at time t to the company i; whereas

$$\pi_0(t) := 1 - \sum_{i=1}^{\mathcal{N}(t)} \pi_i(t), \quad 0 \le t < \infty \tag{2}$$

represents the proportion of wealth invested at time t in a money market with zero interest rate. A portfolio is called *fully invested,* if it never touches the money market, i.e., if $\pi_0(\cdot) \equiv 0$; it is called *long-only,* if $\pi_i(t) \ge 0$ holds for all $i = 0, 1, \cdots, \mathcal{N}(t)$, $0 \le t < \infty$.

The prototypical fully invested, long-only portfolio is the *market portfolio* $\pi(\cdot) \equiv \mu(\cdot)$ of (1). At the other extreme stands the *cash portfolio* $\pi(\cdot) \equiv \kappa(\cdot)$ with $\kappa_i(t) = 0$ for all $i = 1, \cdots, \mathcal{N}(t)$, $0 \le t < \infty$, which never touches the equity market.

When the counting process $\mathcal{N}(\cdot)$ jumps up (after a split) or down (after a merger), the portfolio process behaves as follows:
(i) if two companies merge, the portfolio weight corresponding to the new company's stock is the sum of the portfolio weights corresponding to the two old stocks; whereas
(ii) if a company gets split into two new ones, its weight in the portfolio is partitioned in proportion to the weights of the newly-minted companies.
The formal description is postponed to Section 3.

Suppose now that a small investor, whose actions cannot influence asset prices, starts with initial capital \$1 and invests in the stock market according to a portfolio rule $\pi(\cdot)$. The corresponding *wealth process* $V^\pi(\cdot) = \{V^\pi(t),\ 0 \le t < \infty\}$ takes then values in $(0, \infty)$, satisfies

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_{i=1}^{\mathcal{N}(t)} \pi_i(t) \, \frac{\mathrm{d}X_i(t)}{X_i(t)}, \tag{3}$$

and is not affected when the number of companies changes, i.e., when the counting process $\mathcal{N}(\cdot)$ jumps. For a derivation of (3) with a fixed number of companies, see for instance [8, p.6].

We say that a portfolio $\pi(\cdot)$ *represents an arbitrage opportunity relative to another portfolio* $\rho(\cdot)$ over the time horizon $[0, T]$, for some real number $T > 0$, if

$$\mathbf{P}\big(V^\pi(T) \ge V^\rho(T)\big) = 1, \quad \mathbf{P}\big(V^\pi(T) > V^\rho(T)\big) > 0. \tag{4}$$

In words: over the time-horizon $[0,T]$, the portfolio $\pi(\cdot)$ performs at least as well as $\rho(\cdot)$ with probability one, and strictly better with positive probability.

2.3. **Main Results.** Let us impose some conditions on the parameters of this model. A salient feature of real-world markets is that stocks with smaller market weights tend to have larger drift coefficients (growth rates), so it is not unreasonable to impose the following condition:

Assumption 1.

$$g_{N1} \le \min_{2 \le k \le N} g_{Nk} \quad \text{for every} \quad N \ge 2.$$

We shall also impose the following conditions:

Assumption 2.

$$\sigma := \sup_{\substack{N \ge 2 \\ 1 \le k \le N}} \sigma_{Nk} < \infty, \quad \sigma_0 := \inf_{\substack{N \ge 2 \\ 1 \le k \le N}} \sigma_{Nk} > 0, \quad \sup_{\substack{N \ge 2 \\ 2 \le k \le N}} |g_{Nk}| < \infty, \quad \delta \in \Big(0, \frac{1}{6}\Big).$$

Assumption 3. The probability distribution F of the random variable ξ responsible for splitting companies is supported on the interval $[1/2, 1 - \varepsilon_0]$, where $\varepsilon_0 \in (0, 1/2)$. In other words,

$$\xi \backsim F \quad \Rightarrow \quad \operatorname{ess\,sup} \xi < 1. \tag{5}$$

Assumption 4. The rule for picking companies to be merged is as follows: With $N \geq 3$, we exclude the company which occupies the highest rank in terms of capitalization and choose at random two of the remaining $N-1$ companies according to the uniform distribution over the

$$m_N = \binom{N-1}{2} \tag{6}$$

possible such choices. If two or more companies are tied in terms of capitalization, we resolve the tie *lexicographically,* that is, always in favor of the lowest index.

Assumption 5. The rates of the exponential clocks satisfy, for some real constants c and $\alpha > 0$, the asymptotics

$$\lambda_N \backsim c\, N^{\alpha}\,, \quad N \to \infty\,. \tag{7}$$

Remark 2. This condition is perhaps the most significant one: it ensures that mergers happen with sufficient intensity, so that the number of companies in the model will not only not become infinite in a finite amount of time, but will also have a "tame" temporal growth (cf. Proposition 4.1 below).

As a visualization for the condition (7), suppose there are N companies; then, according to the rules of Assumption 4, there are m_N such possible mergers as in (6). If each pair of companies has its own merger exponential clock Ξ_i with the same parameter λ, and if $\Xi_1, \ldots, \Xi_{m_N}$ are independent, then the earliest merger will happen at the smallest of those exponential clocks; but

$$\min\left(\Xi_1, \ldots, \Xi_{m_N}\right) \backsim \mathcal{E}(m_N \lambda)\,,$$

so $\lambda_N = m_N \lambda \asymp N^2$ as $N \to \infty$. That is, (7) holds in this case with $\alpha = 2$. □

The following two theorems are our main results. They are proved in Section 4.

Theorem 2.1. *Under the Assumptions 1-5, the above market model is free of explosions and can thus be defined on an infinite time-horizon.*

Theorem 2.2. *Under the Assumptions 1-5, no relative arbitrage is possible over any given time horizon* $[0,T]$ *of finite length.*

3. Formal Construction

3.1. **Notation.** For $N \geq 2,\ \delta \in (0,1)$, let

$$\Delta^N_+ := \left\{(z_1, \cdots, z_N) \in \mathbb{R}^N \mid z_1 > 0\,, \cdots,\ z_N > 0\,,\ z_1 + \ldots + z_N = 1\right\},$$

$$\Delta^{N,\delta}_+ := \left\{(z_1, \ldots, z_N) \in \Delta^N_+ \mid z_1 \leq 1 - \delta\,, \cdots,\ z_N \leq 1 - \delta\right\},$$

$$\mathcal{J}^{N,\delta} := \left\{(z_1, \ldots, z_N) \in \Delta^N_+ \mid \max_{1 \leq i \leq N} z_i = 1 - \delta\,\right\}.$$

For $x \in \mathcal{S}$, let $\mathfrak{N}(x)$ be the number of components of x, i.e., the integer $N \geq 2$ for which $x \in (0,\infty)^N$; and with $N = \mathfrak{N}(x)$ we denote by $\mathfrak{z}(x) \in \Delta^N_+$ the vector with components

$$\mathfrak{z}_i(x) := \frac{x_i}{x_1 + \cdots + x_N}\,, \qquad i = 1, \cdots, N\,.$$

For any vector $y \in \mathbb{R}^N$, we denote by $y_{(1)} \geq y_{(2)} \geq \ldots \geq y_{(N)}$ its *ranked components;* in this ranking, ties are resolved lexicographically, always in favor of the lowest index as in [2], [18] and in Assumption 4. Let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

For every $N \geq 2$, we denote by $\mathcal{Q}_N$ the set of all probability distributions on the family of subsets of $\{1, \ldots, N\}$ which consist of exactly two elements.

We say that the market weight process $\mu(\cdot) = \mathfrak{z}\big(X(\cdot)\big)$ is *on the level* N at time t, if $\mu(t) \in \Delta_+^N$. We also denote

$$\mathcal{S} := \bigcup_{N=2}^{\infty} (0,\infty)^N\,, \qquad \mathcal{M} := \bigcup_{N=2}^{\infty} \Delta_+^N\,, \qquad \mathcal{M}^\delta := \bigcup_{N=2}^{\infty} \Delta_+^{N,\delta}. \tag{8}$$

Remark 3. Under the Assumption 4, the probability distributions $\{\mathcal{P}_{\mathfrak{N}(x)}(x)\}_{x\in\mathcal{S}}$ in Section 2 are constructed thus: For any given $x \in \mathcal{S}$ we let $N = \mathfrak{N}(x)$, rank lexicographically the components of the vector x, and consider the smallest index $j \in \{1,\dots,N\}$ such that $x_j \ge x_k$ holds for all $k = 1,\dots,N$. Then $\mathcal{P}_N(x) \in \mathcal{Q}_N$ is the uniform distribution on the family of subsets of $\{1,\dots,N\}\setminus\{j\}$ that contain exactly two elements (there are m_N such subsets, as in (6)). □

3.2. **Fixed number of companies.** First, let us formally introduce CBP-based models with a fixed, finite number of particles. These will serve as building blocks for our ultimate model; in [34], similar preparatory models are referred to as "premodels".

Fix an integer $N \ge 2$, and consider a system of N particles moving on the real line, formally expressed as one $\mathbb{R}^N$−valued process

$$Y(\cdot) = \big\{Y(t),\ 0 \le t < \infty\big\}\,, \qquad Y(t) = \big(Y_1(t),\dots,Y_N(t)\big)'.$$

Consider a filtered probability space $\big(\Omega,\mathcal{F},\mathbf{P}\big)$, $\mathbb{G} = \{\mathcal{G}(t)\}_{0\le t<\infty}$, where the filtration satisfies the *usual conditions* of right-continuity and augmentation by null sets, and let $W(\cdot) = \{W(t),\ 0 \le t < \infty\}$ be standard N-dimensional $\big(\mathbb{G}\,,\mathbf{P}\big)$−Brownian motion.

Definition 1. A *finite system of CBPs with symmetric collisions,* is an $\mathbb{R}^N$−valued process governed by the system of stochastic differential equations

$$\mathrm{d}Y_i(t) \,=\, \sum_{k=1}^{N} \mathbf{1}_{\{Y_i(t)=Y_{(k)}(t)\}}\,\big(g_k\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_i(t)\big)\,, \qquad i = 1,\dots,N, \quad 0 \le t < \infty\,.$$

Here, $g_1,\dots,g_N$ are given real numbers, and $\sigma_1,\dots,\sigma_N$ are given positive real numbers. □

Informally, such a model posits that the kth largest particle moves as a one-dimensional Brownian motion with local drift g_k and local variance σ_k^2. We denote the ranked (in decreasing order) statistics for the components of this system as

$$\max_{1\le i\le N} Y_i(\cdot) =: Y_{(1)}(\cdot) \ge Y_{(2)}(\cdot) \ge \cdots \ge Y_{(N)}(\cdot) := \min_{1\le i\le N} Y_i(\cdot)\,, \tag{9}$$

and set $\Lambda_{(k,k+1)}(\cdot) = \{\Lambda_{(k,k+1)}(t),\ 0 \le t < \infty\}$ for the local time accumulated at the origin by the nonnegative semimartingales $Y_{(k)}(\cdot) - Y_{(k+1)}(\cdot) = \big\{Y_{(k)}(t) - Y_{(k+1)}(t),\, 0 \le t < \infty\big\}$ with $k = 1,\cdots,N-1$ (for notational convenience, we set also $\Lambda_{(0,1)}(\cdot) \equiv \Lambda_{(N,N+1)}(\cdot) \equiv 0$ for all $t \in [0,\infty)$). Then we have

$$\mathrm{d}Y_{(k)}(t) = g_k\mathrm{d}t + \sigma_k\mathrm{d}B_k(t) + \frac{1}{2}\,\mathrm{d}\Lambda_{(k,k+1)}(t) - \frac{1}{2}\,\mathrm{d}\Lambda_{(k-1,k)}(t)\,, \qquad 0 \le t < \infty \tag{10}$$

for the dynamics of the ranked semimartingales in (9), where

$$B_k(\cdot) \,:=\, \sum_{i=1}^{N} \int_0^{\cdot} \mathbf{1}_{\{Y_i(t)=Y_{(k)}(t)\}}\,\mathrm{d}W_i(t)\,, \qquad k = 1,\cdots,N \tag{11}$$

are independent standard Brownian motions by the P. Lévy theorem. We refer to [2], [18, Lemma 1] and [14] for the derivation of (10), as well as to [4] for the existence and uniqueness in distribution of a weak solution to the CBP system of Definition 1. As shown in [16] pathwise uniqueness, thus also existence of a strong solution, also hold for this system up until the first time three particles collide – and this never happens if the mapping $k \mapsto \sigma_k^2$ is concave (cf. [16], [30], [29]).

The *CBP-based market model* with a fixed number N of companies is defined as a collection of N real-valued, strictly positive stochastic processes

$$X_i(\cdot) = \big\{ X_i(t),\ 0 \le t < \infty \big\}, \qquad i = 1, \ldots, N, \quad X_i(t) = e^{Y_i(t)},$$

whose dynamics are given by

$$\mathrm{d} \log X_i(t) \ = \ \sum_{k=1}^{N} \mathbf{1}_{\{X_i(t) = X_{(k)}(t)\}} \big[\, g_k \, \mathrm{d}t + \sigma_k \, \mathrm{d}W_i(t) \,\big] \,, \tag{12}$$

or equivalently

$$\frac{\mathrm{d}X_i(t)}{X_i(t)} \ = \ \sum_{k=1}^{N} \mathbf{1}_{\{X_i(t) = X_{(k)}(t)\}} \left[\Big(g_k + \frac{\sigma_k^2}{2} \Big) \mathrm{d}t + \sigma_k \, \mathrm{d}W_i(t) \right]. \tag{13}$$

In this model, the vector process $\mu(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_N(\cdot) \big)' = \mathfrak{z}\big(X(\cdot) \big)$ of market weights $\mu_i(t) := X_i(t) / \big(X_1(t) + \cdots + X_N(t) \big)$ with $i = 1, \cdots, N$, $0 \le t < \infty$ for its various companies, evolves as an N-dimensional diffusion governed by the system of SDEs

$$\begin{aligned} \mathrm{d} \log \mu_i(t) = & \left[\sum_{k=1}^{N} g_k \, \mathbf{1}_{\{\mu_i(t) = \mu_{(k)}(t)\}} - \sum_{k=1}^{N} g_k \, \mu_{(k)}(t) \ - \frac{1}{2} \sum_{k=1}^{N} \sigma_k^2 \Big(\mu_{(k)}(t) - \mu_{(k)}^2(t) \Big) \right] \mathrm{d}t \\ & + \sum_{k=1}^{N} \sigma_k \left[\mathbf{1}_{\{\mu_i(t) = \mu_{(k)}(t)\}} \, \mathrm{d}W_i(t) - \mu_{(k)}(t) \sum_{\nu=1}^{N} \mathbf{1}_{\{\mu_\nu(t) = \mu_{(k)}(t)\}} \mathrm{d}W_\nu(t) \right], \quad i = 1, \cdots N. \end{aligned} \tag{14}$$

(We derive this system from the general expression in equation (2.4) of [12, Section 2]. Substituting in that expression the concrete values of drift and covariance coefficients for the CBP-based market model of (12) under consideration, we arrive at the dynamics of (14) for the $\log \mu_i(\cdot)$'s.)

Remark 4. In the terminology of [34], Remark 2, the companies in models of this sort are "generic": The characteristics of their capitalizations' dynamics depend entirely on the ranks the companies occupy in the capitalization hierarchy; they are not idiosyncratic (i.e., name- or sector-dependent).

3.3. **Formal construction of the main model.** Let us begin the formal construction of our model. This will take the form of a process $X(\cdot) = \big\{ X(t),\ 0 \le t < \infty \big\}$ on the state-space $\mathcal{S}$ of (8).

For every $N \ge 2$ and every $x = (x_1, \cdots, x_N)' \in (0, \infty)^N$, we construct a probability space $(\Omega^{N,x}, \mathcal{F}^{N,x}, \mathbf{P}^{N,x})$ which contains countably many i.i.d. copies $Y^{N,x,n}(\cdot)$, $n \in \mathbb{N}$ of the solution

$$Y^{N,x}(\cdot) \ = \ \big\{ Y^{N,x}(t), \ \ 0 \le t < \infty \big\} \,, \qquad Y^{N,x}(t) \ = \ \big(Y_1^{N,x}(t), \ldots, Y_N^{N,x}(t) \big)'$$

to the following system of stochastic differential equations:

$$\mathrm{d}Y_i^{N,x}(t) = \sum_{k=1}^{N} \mathbf{1}_{\{Y_i^{N,x}(t) = Y_{(k)}^{N,x}(t)\}} \big(g_{Nk} \, \mathrm{d}t + \sigma_{Nk} \, \mathrm{d}W_i(t) \big) \,, \quad Y_i^{N,x}(0) = \log x_i \,, \quad i = 1, \cdots, N \,. \tag{15}$$

Here $W(\cdot) = \{ W(t),\ t \ge 0 \}$ is a standard $N-$dimensional Brownian motion, and the parameters g_{Nk}, σ_{Nk} satisfy the conditions of Assumptions 1 and 2.

For every $N \ge 2$, we fix a collection of probability distributions $\{ \mathcal{P}_N(x) \}_{x \in \mathcal{S}} \subseteq \mathcal{Q}_N$. This specification will provide the rule for choosing two out of the existing $N = \mathfrak{N}(x)$ companies to merge, when the system is in state $x \in \mathcal{S}$ and an exponential clock rings.

Consider another probability space $(\Omega', \mathcal{F}', \mathbf{P}')$ which contains:

(a) countably many i.i.d. copies $\xi_1, \xi_2, \xi_3, \ldots$ of a random variable ξ with given probability distribution F, which is supported on the interval $[1/2, 1)$ as in Assumption 3;

(b) for each $N \geq 2$, countably many copies $\eta_1^N, \eta_2^N, \ldots$ of an exponential clock η^N with rate λ_N, if this rate is positive (if the rate is zero, as we assume it is for $N = 2$, we let $\eta_1^N = \eta_2^N = \cdots = \infty$);

(c) for each $N \geq 2$, and each probability distribution $\mathfrak{p} \in \mathcal{Q}_N$, countably many i.i.d. copies $\zeta_i(N, \mathfrak{p})$, $i = 1, 2, \ldots$ of a random element $\zeta(N, \mathfrak{p})$, which takes values in the family of subsets of $\{1, \cdots, N\}$ that contain exactly two elements and is distributed according to $\mathfrak{p}$.

- Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the direct product of all these probability spaces. Starting from a point $X(0) = x \in \mathcal{S}$, we let $N_0 = \mathfrak{N}(x)$ be the number of companies extant at $t = 0$, and construct a process $X(\cdot) = \big\{X(t),\ 0 \leq t < \mathcal{T}\big\}$ and a random time $\mathcal{T}$, the "lifetime" of $X(\cdot)$, as follows:

Step (i): For $t \leq \tau_1 \wedge \eta_1^{N_0}$, we define the random vector $X(t) = (X_1(t), \cdots, X_{N_0}(t))'$ with values in $(0, \infty)^{N_0}$ as

$$X_i(t) := \exp\big(Y_i^{N_0,x,1}(t)\big), \quad \text{and} \quad \tau_1 := \inf\big\{\, t \geq 0 \mid \exists \ \ i = 1, \ldots, N_0 : \mu_i^{N_0,x}(t) = 1 - \delta \,\big\}, \tag{16}$$

where

$$\mu_i(t) \equiv \mu_i^{N_0,x}(t) := \frac{X_i(t)}{\sum_{j=1}^{N_0} X_j(t)}\,, \quad 0 \leq t \leq \tau_1 \wedge \eta_1^{N_0}\,, \quad i = 1, \ldots, N_0 \tag{17}$$

is the market weight of company i (we adopt here the usual convention $\inf \emptyset = \infty$). Since $\delta \in (0, 1/2)$, there can be at most one index i with $\mu_i^{N_0,x}(\tau_1) = 1 - \delta$; thus,

$$T_1 \ := \ \tau_1 \wedge \eta_1^{N_0}$$

is the moment of the first jump, or "event", in this setup.

Step (ii): If $\tau_1 \leq \eta_1^{N_0}$, $\tau_1 < \infty$, we pick the unique $i = 1, \ldots, N_0$ such that $\mu_i(\tau_1) = 1 - \delta$, and define the vector $X(\tau_1+) \in (0, \infty)^{N_0+1}$ as follows:

$$X_\nu(\tau_1+) = X_\nu(\tau_1), \ \ \nu = 1, \cdots, i-1\,; \qquad X_\nu(\tau_1+) = X_{\nu+1}(\tau_1), \ \ \nu = i, \cdots, N_0 - 1\,;$$
$$X_{N_0}(\tau_1+) = \xi_1 X_i(\tau_1)\,, \quad X_{N_0+1}(\tau_1+) = (1 - \xi_1) X_i(\tau_1)\,.$$

To wit: at the time τ_1 of (16), company i is split into two new companies, anointed by the names N_0 and $N_0 + 1$; these inherit the capitalization $X_i(\tau_1)$ of their progenitor in proportions ξ_1 and $1 - \xi_1$, respectively. Companies $1, \cdots, i-1$ keep both their names and their capitalizations; whereas the companies formerly known as $i+1, \cdots, N_0$ keep their capitalizations but change their names to $i, \cdots, N_0 - 1$, respectively.

Step (iii): If $\tau_1 > \eta_1^{N_0}$, a subset with two elements $\{i, j\} \subseteq \{1, \ldots, N_0\}$ is selected according to the random variable $\zeta_1\big(N_0, \mathcal{P}_{N_0}(X(\eta_1^{N_0}))\big)$ whose distribution is $\mathcal{P}_{N_0}(X(\eta_1^{N_0})) \in \mathcal{Q}_{N_0}$.

In the event $\mu_i(\eta_1^{N_0}) + \mu_j(\eta_1^{N_0}) \geq 1 - \delta$, we proceed to Step (iv), Case B below. Otherwise, we define the vector $X(\eta_1^{N_0}+) \in (0, \infty)^{N_0 - 1}$ as follows, say with $i < j$:

$$X_\nu(\eta_1^{N_0}+) = X_\nu(\eta_1^{N_0}), \ \nu = 1, \ldots, i-1\,; \qquad X_\nu(\eta_1^{N_0}+) = X_{\nu+1}(\eta_1^{N_0}), \ \nu = i, \ldots, j-2\,;$$
$$X_\nu\big(\eta_1^{N_0} +\big) = X_{\nu+2}\big(\eta_1^{N_0}\big), \ \nu = j-1, \ldots, N_0 - 2\,; \qquad X_{N_0 - 1}\big(\eta_1^{N_0} +\big) = X_i\big(\eta_1^{N_0}\big) + X_j\big(\eta_1^{N_0}\big)\,.$$

Once again, companies $1, \cdots, i-1$ keep both their names and their capitalizations. The erstwhile companies $i+1, \cdots, j-1$ keep their capitalizations but change their names to $i, \cdots, j-2$; whereas the erstwhile companies $j+1, \cdots, N_0$ keep their capitalizations but change their names to $j-1, \cdots, N_0 - 2$. The former companies i and j merge; they create a new company, anointed with the index $N_0 - 1$, which inherits the sum total of their capitalizations.

Step (iv): We let $N_1 = \mathcal{N}(T_1+)$ be the new number of companies extant right after time $T_1 = \tau_1 \wedge \eta_1^{N_0}$, and note that there are three possibilities:

Case A: $N_1 = N_0 + 1$ on the event $\{\tau_1 \leq \eta_1^{N_0},\ \tau_1 < \infty\}$ of a split;

Case B: $N_1 = N_0$ on the event $\{\tau_1 > \eta_1^{N_0},\ \mu_i(\eta_1^{N_0}) + \mu_j(\eta_1^{N_0}) \ge 1-\delta\}$ of a "suppressed" merger;
Case C: $N_1 = N_0 - 1$ on the event $\{\tau_1 > \eta_1^{N_0},\ \mu_i(\eta_1^{N_0}) + \mu_j(\eta_1^{N_0}) < 1-\delta\}$ of a "successful" merger.

We define

$$X_i(t) := \exp\Big(Y_i^{N_1,x_1,2}\big(t-T_1\big)\Big) \qquad \text{for} \qquad T_1 < t \le T_1 + \big(\tau_2 \wedge \eta_2^{N_1}\big)\,. \tag{18}$$

Here $T_1 = \tau_1 \wedge \eta_1^{N_0}$, $x_1 = X(T_1)$, and

$$\tau_2 := \inf\big\{ t > 0 \mid \exists\ i = 1,\ldots,N_1 : \mu_i^{N_1,x_1}(T_1+t) = 1-\delta \big\}\,,$$

where $\mu_i^{N_1,x_1}(t)$ is defined by analogy with (17) as

$$\mu_i(t) \equiv \mu_i^{N_1,x_1}(t) := \frac{X_i(t)}{\sum_{j=1}^{N_1} X_j(t)}\,, \quad T_1 < t \le T_1 + \big(\tau_2 \wedge \eta_2^{N_1}\big)\,, \quad i = 1,\ldots,N_1$$

in terms of the capitalizations in (18). Thus, the time of the second jump in the integer-valued process $\mathcal{N}(\cdot)$ is

$$T_2 \,:=\, T_1 + \big(\tau_2 \wedge \eta_2^{N_1}\big)\,.$$

Step (v): We define similarly the values of the capitalization processes after the second jump. (If this jump corresponds to a merger, then we choose two companies to be merged according to $\zeta_2(N_1, \mathcal{P}_{N_1}(X(\eta_2^{N_1})))$.) Then we define their evolution until the moment T_3 of the third jump, etc.

Remark 5. We shall see in Subsection 4.2 below, that the specification of the probability distributions $\{\mathcal{P}_{\mathfrak{N}(x)}(x)\}_{x\in\mathcal{S}}$ as in Assumption 4 and Remark 3 guarantees that, with three or more companies present, no merger is ever suppressed (i.e., that Case B in (iv) above never occurs).

Step (vi): This construction leads to a piecewise-continuous strong MARKOV process $X(\cdot) = \big\{X(t),\ 0 \le t < \mathcal{T}\big\}$ with state space $\mathcal{S}$, defined on the time interval $[0,\mathcal{T})$ with

$$\mathcal{T} \,:=\, \lim_{m\to\infty} T_m\,. \tag{19}$$

The resulting market-weight process $\mu(\cdot) = \mathfrak{z}\big(X(\cdot)\big)$ of (1) has state space $\mathcal{M}^\delta$ as in (8); in particular, $\max_{1\le i\le \mathcal{N}(t)} \mu_i(t) \le 1-\delta$ holds for all $t \in [0,\infty)$, so the resulting market model is *diverse* in the terminology of [8], Chapter 2. We also note that, in all cases of the above construction, the total capitalization $\mathcal{C}(\cdot)$ in (1) is preserved at each "event-time" T_m, namely

$$\mathcal{C}(T_m+) \,=\, \mathcal{C}(T_m)\,, \quad \forall\ m \in \mathbb{N}\,.$$

Definition 2. We say that the so-constructed model *admits explosions,* if $\mathbf{P}\,(\mathcal{T} = \infty) < 1$. Otherwise, the model is said to be *free of explosions.* □

Theorem 2.1 guarantees that $\mathbf{P}\,(\mathcal{T} = \infty) = 1$ holds under Assumptions 1-5. In the absence of explosions the process $X(\cdot)$ is defined on all of $[0,\infty)$, and we denote by $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ the smallest filtration to which $X(\cdot)$ is adapted and which satisfies the usual conditions.

Remark 6. In addition to being diverse, the model just constructed has intrinsic relative variance (equivalently, excess growth rate for market portfolio)

$$\gamma_*^\mu(t) \,=\, \frac{1}{2}\left(\sum_{k=1}^{N} \sigma_{Nk}^2\, \mu_{(k)}(t)\big(1-\mu_{(k)}(t)\big)\right)\Bigg|_{N=\mathcal{N}(t)}\,, \qquad 0 \le t < \infty\,,$$

which is bounded away from zero, namely $\gamma_*^\mu(\cdot) \ge \big(\sigma_0^2\,\delta\big)/2 > 0$; we owe this observation to Dr. Robert FERNHOLZ. See Proposition 3.1 in [11], or Example 11.1 in [12], for the significance of such a positive lower bound in the context of arbitrage relative to the market portfolio with a fixed number of companies.

3.4. **Portfolios and Associated Wealth processes.** Let us discuss portfolios and the wealth processes they generate. A *portfolio* $\pi(\cdot)$ is an $\mathbb{F}-$progressively measurable process $\pi(\cdot) = \big\{\pi(t),\ 0 \le t < \infty\big\}$, $\pi(t) = \big(\pi_1(t), \dots, \pi_{\mathcal{N}(t)}(t)\big)'$ with

$$\sup_{\substack{1 \le i \le \mathcal{N}(t) \\ 0 \le t < \infty}} |\pi_i(t)| \le K_\pi$$

valid almost surely, for some constant $K_\pi \in [0, \infty)$. When the integer-valued process $\mathcal{N}(\cdot)$ suffers a downward or an upward jump, this portfolio must behave according to the rules described informally in Section 2. We formalize these rules presently.

(A) Assume that at time t, the ith and jth companies merge into one company, which is then anointed with index $N-1$, where $N = \mathcal{N}(t+)$. The new portfolio weights are

$$\pi_\nu(t+) = \pi_\nu(t), \quad \nu = 1, \cdots, i-1\,; \qquad \pi_\nu(t+) = \pi_{\nu+1}(t), \quad \nu = i, \cdots, j-2\,;$$
$$\pi_\nu(t+) = \pi_{\nu+2}(t), \quad \nu = j-1, \cdots, N-2\,; \qquad \pi_{N-1}(t+) = \pi_i(t) + \pi_j(t).$$

In words: companies not involved in the merger are assigned the same portfolio weights, under their new appellations if necessary; whereas the newly minted company $N-1$ inherits the sum of the portfolio weights formerly assigned to its two parent companies.

(B) Assume that at time t the ith company, with capitalization $X_i(t)$, is split into two companies (anointed with indices N and $N+1$, where $N = \mathcal{N}(t+)$). The new portfolio weights are

$$\pi_\nu(t+) = \pi_\nu(t), \quad \nu = 1, \cdots, i-1\,; \qquad \pi_\nu(t+) = \pi_{\nu+1}(t), \quad \nu = i, \cdots, N-1\,;$$
$$\pi_N(t+) = \pi_i(t)\frac{X_N(t+)}{X_i(t)}\,, \qquad \pi_{N+1}(t+) = \pi_i(t)\frac{X_{N+1}(t+)}{X_i(t)}\,.$$

Once again, companies not involved in the split keep their weights in the portfolio, under their new appellations if necessary; whereas each of the two newly created companies N and $N+1$ inherits the weight in the portfolio of the parent company, in proportion to its currently assigned capitalization.

The corresponding wealth process $V^\pi(\cdot) = \{V^\pi(t),\ 0 \le t < \infty\}$ is continuous, $\mathbb{F}-$adapted, has values in $(0, \infty)$, and is governed for each integer $m \in \mathbb{N}_0$ by the dynamics

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_{i=1}^{N_m} \pi_i(t)\,\frac{\mathrm{d}X_i(t)}{X_i(t)}\,, \qquad t \in (T_m, T_{m+1})\,, \quad V^\pi(0) = 1 \tag{20}$$

and with $T_0 = 0$. Quite clearly, we have $V^\kappa(\cdot) \equiv 1$ for the cash portfolio; and $V^\mu(\cdot) \equiv \mathcal{C}(\cdot)/\mathcal{C}(0)$ for the market portfolio of (1). As mentioned before, the amount $\pi_0(t)V^\pi(t)$ in the notation of (2) is invested in the money market at time t.

4. Proofs

4.1. **Subexponential Tail.** We state and prove the following crucial Proposition: the distribution of the maximum number of companies over any finite time-interval has a tail which is lighter than any exponential tail.

Proposition 4.1. *Under Assumptions 1-5, for any $T \in (0, \infty)$ we have:*

$$\lim_{u \to \infty} \frac{1}{u}\left[-\log \mathbf{P}_x\Big(\max_{0 \le t \le T} \mathcal{N}(t) > u\Big)\right] = \infty.$$

Equivalently, for all $c \in (0, \infty)$ we have

$$\mathbf{E}\left[\exp\left(c \max_{0 \le t \le T} \mathcal{N}(t)\right)\right] < \infty\,. \tag{21}$$

Theorem 2.1 follows trivially from this Proposition: if the maximal number of companies over the time-horizon $[0,T]$ has this property, then it is a.s. finite, which is another way of saying that the counting process $\mathcal{N}(\cdot)$ does not explode. Theorem 2.2 also uses this fact, but in subtler ways; its proof is postponed until Subsection 4.4.

We shall find the following notation convenient: Consider the "event-time" random sequence $\{N_m\}_{m\geq 0}$ with $N_m = \mathcal{N}(T_m+),\ m \geq 0$ and the jump times $\{T_m\}_{m\geq 0}$ as in (19), (20). Recall that N_m is the level of the process $\mu(\cdot)$ of market weights; in other words, the number of companies extant during the time interval (T_m, T_{m+1}) between the mth and the $(m+1)$st jumps of the integer-valued process $\mathcal{N}(\cdot)$.

4.2. **Preliminary Remarks.** The idea of the proof of Proposition 4.1 is as follows. We say that a *double jump upward* happens at step m, if $N_m = N_{m-1}+1$ and $N_{m+1} = N_m + 1$. If $N_m = N$, then we say that this is a *double jump upward from level* $N-1$ *to level* $N+1$ at step m; we denote this event by $A(m,N)$.

Suppose we start from level N, and the maximal number of companies during the time interval $[0,T]$ is larger than or equal to $2L$, where $L > N$ is some large number. Then it takes time less than T to get from the level L to the level $2L$; this will require at least $L-1$ double jumps upward, for instance, one from L to $L+2$, another from $L+1$ to $L+3$, etc. Note that such double jumps upward may "overlap", when there are three or more consecutive jumps upward. But the probability of a double jump upward from $N-1$ to $N+1$ is small for large N. Indeed, immediately after the first jump, from $N-1$ to N, the top market weight will be less than or equal to $1-\delta_0$. In other words, the process $\mu(\cdot)$ of market weights will "stay away" from the threshold $1-\delta$, which it must hit before the exponential clock $\mathcal{E}(\lambda_N)$ rings; but from Lemma 5.1 in the Appendix, the probability of this event is at most

$$p_N \,:=\, 2\left(\frac{(1-\delta_0)\vee(1/2)}{1-\delta}\right)^{\sigma^{-1}\lambda_N^{1/2}} \,=\, 2\exp\left(-\alpha_1\lambda_N^{1/2}\right), \tag{22}$$

where $\alpha_1 := \big(\log(1-\delta) - \log\big((1-\delta_0)\vee(1/2)\big)\big)\,\sigma^{-1} > 0$.

The detailed proof of Proposition 4.1 will be given in Subsection 4.3 below. We shall fix the number of steps u and claim that it is unlikely for the process to perform $L-1$ double jumps upward within u steps. But if the process gets to the level $2L$ in $M \geq u$ steps, then there will be a lot of jumps downward, at independent exponential random times. Since there will be a lot of these random times, we can apply Large Deviation Theory and argue that their sum is very likely to be greater than T.

4.2.1. *Leaving a Given Level.* The process $\mu(\cdot) = \mathfrak{z}\big(X(\cdot)\big)$ of market weights evolves as follows: as long as it stays on the $(N-1)-$dimensional manifold $\Delta_+^{N,\delta}$ (that is, "on the level N"), the process $\mu(\cdot)$ evolves as an N-dimensional diffusion governed by the system (14) of SDEs. *How does the process* $\mu(\cdot)$ *leave the level* N*?* There are two possibilities:

(i) An exponential clock $\eta_j^N \backsim \mathcal{E}(\lambda_N)$ rings; by construction, the random variable η_j^N is independent of the diffusion given by the system of SDEs in (14). Then we choose randomly two companies to merge, *excluding the top one;* this requirement is essential for ensuring than the merger will not be suppressed (i.e., with this proviso we never find ourselves in Case B (iv) of Subsection 3.3).

Note that, of the chosen companies, the one with the biggest market weight will occupy the second place at best, so its market weight will be no more than $1/2$; whereas the other will occupy the third place at best, so its market weight will not exceed $1/3$. Therefore, the market weight of the amalgamated company will not exceed $5/6$, a number smaller than $1-\delta$ because we have $\delta < 1/6$ from Assumption 2, *and so the merger will not be suppressed.* Moreover, *all* of the new market weights will be bounded away from the threshold $1-\delta$, so it will take some time for *any* company that emerged after the merger to hit this threshold.

(ii) The market weight of one of N companies, say of the ith one, hits at some time T the level $1-\delta$; thus $\mu_i(T) = 1-\delta$ and $\sum_{j\neq i}\mu_j(T) = \delta$. Then we pick a random variable $\xi \backsim F$, independent of the past, and split the ith company into two new companies: these are assigned market weights $\xi\mu_i(T)$ and $(1-\xi)\mu_i(T)$. Since $1/2 \le \xi \le 1-\varepsilon_0$, each of the resulting two new market weights is at most $(1-\delta)(1-\varepsilon_0)$; whereas all the other companies, those unaffected by the split, have market weights bounded from above by δ. Because $\varepsilon_0 \in (0, 1/2)$ and $\delta \in (0, 1/6)$, we have $\delta < (1-\delta)/2 < (1-\delta)(1-\varepsilon_0) < 1-\delta$, so again *all* of the new market weights are bounded away from the threshold $1-\delta$.

More precisely, for the time $\boldsymbol{\tau}$ of any upward jump in the process $\mathcal{N}(\cdot)$, we have: $\mu(\boldsymbol{\tau}+) \in \Delta_+^{N',\delta_0}$ for some integer $N' \ge 2$ and with

$$\delta_0 \,=\, \big(1-(1-\delta)(1-\varepsilon_0)\big) > \delta\,.$$

Immediately after any upward jump, the market weight process is in $\mathcal{M}^{\delta_0}$ for some $\delta_0 > \delta$; in other words, we have $\mu_{(1)}(\boldsymbol{\tau}+) \le 1-\delta_0$ with $1-\delta_0 < 1-\delta$.

4.2.2. *Jumping Upwards, rather than Downwards.* Let us obtain an upper bound for the conditional probability $\mathbf{P}_x\big(\tau_m \le \eta_m^{N_{m-1}} \big| N_{m-1} = N\big)$ that the mth jump will be upward rather than downward, given that during the time-interval (T_{m-1}, T_m) the market weight process $\mu(\cdot) = \mathfrak{z}\big(X(\cdot)\big)$ is at a given level $N \in \mathbb{N}$. On the event $\{N_{m-1} = N\}$ and during the time-interval (T_{m-1}, T_m) with $T_m = T_{m-1} + (\tau_m \wedge \eta_m^N)$, the process of log-capitalizations evolves as a system of competing Brownian particles with drifts $g_k = g_{Nk}$ and variances $\sigma_k^2 = \sigma_{Nk}^2$, for $k = 1, \ldots, N$.

First, we consider $m = 1$; by the comparison lemma from the Appendix, this probability is bounded from above by

$$2\left(\frac{\mu_{(1)}(0) \vee \frac{1}{2}}{1-\delta}\right)^{\sigma^{-1}\lambda_N^{1/2}}\,,$$

because $\sigma \ge \overline{\sigma} := \max_{1\le k\le N}(\sigma_{Nk})$ in the notation of Assumption 2. Now, back to the case of general m, we have from the strong Markov property

$$\mathbf{P}\big(\tau_m \le \eta_m^{N_{m-1}} \big| N_{m-1} = N\big) \;\le\; 2\,\mathbf{E}\left(\frac{\mu_{(1)}(T_{m-1}) \vee \frac{1}{2}}{1-\delta}\right)^{\sigma^{-1}\lambda_N^{1/2}}. \tag{23}$$

4.2.3. *A Couple of Auxiliary Estimates.*

Lemma 4.2. *Fix $m \ge 1$, $N \ge 3$. Then, with p_N as in (22), we have the following estimate for the probability of a double upward jump:*

$$\mathbf{P}\big(A(m,N) \mid N_{m-1} = N-1\big) \equiv \mathbf{P}\big(N_{m+1} = N+1 \mid N_m = N,\ N_{m-1} = N-1\big) \le p_N\,.$$

Proof. If $N_{m-1} = N-1$ and $N_m = N$, then $\mu_{(1)}(T_m+) = \mathfrak{z}\big(X(T_m+)\big) \le 1-\delta_0$. Therefore, from (23) we have

$$\mathbf{P}(N_{m+1} = N+1 \mid N_m = N,\ N_{m-1} = N-1)$$
$$= \mathbf{P}\big(\tau_{m+1} \le \eta_{m+1}^{N_m} \big| N_m = N, N_{m-1} = N-1\big) \le 2\left(\frac{(1-\delta_0) \vee (1/2)}{1-\delta}\right)^{\sigma^{-1}\lambda_N^{1/2}} = p_N$$

in the notation of (22). ∎

An immediate corollary of this last Lemma, is

$$\mathbf{P}\big(N_{m+1} = N+1,\ N_m = N \mid N_{m-1} = N-1\big) \le \mathbf{P}\big(N_{m+1} = N+1 \mid N_m = N,\ N_{m-1} = N-1\big) \le p_N .$$

Lemma 4.3. *For every* $m \ge 1,\ N \ge 3$ *and* $A \in \mathcal{F}(T_m)$*, we have:*

$$\mathbf{P}\big(N_{m+1} = N+1 \mid N_m = N,\ N_{m-1} = N-1,\ A\big) \le p_N .$$

Proof. Indeed, let us show that for $x \in (0,\infty)^N$ with $\mathfrak{z}(x) \in \Delta_+^{N,\delta_0}$, we have:

$$\mathbf{P}\big(N_{m+1} = N+1 \mid X(T_m+) = x,\ N_m = N,\ N_{m-1} = N-1,\ A\big) \le p_N .$$

This follows from the estimate of the previous lemma, and from the fact that τ_{m+1} is a function of the Brownian motions driving the system $Y^{N_m,x_m,m+1}(\cdot) = Y^{N,x,m+1}(\cdot)$ and of the initial condition $x_m = X(T_m+)$. By construction, the driving Brownian motions of the system $Y^{N_m,x_m,m+1}(\cdot) = Y^{N,x,m+1}(\cdot)$ are independent of $\mathcal{F}(T_m)$, and this implies

$$\begin{aligned}&\mathbf{P}\left(N_{m+1} = N+1 \mid X(T_m+) = x,\ N_m = N,\ N_{m-1} = N-1,\ A\right)\\ &= \mathbf{P}\left(N_{m+1} = N+1 \mid X(T_m+) = x,\ N_m = N\right) \le p_N .\end{aligned}$$

Now, if $N_{m-1} = N-1$ and $N_m = N$, then we have $\mathfrak{z}\big(X(T_m+)\big) = \mathfrak{z}(x) \in \Delta_+^{N,\delta_0}$; and using the strong Markov property we complete the proof. □

An immediate corollary is that for fixed $m_1 < m_2 < \ldots < m_j < m$, we have:

$$\mathbf{P}\big(A(m,N) \mid A(m_1,N_1), A(m_2,N_2), \ldots, A(m_j,N_j)\big) \le p_N .$$

4.3. **Proof of Proposition 4.1.** With $N = \mathfrak{N}(x)$, let us estimate the probability that the market weight process $\mu(\cdot) = \mathfrak{z}\big(X(\cdot)\big)$ rises from level N to the level $2L$ during the time-interval $(0,T)$, where $L > N$. First, the process has to reach the level L; it will get there fork the first time as a result of a split, and immediately after the jump it will be in Δ_+^{L,δ_0}. Then it will have time less than T to reach the level $2L$. For each $n = 2, 3, \cdots$, the random variable

$$\Theta_n := \inf\big\{t \ge 0 : \mathcal{N}(t) = n\big\}$$

will denote the first time the counting process $\mathcal{N}(\cdot)$ of our model hits the nth level (that is, the first time n companies are extant). Suppose we are able to establish the following estimate: that for every $\beta > 0$, there exist $L_0 > N$ such that for every $L > L_0$ and $y \in \Delta_+^{L,\delta_0}$ we have:

$$\mathbf{P}_y\big(\Theta_{2L} \le T\big) \le e^{-\beta L}. \tag{24}$$

To get from x to the level $2L$ in time less than or equal to T, the process needs first to get to the level L, passing to some point $y \in \Delta_+^{L,\delta_0}$; then starting from this point, it has to reach the level $2L$ during the remaining time (which is of course smaller than T) . Therefore, integrating by $x \in \Delta_+^{L,\delta_0}$ with respect to the distribution of $\mu\big(\Theta_L+\big)$ and using the strong Markov property, we would get

$$\mathbf{P}_x\big(\Theta_L \le T\big) \le e^{-\beta L},$$

and this would complete the proof.

• Thus, let us try to estimate the $\mathbf{P}_y-$probability of the event $\{\Theta_{2L} \le T\}$ in (24), for $y \in \Delta_+^{L,\delta_0}$. Suppose that it took the process of market weights M jumps to reach the level $2L$; we shall try to find a number $u > L$ such that the event $\{M \le 3u, \Theta_{2L} \le T\}$ is unlikely and the event $\{M > 3u, \Theta_{2L} \le T\}$ is also unlikely. Let us introduce a new piece of notation:

$$\underline{\lambda}_L := \min(\lambda_L, \ldots, \lambda_{2L-1}), \qquad \overline{\lambda}_L := \max(\lambda_L, \ldots, \lambda_{2L-1}).$$

On the event $\{M > 3u,\ \Theta_{2L} \le T\}$, there are

$$\frac{M-L}{2} > \frac{3u-u}{2} = u$$

downward jumps. If a jump occurrs from the level i to the level $i-1$, it takes time $\eta \backsim \mathcal{E}(\lambda_i)$. Since $\lambda_i \le \overline{\lambda}_{2L}\,,\ i = 3, \ldots, 2L$, then we have $\eta \ge \lambda_i\, \overline{\lambda}_{2L}^{-1}\, \eta \backsim \mathcal{E}(\overline{\lambda}_{2L})$. All these exponential jump times are independent, so there exist i.i.d. $\mathcal{E}(\overline{\lambda}_{2L})$ random variables $\eta_1, \ldots, \eta_u$ such that $\eta_1 + \ldots + \eta_u \le T$. We can write this as

$$\frac{\widetilde{\eta}_1 + \ldots + \widetilde{\eta}_u}{u} \le \frac{T\overline{\lambda}_{2l}}{u},$$

where $\widetilde{\eta}_i = \overline{\lambda}_{2L}^{-1}\eta_i \backsim \mathcal{E}(1),\ i = 1, \ldots, u$. By Large Deviation theory, see for example [7, Exercise 2.2.23(c), p.35], this event has probability at most

$$\exp\left(-u\,\mathcal{H}\left(\frac{T\overline{\lambda}_{2L}}{u}\right)\right), \qquad \mathcal{H}(s) := s - 1 - \log s\,, \qquad s \in (0, \infty),$$

provided $u > L \vee \big(T\,\overline{\lambda}_{2L}\big)$, thus

$$\mathbf{P}_y\big(M > 3u,\, \Theta_{2L} \le T\big) \le \exp\left(-u\,\mathcal{H}\left(\frac{T\overline{\lambda}_{2L}}{u}\right)\right). \tag{25}$$

• Now, let us estimate the probability $\mathbf{P}_y\big(M \le 3u,\ \Theta_{2L} \le T\big)$. In order to reach the level $2L$ starting from L in no more than $3u$ jumps, we need to have at least $L-1$ double jumps upward, discussed above. One of these double jumps is from level L to level $L+2$, occurring at step m_1. Another is from level $L+1$ to level $L+3$, occurring at step m_2, etc., up to a double jump upward from level $2L-2$ to level $2L$, occurring at step m_{L-1}. We have: $1 \le m_1 < m_2 < \ldots < m_{L-1} \le 3u - 1$. We can choose this subset $\{m_1, \ldots, m_{L-1}\} \subseteq \{1, \ldots, 3u-1\}$ in

$$\binom{3u-1}{L-1} \le \binom{3u}{L} \le \frac{(3u)^L}{L!}$$

different ways. For each combination of double steps upward, the probability that it can occur is dominated by

$$\begin{aligned} p_L \ldots p_{2L-1} &= 2^{L-1}\exp\left(-\alpha_1\left(\lambda_L^{1/2} + \ldots + \lambda_{2L-1}^{1/2}\right)\right) \le 2^{L-1}\exp\left(-\alpha_1(L-1)\underline{\lambda}_L^{1/2}\right) \\ &\le 2^L\exp\left(-\frac{1}{2}\alpha_1 L\underline{\lambda}_L^{1/2}\right). \end{aligned}$$

Indeed, fix a subset $\{m_1, \ldots, m_{L-1}\} \subseteq \{1, \ldots, 3u-1\}$. Then this probability is no more than

$$\begin{aligned} &\mathbf{P}(A(m_1, L+1), A(m_2, L+2), \ldots, A(m_{L-1}, 2L-1)) \\ &= \mathbf{P}(A(m_{L-1}, 2L-1) \mid A(m_{L-2}, 2L-2), \ldots, A(m_2, L+2), A(m_1, L+1)) \\ &\cdot \mathbf{P}(A(m_{L-2}, 2L-2) \mid A(m_{L-3}, 2L-3), \ldots, A(m_2, L+2), A(m_1, L+1)) \\ &\cdot \mathbf{P}(A(m_2, L_2) \mid A(m_1, L_1)) \le p_{L+1} \ldots p_{2L-1}\,, \end{aligned}$$

and thus

$$\mathbf{P}_y\big(M \le 3u,\ \Theta_{2L} \le T\big) \le \frac{(3u)^L}{L!}2^L\exp\left(-\frac{1}{2}\alpha_1 L\underline{\lambda}_L^{1/2}\right). \tag{26}$$

• It follows from (25) and (26) that the probability of the event which we would like to estimate, as in (24), is

$$\mathbf{P}_y\left(\Theta_{2L} \le T\right) \le \Sigma_1 + \Sigma_2 := \frac{(3u)^L}{L!}2^L\exp\left(-\frac{1}{2}\alpha_1 L\underline{\lambda}_L^{1/2}\right) + \exp\left(-u\,\mathcal{H}\left(\frac{T\overline{\lambda}_{2L}}{u}\right)\right).$$

Here, $u > L$. Note that $\lambda_L \backsim cL^\alpha$ as $L \to \infty$, so $\overline{\lambda}_L \backsim 2^\alpha cL^\alpha$ and $\underline{\lambda}_L \backsim cL^\alpha$ as $L \to \infty$.

Let $u = kL^{\alpha \vee 1}$ for large enough $k > 0$. The function $\mathcal{H}(\cdot)$ satisfies $\mathcal{H}(s) \geq -(1/2)\log s$ for $s \in (0, s_0)$ for some constant $s_0 \in (0,1)$. Therefore, for large enough L we have

$$\mathcal{H}\left(\frac{T\overline{\lambda}_{2L}}{u}\right) \geq \frac{1}{2}\log\frac{u}{T\overline{\lambda}_{2L}} \geq \frac{1}{2}\log\frac{k/2}{T2^{\alpha}c} =: k_0 \qquad \text{and} \qquad \Sigma_2 := \exp\left(-u\,\mathcal{H}\left(\frac{T\overline{\lambda}_{2L}}{u}\right)\right) \leq e^{-kk_0L^{1\vee\alpha}}.$$

By taking k large enough, we can make kk_0 as large as we want. This completes the proof for this summand. The other summand

$$\Sigma_1 := \frac{(3u)^L}{L!}\,2^L \exp\left(-\frac{1}{2}\alpha_1 L \underline{\lambda}_L^{1/2}\right)$$

decreases faster than any $e^{-\beta L}$ as $L \to \infty$ for any fixed $\beta > 0$, because $\underline{\lambda}_L \backsim cL^{\alpha}$ and

$$\log\Sigma_2 = L\log(3u) - \log(L!) + L\log 2 - \frac{1}{2}\alpha_1 L\underline{\lambda}_L^{1/2}$$

is asymptotically smaller as $L \to \infty$ than

$$(\alpha \vee 1)L\log L - L\log L - \frac{1}{2}\alpha_1 c L^{1+\alpha/2} \backsim -\frac{\alpha_1 c}{2}L^{1+\alpha/2}.$$

This establishes the bound of (24), so the proof of the Proposition is complete. □

4.4. **Proof of Theorem 2.2.** The general philosophy of the proof is this: We shall try to find an *equivalent martingale measure,* that is, a probability measure $\mathbf{Q}$ on $\mathcal{F}$ with the following properties:

(i) $\mathbf{Q} \backsim \mathbf{P}$ on $\mathcal{F}(t)$ for each $0 \leq t < \infty$;

(ii) for every portfolio $\pi(\cdot)$, the wealth process $V^{\pi}(\cdot)$ is a $\mathbf{Q}-$martingale.

Suppose this is done; take two portfolios $\pi(\cdot)$ and $\rho(\cdot)$, and assume for a moment $\pi(\cdot)$ allows an arbitrage opportunity relative to $\rho(\cdot)$ on a given time horizon $[0,T]$ with $T \in (0,\infty)$. Then (4) holds with respect to the measure $\mathbf{P}$, and therefore with respect to the measure $\mathbf{Q}$ as well. But

$$V^{\pi}(\cdot) - V^{\rho}(\cdot) \;=\; \left\{V^{\pi}(t) - V^{\rho}(t),\; 0 \leq t < \infty\right\}$$

is a $\mathbf{Q}$-martingale with initial value zero, so $\mathbf{E}^{\mathbf{Q}}\big(V^{\pi}(T) - V^{\rho}(T)\big) = 0$ in contradiction to (4). This contradiction completes the proof, that arbitrage in not possible.

For a given, fixed number of companies, an equivalent martingale measure is constructed thus: a Girsanov change of measure ensures that each capitalization process is a martingale with respect to the new measure, and the wealth process is a stochastic integral with these processes as integrators; as a result, the wealth process is also a martingale with respect to the new measure. But here the number of extant companies fluctuates, so we shall carry out a Girsanov construction up to the first jump, then carry out the same construction with the new number of stocks up to the second jump, and so on. We do this in a number of steps, as follows.

Step 1: First, as a preliminary step, let us consider the CBP-based market model from Subsection 3.2 with the dynamics of (15) and under an appropriate filtration $\mathbb{G} = \{\mathcal{G}(t)\}_{0 \leq t < \infty}$ that satisfies the usual conditions. Each of the processes $\Upsilon_i(\cdot) = \{\Upsilon_i(t),\ 0 \leq t < \infty\},\ i = 1, \dots, N$, given by

$$\Upsilon_i(t) := \int_0^t \frac{\mathrm{d}X_i(s)}{X_i(s)} = \sum_{k=1}^{N}\int_0^t \mathbf{1}_{\{X_i(t) = X_{(k)}(t)\}}\left(g_k\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_i(t)\right), \quad 0 \leq t < \infty \tag{27}$$

can be turned into a martingale by the change of measure

$$\mathbf{Q}(A) = \mathbf{P}\big(Z(T)\mathbf{1}_A\big), \quad A \in \mathcal{G}(T), \qquad 0 \leq T < \infty\,.$$

Here

$$Z(T) = \exp\left(-M(T) - \frac{1}{2}\langle M\rangle(T)\right), \quad 0 \leq T < \infty\,,$$

where the $\mathbf{P}-$martingale $M(\cdot) = \{M(T),\ 0 \le T < \infty\}$ is given by

$$M(T) = \sum_{k=1}^N \left(\frac{g_k}{\sigma_k}\right) \sum_{i=1}^N \int_0^T \mathbf{1}_{\{X_i(t)=X_{(k)}(t)\}} \mathrm{d}W_i(t), \qquad 0 \le T < \infty.$$

The quadratic variation of $M(\cdot)$ is

$$\langle M\rangle(T) \; = \; \sum_{k=1}^N \left(\frac{g_k}{\sigma_k}\right)^2 \sum_{i=1}^N \int_0^T \mathbf{1}_{\{X_i(t)=X_{(k)}(t)\}} \, \mathrm{d}t \;\le\; TN \max_{1\le k\le N} \left(\frac{g_k}{\sigma_k}\right)^2, \tag{28}$$

so using the Novikov condition [21, Proposition 3.5.12] we see that $Z(\cdot)$ is a $\mathbf{P}-$martingale, and thus $\mathbf{Q}$ a probability measure. Also, the quadratic variation of $\Upsilon_i(\cdot)$ is given by

$$\langle \Upsilon_i\rangle(T) \; = \; \sum_{k=1}^N \sigma_k^2 \int_0^T \mathbf{1}_{\{X_i(t)=X_{(k)}(t)\}} \, \mathrm{d}t \;\le\; T \max_{1\le k\le N} \sigma_k^2\,,$$

whereas $\langle \Upsilon_i(\cdot), \Upsilon_j(\cdot)\rangle \equiv 0$ holds for $1 \le i \ne j \le N$, because $W_i(\cdot)$ and $W_j(\cdot)$ are independent.

It is clear from this discussion that $X_i(\cdot) = \exp\big(Y_i(\cdot) - (1/2)\langle Y_i\rangle(\cdot)\big)\,,\ i = 1, \cdots, N$ are martingales (with zero cross-variations) under the probability measure $\mathbf{Q}$, which thus earns the appellation of Equivalent Martingale Measure (EMM) for the model of Subsection 3.2.

Remark 7. It follows now easily, that the CBP-based market model of Subsection 3.2 is not diverse. For if this model were diverse on some time-horizon $[0, T]$ of finite length, then Proposition 6.2 of [12] would proscribe for it EMMs, such as the probability measure $\mathbf{Q}$ just constructed.

Step 2: Now, let $M^{N,x,n}(\cdot) = \{M^{N,x,n}(t),\ 0 \le t < \infty\}$ be the same martingale $M(\cdot)$ for the copy of a CBP-based market model

$$\Big(\exp\big(Y_1^{N,x,n}(\cdot)\big), \ldots, \exp\big(Y_N^{N,x,n}(\cdot)\big)\Big)'$$

from Subsection 3.3. This model has parameters $g_k = g_{Nk},\ \sigma_k = \sigma_{Nk},\ k = 1, \ldots, N$, the initial condition is x, and all the processes $\{M^{N,x,n}(\cdot)\}_{n\in\mathbb{N}}$ are independent. Also, denote by

$$\Upsilon_i^{N,x,n}(\cdot), \quad i = 1, \ldots, N,$$

the processes $\Upsilon_i(\cdot)$ of (27) for this copy of the market model. Slightly abusing notation, we define

$$M(t) = \sum_{m=0}^{\infty} M^{N_m,x_m,m+1}(t\wedge T_{m+1} - t\wedge T_m), \quad 0 \le t < \infty \tag{29}$$

in the notation of Subsection 3.3. This is an $\mathbb{F} = \{\mathcal{F}(t)\}_{t\ge 0}-$local martingale, with localizing sequence $\{T_m\}_{m\in\mathbb{N}_0}$ and quadratic variation

$$\langle M\rangle(T) \; = \; \sum_{m=0}^{\infty} \langle M^{N_m,x_m,m+1}\rangle(T\wedge T_{m+1} - T\wedge T_m)\,.$$

Step 3: Let us verify the Novikov condition

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\langle M\rangle(T)\right)\right] \; < \; \infty\,, \qquad 0 \le T < \infty \tag{30}$$

of [21, Proposition 3.5.12]. The expression of (28) leads to the estimate

$$\langle M\rangle(T) \le \sum_{m=0}^{\infty} \big(T\wedge T_{m+1} - T\wedge T_m\big)\, N_m \max_{1\le k\le N_m} \left(\frac{g_{N_m k}}{\sigma_{N_m k}}\right)^2.$$

Since Assumption 2 implies that

$$\frac{|g_{N_m k}|}{\sigma_{N_m k}} \leq C < \infty \quad \text{holds for all } \; m \geq 0\,,\; k \geq 1\,,$$

we get

$$\langle M \rangle(T) \,\leq\, C^2 T \max_{0\leq t\leq T} N(t)\,, \tag{31}$$

and so the left-hand side in (30) can be estimated as

$$\mathbf{E}\left(\exp\left[\frac{C^2 T}{2}\cdot \max_{0\leq t\leq T} \mathcal{N}(t)\right]\right).$$

But this quantity is finite for any $T \in (0,\infty)$ because of (21) from Proposition 4.1, establishing the Novikov condition (30). Then the stochastic exponential

$$Z(\cdot) = \big\{Z(t),\; 0 \leq t < \infty\big\}, \qquad Z(t) = \exp\left[-M(t) - \frac{1}{2}\langle M \rangle(t)\right] \tag{32}$$

is a $\mathbf{P}-$martingale, and we can define a new probability measure $\mathbf{Q}$ that satisfies

$$\mathrm{d}\mathbf{Q} = Z(T)\,\mathrm{d}\mathbf{P} \;\; \text{on} \;\; \mathcal{F}(T), \;\; \text{for each} \;\; T \in [0,\infty)\,.$$

Step 4: We shall show now that, under the new measure, the wealth process $V^\pi(\cdot)$ is a martingale for any portfolio $\pi(\cdot)$. (This is very clearly the case for the cash portfolio, as $V^\kappa(\cdot) \equiv 1$.) We write the equation for $V^\pi(\cdot)$ in the form

$$\int_0^T \frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} \;=\; \sum_{m=0}^{\infty}\sum_{i=1}^{N_m}\int_0^{T\wedge T_{m+1}-T\wedge T_m} \pi_i(T_m+t)\,\mathrm{d}\Upsilon^{N_m,x_m,m+1}(t)\,, \qquad 0 \leq T < \infty\,.$$

Under the measure $\mathbf{Q}$, the process $\Upsilon^{N_m,x_m,m+1}(\cdot)$ given by

$$\Upsilon_i^{N_m,x_m,m+1}(t) \;=\; \int_0^{t\wedge T_{k+1}-t\wedge T_k} \frac{\mathrm{d}X_i^{N_m,x_m,m+1}(s)}{X_i^{N_m,x_m,m+1}(s)}\,, \quad i = 1,\ldots,N_m$$

is a local martingale. Therefore, the process $L(\cdot) = \{L(T), T \geq 0\}$ given by

$$L^\pi(T) \;:=\int_0^T \frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} \;=\; \sum_{m=0}^{\infty}\sum_{i=1}^{N_m}\int_0^{T\wedge T_{m+1}-T\wedge T_m} \pi_i(T_m+t)\,\mathrm{d}\Upsilon_i^{N_m,x_m,m+1}(t)$$

is an $(\mathbb{F},\mathbf{Q})-$local martingale, and we have

$$V^\pi(T) = \exp\left(L^\pi(T) - \frac{1}{2}\langle L^\pi\rangle(T)\right).$$

If we can establish the Novikov condition

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\langle L^\pi\rangle(T)\right)\right] < \infty\,, \tag{33}$$

then it will turn out that $V^\pi(\cdot)$ is an $(\mathbb{F},\mathbf{Q})-$martingale, as indeed we set out to establish at the start of this proof.

But the processes $\Upsilon_i^{N_m,x_m,m+1}(\cdot)$, $i = 1,\cdots,N_m$ have zero cross-variations, and quadratic variations

$$\langle \Upsilon_i^{N_m,x_m,m+1}\rangle(T) \max_{1\leq k\leq N} \sigma_{Nk}^2\,;$$

therefore, the process $\Upsilon_\pi^{N_m,x_m,m+1}(\cdot) = \big\{\Upsilon_\pi^{N_m,x_m,m+1}(T),\ 0 \le T < \infty\big\}$ given by

$$\Upsilon_\pi^{N_m,x_m,m+1}(T) := \sum_{i=1}^{N_m}\int_0^T \pi_i(T_m+t)\,\mathrm{d}\Upsilon_i^{N_m,x_m,m+1}(\cdot),$$

has quadratic variation

$$\langle \Upsilon_\pi^{N_m,x_m,m+1}\rangle(T) \le N_m T K_\pi \cdot \max_{1\le k\le N_m}\sigma^2_{N_m k}\,.$$

Indeed, $|\pi_i(t)| \le K_\pi < \infty$ holds for all $0 \le t < \infty$, $i = 1, \cdots, \mathcal{N}(t)$, therefore

$$\langle L^\pi\rangle(T) = \sum_{k=0}^{\infty}\langle \Upsilon_\pi^{N_k,x_k,k+1}\rangle\big(T\wedge T_{k+1} - T\wedge T_k\big) \le TK_\pi\cdot \max_{0\le t\le T}\mathcal{N}(t)\cdot \max_{\substack{N\ge 2\\ 1\le k\le N}}\sigma^2_{Nk}\,.$$

Since $\sigma_{Nk} \le \sigma$ holds for $N \ge 2,\ 1 \le k \le N$, the property (33) follows from (21). □

4.5. **Some Open Questions.** (1) The above proof used the boundedness of the portfolio $\pi(\cdot)$ in a crucial way. It would be interesting to see whether arbitrage in this (or in a related) model with splits and mergers might exist with more general, unbounded portfolios.

(2) The estimate (31) also gives the bound

$$\langle M\rangle(T) - \langle M\rangle(t) \le C^2 T\cdot \max_{t\le\theta\le T}\mathcal{N}(\theta)$$

for every $t \in (0, T)$. From the theory of BMO martingales, in order to show that the exponential process $Z(\cdot)$ of (32), (29) is a martingale, it suffices to show that $\mathbf{E}\big(\max_{t\le\theta\le T}\mathcal{N}(\theta)\,|\,\mathcal{F}(t)\big)$, $0 \le t \le T$ is uniformly bounded. If this can be done, it might obviate the need to establish the sub-exponential bound of Proposition 4.1.

(3) It would be very interesting to decide whether absence of arbitrage, say with respect to the market portfolio, survives when one begins to constrain the splits and/or mergers that can happen over a given period of time, or along genealogies of companies produced by any given split (mandating, say, that the resulting companies cannot be touched for a certain amount of time); we believe not, but this issue remains to be settled.

5. Appendix

Lemma 5.1. *Consider a CBP-based market model as described in Definition* 1 *of Subsection* 3.2. *Let*

$$\tau := \inf\big\{t \ge 0 :\ \exists\ \ i = 1,\ldots,N\,,\ \ s.t.\ \ \mu_i(t) = 1-\delta\,\big\}\,,\qquad \overline{\sigma} := \max_{1\le k\le N}\sigma_k$$

and assume in the manner of (1) *that*

$$g_1 \le \min_{2\le k\le N} g_k\,.$$

Then for an independent random variable η, exponentially distributed with parameter $\lambda > 0$, we have

$$\mathbf{P}(\tau < \eta) \le 2\left(\frac{\mu_{(1)}(0)\vee\frac{1}{2}}{1-\delta}\right)^{\overline{\sigma}^{-1}\lambda^{1/2}}. \tag{34}$$

The idea behind the argument of the proof is as follows. We can rewrite the stopping time τ as $\tau = \inf\big\{t \ge 0 : \mu_{(1)}(t) = 1-\delta\big\}$; indeed, whenever a market weight reaches $1-\delta$, then it gets also to be the largest market weight, because $1-\delta > 1/2$. The process $\log\mu_{(1)}(\cdot)$ reflects on $\log\mu_{(2)}(\cdot)$, as follows from (10); it behaves like an Itô process, but when it collides with $\log\mu_{(2)}(\cdot)$ a positive local time term emerges. Now, we would like to replace $\log\mu_{(1)}(\cdot)$ with something larger. Consider a similar process $U(\cdot)$, now reflected at $\log(1/2)$; the second top market weight never gets above $1/2$, and for this new reflection pattern the logarithm of the top market weight will reflect earlier, so the resulting reflected process $U(\cdot)$ will be larger. In particular, we shall have $\tau \geq \widetilde{\tau} := \inf\big\{t \geq 0 : U(t) = 1 - \delta\big\}$; and the probability that the exponential clock (which is responsible for mergers) rings later than τ (the time of a split), will be smaller than the probability that this exponential clock rings later than $\widetilde{\tau}$. But $U(\cdot)$ is plain reflected Brownian motion, so we can calculate this latter probability explicitly.

Proof: Let us derive an equation for the dynamics of $\log \mu_{(1)}(\cdot)$. We recall the expression for the dynamics of $\log \mu_i(\cdot)$ from (14), and denote by $\Lambda_{(k,\ell)}(\cdot) = \{\Lambda_{(k,\ell)}(t),\ t \geq 0\}$ the local time at the origin of the continuous semimartingale

$$\log \mu_{(k)}(\cdot) - \log \mu_{(\ell)}(\cdot) = Y_{(k)}(\cdot) - Y_{(\ell)}(\cdot)\,, \qquad \text{for} \quad 1 \leq k < \ell \leq N\,.$$

We have $\Lambda_{(k,\ell)}(\cdot) \equiv 0$ if $\ell - k \geq 2$, see [18, Lemma 1], as well as

$$\mathrm{d} \log \mu_{(1)}(t) = \sum_{i=1}^{N} \mathbf{1}_{\{\mu_i(t) = \mu_{(1)}(t)\}}\, \mathrm{d} \log \mu_i(t) + \frac{1}{2}\, \mathrm{d}\Lambda_{(1,2)}(t) \tag{35}$$

from [3]. We also note from [18] that the set $\{t \geq 0 \mid \mu_{(k)}(t) = \mu_{(1)}(t)\} = \{t \geq 0 \mid Y_{(k)}(t) = Y_{(1)}(t)\}$ has zero Lebesgue measure, for $k = 2, \ldots, N$. Therefore, we can rewrite (35) as

$$\begin{aligned} \mathrm{d} \log \mu_{(1)}(t) &= \beta(t)\, \mathrm{d}t + \sigma_1\, \mathrm{d}B_1(t) - \sum_{k=1}^{N} \sigma_k \mu_{(k)}(t) \mathrm{d}B_k(t) + \frac{1}{2}\, \mathrm{d}\Lambda_{(1,2)}(t) \\ &= \beta(t)\mathrm{d}t + \sqrt{a(t)}\, \mathrm{d}V(t) + \frac{1}{2}\, \mathrm{d}\Lambda_{(1,2)}(t)\,. \end{aligned}$$

Here we are recalling the notation of (11), we are denoting by $V(\cdot) = \{V(t),\ t \geq 0\}$ yet another one-dimensional standard $\big\{\mathcal{F}(t)\big\}_{t \geq 0}$–Brownian motion, and we are using the notation

$$\beta(t) \;:=\; g_1 - \sum_{k=1}^{N} g_k \mu_{(k)}(t) - \frac{1}{2} \sum_{k=1}^{N} \sigma_k^2 \left(\mu_{(k)}(t) - \mu_{(k)}^2(t)\right),$$

$$a(t) \;:=\; \sigma_1^2 \big(1 - \mu_{(1)}(t)\big)^2 + \sum_{k=2}^{N} \sigma_k^2\, \mu_{(k)}^2(t) > 0\,.$$

The following estimates hold for these quantities:

$$\begin{aligned} \beta(t) &= g_1\big(1 - \mu_{(1)}(t)\big) - \frac{1}{2}\sigma_1^2\big(\mu_{(1)}(t) - \mu_{(1)}^2(t)\big) - \sum_{k=2}^{N} \left[g_k\, \mu_{(k)}(t) + \frac{1}{2}\, \sigma_k^2 \left(\mu_{(k)}(t) - \mu_{(k)}^2(t)\right) \right] \\ &\leq g_1\big(1 - \mu_{(1)}(t)\big) - \sum_{k=2}^{N} g_k \mu_{(k)}(t) \;\leq\; g_1\big(1 - \mu_{(1)}(t)\big) - \min_{2 \leq k \leq N} g_k \cdot \sum_{k=2}^{N} \mu_{(k)}(t) \\ &= \Big(g_1 - \min_{2 \leq k \leq N} g_k\Big)\big(1 - \mu_{(1)}(t)\big) \leq 0\,; \\ a(t) &\leq \sigma_1^2 + \max_{2 \leq k \leq N} \sigma_k^2 \leq 2\,\overline{\sigma}^2\,. \end{aligned}$$

Let us make a time-change: using Lemma 2 from [28] for $\sigma = 1$, we have

$$\mathrm{d}Z(s) = \gamma(s)\, \mathrm{d}s + \mathrm{d}\overline{V}(s) + \mathrm{d}\overline{\Lambda}(s)\,,$$

where $\overline{V}(\cdot) = \big\{\overline{V}(s),\ s \geq 0\big\}$ is yet another standard Brownian motion and

$$Z(\cdot) = \{Z(s), s \geq 0\}\,, \quad Z(s) = \log \mu_{(1)}(T(s))\,,$$

$$\overline{\Lambda}(\cdot) = \left\{\overline{\Lambda}(s), s \ge 0\right\}, \quad \overline{\Lambda}(s) = \frac{1}{2}\,\Lambda_{(1,2)}(T(s)),$$

$$T(s) = \inf\{t \ge 0 \mid \Delta(t) \ge s\}, \qquad \Delta(t) := \int_0^t a(s)\mathrm{d}s, \quad t \ge 0, \qquad \gamma(s) = \frac{\beta(T(s))}{a(T(s))}.$$

We shall show the comparison $Z(\cdot) \le Z_0(\cdot)$, where $Z_0(\cdot) = \{Z_0(s),\ s \ge 0\}$ is a one-dimensional Brownian motion with zero drift and unit dispersion, starting from $\log \mu_{(1)}(0)$ and reflected at $\log(1/2)$, namely

$$\mathrm{d}Z_0(s) = \mathrm{d}\overline{V}(s) + \mathrm{d}\Lambda^0(s).$$

Here $\Lambda^0(\cdot) = \{\Lambda^0(s),\ s \ge 0\}$ is the local time of this reflecting Brownian motion at the site $\log(1/2)$.

The proof proceeds along the same lines as in [19, Chapter 6, Theorem 1.1], but with some adjustments which are necessary because of the local time terms. Let us recall the essential steps of the proof. We define $\psi(x) := x_+^3$ for $x \in \mathbb{R}$; then $\psi \in C^2(\mathbb{R})$, and we have

$$\psi(Z(s) - Z_0(s)) = \int_0^s \psi'(Z(u) - Z_0(u))\gamma(u)\mathrm{d}u + \int_0^s \psi'(Z(u) - Z_0(u))(\mathrm{d}\overline{\Lambda}(u) - \mathrm{d}\Lambda^0(u)).$$

This does not contain stochastic integrals, because in the expression for $Z(s) - Z_0(s)$ they cancel out. But $\gamma(s) \le 0$, and when $Z(s) > Z_0(s)$ we have $Z(s) > \log(1/2)$, therefore $\mathrm{d}\overline{\Lambda}(s) = 0$. On the other hand, we always have $\psi'(x) = 3x_+^2 \ge 0$, so $\psi(Z(s)-Z_0(s)) \le 0$, and this implies $Z(s) \le Z_0(s)$.

Let $\tau_0 := \inf\{t \ge 0 \mid Z_0(t) = \log(1-\delta)\}$. We note that $\Delta(\tau)$ is the hitting time by $Z(\cdot)$ of the level $\log(1-\delta)$, so we have $\tau_0 \le \Delta(\tau)$. But $\Delta'(t) = a(t) \le 2\,\overline{\sigma}^2$, thus $\tau \ge \tau_0/(2\,\overline{\sigma}^2)$ and

$$\mathbf{P}(\tau < \eta) \le \mathbf{P}(\tau_0 < 2\,\overline{\sigma}^2\eta).$$

Using [5, Part II, Section 3, formula 1.1.2] and the fact that $2\,\overline{\sigma}^2\eta$ is exponentially distributed with parameter $(\lambda\,\overline{\sigma}^{-2})/2$, we get

$$\mathbf{P}\big(\tau_0 < 2\,\overline{\sigma}^2\eta\big) = \mathbf{P}_{\tilde{x}}\Big(\sup_{0 \le s \le 2\,\overline{\sigma}^2\eta} |\overline{B}(s)| \ge y\Big) = \frac{\mathrm{ch}\left(\tilde{x}\sqrt{\lambda}\,\overline{\sigma}^{-1}\right)}{\mathrm{ch}\left(\tilde{y}\sqrt{\lambda}\,\overline{\sigma}^{-1}\right)}$$

with $\tilde{y} = \log(1-\delta) - \log(1/2)$, $\tilde{x} = \big(\log(\mu_{(1)}(0)) - \log(1/2)\big)^+$. From the elementary inequality $(e^z/2) \le \mathrm{ch}\,z \le e^z$, $z \ge 0$ we conclude

$$\frac{\mathrm{ch}\left(\tilde{x}\sqrt{\lambda}\,\overline{\sigma}^{-1}\right)}{\mathrm{ch}\left(\tilde{y}\sqrt{\lambda}\,\overline{\sigma}^{-1}\right)} \le 2\exp\Big(-\big(\tilde{y} - \tilde{x}\big)\sqrt{\lambda}\,\overline{\sigma}^{-1}\Big),$$

and it is then straightforward to rewrite the right-hand side as (34). This completes the proof. □

References

[1] Francesco Audrino, Robert Fernholz, and Roberto G. Ferretti. A forecasting model for stock market diversity. *Annals of Finance*, 3(2):213–240, 2007.

[2] Adrian D. Banner, Robert Fernholz, and Ioannis Karatzas. Atlas models of equity markets. *Ann. Appl. Probab.*, 15(4):2296–2330, 2005.

[3] Adrian D. Banner and Raouf Ghomrasni. Local times of ranked continuous semimartingales. *Stochastic Process. Appl.*, 118(7):1244–1253, 2008.

[4] R. F. Bass and É. Pardoux. Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields*, 76(4):557–572, 1987.

[5] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian motion—facts and formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[6] Sourav Chatterjee and Soumik Pal. A phase transition behavior for Brownian motions interacting through their ranks. *Probab. Theory Related Fields*, 147(1-2):123–159, 2010.

[7] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[8] Robert Fernholz. *Stochastic portfolio theory*, volume 48 of *Applications of Mathematics (New York).* Springer-Verlag, New York, 2002. Stochastic Modelling and Applied Probability.

[9] Robert Fernholz, Tomoyuki Ichiba, and Ioannis Karatzas. A second-order stock market model. *Annals of Finance*, 9(3):439–454, 2013.
[10] Robert Fernholz, Tomoyuki Ichiba, and Ioannis Karatzas. Two Brownian particles with rank-based characteristics and skew-elastic collisions. *Stochastic Process. Appl.*, 123(8):2999–3026, 2013.
[11] Robert Fernholz and Ioannis Karatzas. Relative arbitrage in volatility-stabilized markets. *Annals of Finance*, 1:149–177, 2005.
[12] Robert Fernholz and Ioannis Karatzas. Stochastic portfolio theory: an overview. *Handbook of Numerical Analysis*, 15:89–167, 2009.
[13] Robert Fernholz, Ioannis Karatzas, and Constantinos Kardaras. Diversity and relative arbitrage in equity markets. *Finance Stoch.*, 9(1):1–27, 2005.
[14] Tomoyuki Ichiba. *Topics in multi-dimensional diffusion theory: Attainability, reflection, ergodicity and rankings*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Columbia University.
[15] Tomoyuki Ichiba and Ioannis Karatzas. On collisions of Brownian particles. *Ann. Appl. Probab.*, 20(3):951–977, 2010.
[16] Tomoyuki Ichiba, Ioannis Karatzas, and Mykhaylo Shkolnikov. Strong solutions of stochastic equations with rank-based coefficients. *Probab. Theory Related Fields*, 156(1-2):229–248, 2013.
[17] Tomoyuki Ichiba, Soumik Pal, and Mykhaylo Shkolnikov. Convergence rates for rank-based models with applications to portfolio theory. *Probability Theory and Related Fields*, pages 1–34, 2012.
[18] Tomoyuki Ichiba, Vassilios Papathanakos, Adrian Banner, Ioannis Karatzas, and Robert Fernholz. Hybrid atlas models. *Ann. Appl. Probab.*, 21(2):609–644, 2011.
[19] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.
[20] Ioannis Karatzas, Soumik Pal, and Mykhaylo Shkolnikov. Systems of brownian particles with asymmetric collisions. 2012. Preprint. Available at arXiv:1210.0259v1.
[21] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
[22] Constantinos Kardaras. Balance, growth and diversity of financial markets. *Annals of Finance*, 4(3):369–397, 2008.
[23] Jorg R. Osterrieder and Thorsten Rheinlander. Arbitrage opportunities in diverse markets via a non-equivalent measure change. *Annals of Finance*, 2:287–301, 2006.
[24] Soumik Pal and Jim Pitman. One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab.*, 18(6):2179–2207, 2008.
[25] Soumik Pal and Mykhaylo Shkolnikov. Concentration of measure for systems of brownian particles interacting through their ranks. 2010. Preprint. Available at arXiv:1011.2443.
[26] Johannes Ruf and Wolfgang Runggaldier. A systematic approach to constructing market models with arbitrage. 2013. Preprint. Available on arXiv:1309.1988.
[27] Andrey Sarantsev. Explicit rates of exponential convergence for reflected brownian motion with jumps in the positive orthant and for competing levy particles. 2013. Preprint. Available at arXiv:1305.1653.
[28] Andrey Sarantsev. On a class of diverse market models. 2013. To appear in Annals of Finance. Available at arXiv:1301.5941.
[29] Andrey Sarantsev. Infinite systems of competing brownian particles: Existence, uniqueness and convergence results. 2014. Preprint. Available at arXiv:1403.4229.
[30] Andrey Sarantsev. Infinite systems of competing brownian particles: Stationary distributions for the gap process. 2014. Preprint. Available at arXiv:1309.2621.
[31] Andrey Sarantsev. Triple and multiple collisions of competing brownian particles. 2014. Preprint. Available at arXiv:1401.6255.
[32] Mykhaylo Shkolnikov. Competing particle systems evolving by interacting Lévy processes. *Ann. Appl. Probab.*, 21(5):1911–1932, 2011.
[33] Winslow Strong. Fundamental theorems of asset pricing for piecewise semimartingales of stochastic dimension. 2011. Preprint. Available on arXiv:1112.5340.
[34] Winslow Strong and Jean-Pierre Fouque. Diversity and arbitrage in a regulatory breakup model. *Annals of Finance*, 7:349–374, 2011.

INTECH Investment Management LLC, One Palmer Square, Suite 441, Princeton, NJ 08542; Department of Mathematics, Columbia University, New York, NY 10027, USA
E-mail address: ik@enhanced.com; ik@math.columbia.edu

Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195-4350
E-mail address: ansa1989@math.washington.edu