# $\mathbf{D}_p$-Generated Indices

Measures of diversity generate diversity weighted indices. Entropy is a classical measure of diversity, but it is certainly not the only one. Another diversity measure, $\mathbf{D}_p$, is discussed in Example 4.3 in the paper *Portfolio Generating Functions* by Robert Fernholz. Adjustment of $p$ makes it possible to control the excess return and tracking error of the corresponding diversity weighted index. For example, $p$ can be set so that the tracking error between the diversity weighted index and the base index is about the same as the tracking error between the S&P 500 and the Russell 1000.

The definition of the $\mathbf{D}_p$ measure is

$$\mathbf{D}_p = \Big( \sum_{i=1}^{n} \pi_i^p \Big)^{1/p},$$

where $\pi_i$, for $i = 1, \ldots, n$, are the weights of the cap-weighted base index. $\mathbf{D}_p$ has a minimum of 1 if all the capital is concentrated in one name, and a maximum of $n^{(1-p)/p}$ if all the weights are equal. The diversity weights will be $\eta_i = (\pi_i/\mathbf{D}_p)^p$, for $i = 1, \ldots, n$. As $p$ decreases from 1 to 0 the "leverage" increases, with the corresponding diversity weighted index cap weighted for $p = 1$ and equal weighted for $p = 0$. In the attached chart, the weight ratios for several values of $p$ are shown, with the entropy weight ratios given for reference by a dotted line.

The value of $p$ for which the tracking error will about the same as that of the S&P 500 versus the Russell 1000 is $p = .7$ for 500 stocks and $p = .8$ for 1000 stocks. The Sharpe ratios and information ratios for these diversity weighted indices as well as the entropy weighted indices appear in the table below. In each case, the base indices are generated by the largest U.S. exchange traded stocks over the period from 1965 to 1994.

| *Strategy* | *I.R. actual* | *I.R. theoretical* | *S.R.* |
|---|---|---|---|
| Entropy 500 | $29/82 = .35$ | $21/82 = .25$ | .28 (.26) |
| $\mathbf{D}_p$ 500 ($p = .7$) | $44/129 = .34$ | $35/129 = .27$ | .29 (.26) |
| Entropy 1000 | $39/109 = .35$ | $33/109 = .30$ | .30 (.28) |
| $\mathbf{D}_p$ 1000 ($p = .8$) | $45/132 = .34$ | $41/132 = .31$ | .31 (.28) |

(Rounding may affect the apparent ratios presented here.) No trading costs have been deducted—the turnover for the entropy strategies is about 1.5 times that of the base index, and for the $\mathbf{D}_p$ strategies it is about twice that of the base. The interpretation of this table for the Entropy 500 strategy, for example, is as follows:

*I.R. actual:* This is the actual logarithmic information ratio over the period from 1965 to 1994. The excess log-return was 29 b.p. with a log tracking error of 82 b.p. for a ratio of .35.

*I.R. theoretical:* This is the logarithmic information ratio after correction for the (positive) drift in the diversity measurement, in this case, the entropy. Here we have 21 b.p. of excess log-return with the same 82 b.p. tracking error, for a ratio of .25. We assume here that the dividend rates for the diversity weighted and cap-weighted portfolios are the same, which currently appears to be the case.

*S.R.:* This gives the Sharpe ratio of the strategy. Note that by its very nature, the Sharpe ratio is not logarithmic and is based on actual returns without correction for diversity drift. The .28 (.26) means that the Sharpe ratio of the Entropy 500 strategy was .28 while that of the cap-weighted base index was .26.

The $\mathbf{D}_p$ strategies described above should generate about 40 b.p. a year in excess returns (non-logarithmic) for the $\mathbf{D}_p$ 500 and 48 b.p. a year for the $\mathbf{D}_p$ 1000, both with about the same tracking error as the S&P 500 versus the Russell 1000.

Robert Fernholz
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