

# Factorization of Equity Returns

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## Abstract

The relative return of an equity portfolio with respect to the market is factored into three components. The factorization separates the effects due to change in the distribution of capital in the market, to change in rank of the stocks in the portfolio, and to dividends. The factorization is of the nature of an accounting identity, and can be used to isolate the company size factor from other factors that affect equity returns. The results are applied to analyze the return structure of the S&P/Barra Value Index.

*Key words:* Factorization of returns, portfolio generating function, value stocks.

*JEL classification:* G11.

# 1 Introduction

It has been understood for some time that the size of companies can affect stock returns, and hence portfolio performance (see Banz (1981), Reinganum (1981), Fama and French (1992,1993,1995,1996), and Fernholz and Garvy (1999)). Usually the size factor is estimated by statistical factor analysis, and this type of analysis can be complicated by correlation among the many factors that affect stock returns. Here we propose a method of factorization of returns that is of the nature of an accounting identity and will isolate the size factor from other factors that affect stock and portfolio returns.

Fernholz (2001) showed that in a continuously traded equity market certain types of mathematical functions generate portfolios. The return on such a portfolio can be factored naturally into three components: a component that measures the effect of changes in the distribution of capital in the market, a component that measures the effect of changes in rank among the stocks in the market, and a component that measures the effect of dividend payments. In this paper we carry out the factorization of Fernholz (2001) for arbitrary equity portfolios over a single period of time (cf. Fernholz (1998)).

The component that measures the effect of changes in the distribution of capital in the market captures any impact that the ebb and flow of capital between large and small stocks may have on the portfolio return. The size effect as it is traditionally measured is a one-dimensional approximation of this distributional component.

The stocks in the market can be ranked by capitalization, with the highest capitalization first. If a stock rises in rank it is likely to have had higher returns than other stocks close to it in capitalization. Hence, the rank component of stock returns measures the relative performance of the stock with respect to other stocks of similar size.

If companies distribute earnings, this has an obvious effect on stock returns but is not reflected in the capitalization of the stock. Hence, a factor that accounts for dividends and other such distributions will be present in the relative return.

As an application of the proposed factorization, we analyze the relative performance of the S&P/Barra Value Index versus the S&P 500 Index over the period from 1975 to 1998. The results of the factorization show that the dividend component most favored the value stocks, but was nullified by a negative rank component. The distributional component has also affected the relative returns of value stocks, particularly in the late 1990s.

In order to simplify our presentation, we shall assume that we operate in an equity market in which there are no takeovers, mergers, or spinoffs, in which no companies enter or leave the market, and in which the number of shares of each company does not change. This is a reasonable approximation of reality over a short enough period of time, and a longer period can be broken up into a sequence of short periods.

## 2 Factorization of stock returns

Suppose that  $\mathcal{M}$  is an equity market composed of  $n$  stocks  $\mathcal{X}_1, \dots, \mathcal{X}_n$  with capitalizations  $X_1 > X_2 > \dots > X_n$  in decreasing order. Then the total capitalization of  $\mathcal{M}$  is

$$M = X_1 + \dots + X_n. \tag{2.1}$$

Let  $\mu_i$  denote the market weight of  $X_i$ , so

$$\mu_i = X_i/M,$$

for  $i = 1, \dots, n$ , with  $\mu_1 + \dots + \mu_n = 1$ . In this case,

$$\mu_1 > \mu_2 > \dots > \mu_n,$$

and we shall refer to the decreasing family of market weights  $\{\mu_1, \dots, \mu_n\}$  as the *capital distribution* of  $\mathcal{M}$ .

Suppose that a period of time  $t$  passes and the stock capitalizations change from  $X_i$  to  $X'_i$  for all  $i$ . We shall denote this change in from  $X_i$  to  $X'_i$  over the time  $t$  by

$$X_i \xrightarrow{t} X'_i,$$

for  $i = 1, \dots, n$ . During the same period of time  $t$  the market capitalization undergoes the transition

$$X_1 + \dots + X_n \xrightarrow{t} X'_1 + \dots + X'_n,$$

or,

$$M \xrightarrow{t} M',$$

where

$$M' = X'_1 + \dots + X'_n.$$

The changes in stock capitalizations induce a transformation of the market weights

$$\mu_i \xrightarrow{t} \mu'_i,$$

where

$$\mu'_i = X'_i/M',$$

for  $i = 1, \dots, n$ .

It is unlikely that the stock capitalizations  $X'_1, \dots, X'_n$  after time  $t$  will still be in descending order, but if we assume that the changes in capitalization have continuous probability distributions, then with probability 1 there exists a permutation  $p$  of the integers 1 through  $n$  such that

$$X'_{p(1)} > X'_{p(2)} > \dots > X'_{p(n)}.$$

This ranking produces a new capital distribution  $\{\mu'_{p(1)}, \dots, \mu'_{p(n)}\}$ , where  $\mu'_{p(i)}$  is in the  $i$ -th rank of the capital distribution, the rank previously occupied by  $\mu_i$ .

We can factor the transformation from  $\mu_i$  to  $\mu'_i$  into the composition of transformations  $d$  and  $r$  with

$$\begin{array}{ccc} \mu_i & & \\ \downarrow d & \searrow t & \\ \mu'_{p(i)} & \xrightarrow{r} & \mu'_i \end{array} \quad (2.2)$$

for each stock  $\mathcal{X}_i$ . Here

$$\mu_i \xrightarrow{d} \mu'_{p(i)}$$

represents the change in capital distribution from  $\{\mu_1, \dots, \mu_n\}$  to  $\{\mu'_{p(1)}, \dots, \mu'_{p(n)}\}$ , and

$$\mu'_{p(i)} \xrightarrow{r} \mu'_i$$

represents the change in rank of  $\mathcal{X}_i$  within the capital distribution from the  $i$ -th rank to the  $p^{-1}(i)$ -th rank, where  $\mu'_i$  resides in  $\{\mu'_{p(1)}, \dots, \mu'_{p(n)}\}$ .

Let us assume for the moment that there are no dividend payments over the period, so the (logarithmic) return of  $\mathcal{X}_i$  is

$$\log(X'_i/X_i),$$

and the return of  $\mathcal{M}$  over the period is

$$\log(M'/M).$$

In this case the relative return of  $\mathcal{X}_i$  with respect to the market is

$$\log(X'_i/X_i) - \log(M'/M) = \log(\mu'_i/\mu_i).$$

The diagram (2.2) corresponds to the factorization

$$\mu'_i/\mu_i = (\mu'_{p(i)}/\mu_i)(\mu'_i/\mu'_{p(i)}),$$

and this produces the decomposition of the return

$$\log(\mu'_i/\mu_i) = \log(\mu'_{p(i)}/\mu_i) + \log(\mu'_i/\mu'_{p(i)}), \quad (2.3)$$

for  $i = 1, \dots, n$ . Hence the relative return of  $\mathcal{X}_i$  can be split into a component due to distributional change

$$\log(\mu'_{p(i)}/\mu_i),$$

and a component due to change in the rank of  $\mathcal{X}_i$  in the market,

$$\log(\mu'_i/\mu'_{p(i)}).$$

If the stocks pay dividends, then the return over the time period  $t$  also has a dividend component. Let  $\delta_i$  represent the (logarithmic) dividend rate of  $\mathcal{X}_i$  over the period, so the total value of  $\mathcal{X}_i$  including dividends is  $X'_i e^{\delta_i}$ . Let  $\delta$  represent dividend rate for the market. In this case the value of the market with dividends after time  $t$  is

$$M' e^\delta = X'_1 e^{\delta_1} + \dots + X'_n e^{\delta_n},$$

so

$$\delta = \log(\mu'_1 e^{\delta_1} + \dots + \mu'_n e^{\delta_n}).$$

It follows that the total return of  $\mathcal{X}_i$  over the time period  $t$  is

$$\log(X'_i/X_i) + \delta_i,$$

and that the relative return of  $\mathcal{X}_i$  with respect to the market is

$$\log(\mu'_i/\mu_i) + (\delta_i - \delta).$$

Hence, the decomposition (2.3) becomes

$$\log(\mu'_i/\mu_i) = \log(\mu'_{p(i)}/\mu_i) + \log(\mu'_i/\mu'_{p(i)}) + (\delta_i - \delta). \quad (2.4)$$

The decomposition (2.4) shows that the relative return of  $\mathcal{X}_i$  contains a distributional component, a rank component, and a dividend component. In simple terms, this means that the relative return of a particular stock will tend to be higher if the stock resides in a part of the capital distribution that increases in weight, if the stock moves to a higher ranking in the capital distribution, or if the stock pays a higher than average dividend over the period. Although the three components we consider may be similar to those found in the literature, our decomposition is closer to an accounting identity than to the customary statistical factor analysis (see, e.g., Ross (1976) and Fama and French (1992,1993,1995,1996)). The distributional component captures the entire effect of the ebb and flow of capital between large and small stocks in the market, whereas the size factor in regression analysis must project this effect onto a one-dimensional subspace.

### 3 Factorization of portfolio returns

In this section we shall extend the results of the previous section to portfolios of stocks. As in the previous section, let  $\mathcal{M}$  be a market of  $n$  stocks  $\mathcal{X}_1, \dots, \mathcal{X}_n$  with corresponding capitalizations  $X_1 > \dots > X_n$ .

**Definition 3.1.** A *portfolio* in  $\mathcal{M}$  is an element  $\mathbf{v} = (v_1, \dots, v_n)$  of  $\mathbb{R}^n$  such that

$$v_1 X_1 + \dots + v_n X_n > 0.$$

This definition deviates somewhat from most conventional definitions of a portfolio. Here the coordinate  $v_i$  of the portfolio  $v$  represents the fraction of the total shares outstanding for  $\mathcal{X}_i$  that are held in  $\mathbf{v}$ . Hence, the *value* of the portfolio is

$$V = v_1 X_1 + \dots + v_n X_n, \quad (3.1)$$

and we have assumed in Definition 3.1 that this value is positive. The portfolio with  $v_i = 1$ , for  $i = 1, \dots, n$  is called the *market portfolio* and its value is given by (2.1).

The *portfolio weights* are

$$\pi_i = v_i X_i / V,$$

for  $i = 1, \dots, n$ . We can calculate the *weight ratios*

$$\pi_i / \mu_i = v_i M / V,$$

for  $i = 1, \dots, n$ . Hence, the weight ratios are proportional to the  $v_i$  with constant of proportionality  $M/V$ .

**Definition 3.2.** Let  $\mathbf{v}$  be a portfolio and suppose that  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Let  $p$  be a permutation of the integers from 1 to  $n$  such that

$$x_{p(1)} \geq x_{p(2)} \geq \dots \geq x_{p(n)}.$$

Then the function  $\mathbf{S}$  defined by

$$\mathbf{S}(x) = v_1 x_{p(1)} + \cdots + v_n x_{p(n)}$$

is called the *generating function* of  $\mathbf{v}$ . The function  $\mathbf{R}$  defined by

$$\mathbf{R}(x) = \frac{v_1 x_1 + \cdots + v_n x_n}{v_1 x_{p(1)} + \cdots + v_n x_{p(n)}}$$

is called the *rank sensitivity function* of  $\mathbf{v}$

Suppose that time  $t$  passes and the stock capitalizations change from  $X_i$  to  $X'_i$ , as in the previous section. Then the value of  $\mathbf{v}$  undergoes the transition

$$v_1 X_1 + \cdots + v_n X_n \xrightarrow{t} v_1 X'_1 + \cdots + v_n X'_n,$$

or,

$$V \xrightarrow{t} V',$$

where

$$V' = v_1 X'_1 + \cdots + v_n X'_n,$$

and we shall again assume that  $V' > 0$ . Let us assume for now that there are no dividends paid over the period we are considering. In this case, the transition from  $V$  to  $V'$  produces a portfolio return of

$$\log(V'/V).$$

Consider now the transition

$$V/M \xrightarrow{t} V'/M',$$

which is equivalent to

$$v_1 \mu_1 + \cdots + v_n \mu_n \xrightarrow{t} v_1 \mu'_1 + \cdots + v_n \mu'_n \quad (3.2)$$

In a manner similar to (2.2), the transformation (3.2) can be factored

$$\begin{array}{ccc} v_1 \mu_1 + \cdots + v_n \mu_n & & \\ \downarrow d & \searrow t & \\ v_1 \mu'_{p(1)} + \cdots + v_n \mu'_{p(n)} & \xrightarrow{r} & v_1 \mu'_1 + \cdots + v_n \mu'_n, \end{array} \quad (3.3)$$

and we shall assume that  $v_1 \mu'_{p(1)} + \cdots + v_n \mu'_{p(n)} > 0$ . The transformation  $d$  is due to the change in the capital distribution and is equivalent to

$$\mathbf{S}(\mu) \xrightarrow{d} \mathbf{S}(\mu')$$

where  $\mu = (\mu_1, \dots, \mu_n)$  and  $\mu' = (\mu'_1, \dots, \mu'_n)$ . The transformation  $r$  corresponds to changes in the ranking of the stocks after time  $t$ .

Following the diagram (3.3), the relative return of the  $\mathbf{v}$  with respect to the market can be decomposed as

$$\begin{aligned}
\log(V'/V) - \log(M'/M) &= \log\left(\frac{V'/M'}{V/M}\right) \\
&= \log\left(\frac{v_1\mu'_1 + \cdots + v_n\mu'_n}{v_1\mu_1 + \cdots + v_n\mu_n}\right) \\
&= \log\left(\frac{v_1\mu'_{p(1)} + \cdots + v_n\mu'_{p(n)}}{v_1\mu_1 + \cdots + v_n\mu_n}\right) + \log\left(\frac{v_1\mu'_1 + \cdots + v_n\mu'_n}{v_1\mu'_{p(1)} + \cdots + v_n\mu'_{p(n)}}\right) \\
&= \log(\mathbf{S}(\mu')/\mathbf{S}(\mu)) + \log \mathbf{R}(\mu').
\end{aligned} \tag{3.4}$$

The first term in (3.4),

$$\log(\mathbf{S}(\mu')/\mathbf{S}(\mu)), \tag{3.5}$$

measures the effect on the portfolio of the change in the capital distribution from  $\{\mu_1, \dots, \mu_n\}$  to  $\{\mu'_{p(1)}, \dots, \mu'_{p(n)}\}$ . The second term,  $\log \mathbf{R}(\mu')$ , measures the effect of the changes in rank among the stocks.

**Remark.** The decomposition (3.4), as well as the interpretation of the terms in it, is similar to the continuous-time versions presented in Fernholz (1999) and Fernholz (2001). The generating function  $\mathbf{S}$  corresponds to the generating functions used there to construct dynamic continuous-time portfolios, and  $\log \mathbf{R}$  corresponds to the drift function  $\Theta$ .

Since the  $v_i$  are proportional to the weight ratios for the portfolio, we see that (3.5) will be positive if, on average,  $\mathbf{v}$  is overweighted in those parts of the capital distribution that increase, and underweighted in those that decrease. Any effect that the ebb and flow of capital between large and small stocks may have on the portfolio will be captured this factor: it is a comprehensive size factor. The second term,  $\log \mathbf{R}(\mu')$ , will be positive if  $\mathbf{v}$  is overweighted in stocks that move up in rank. This could be interpreted as a stock-picking component: “good” stocks tend to rise in rank.

If the stocks pay dividends, then the return over the period also has a dividend component, as in the previous section. In this case the value of  $\mathbf{v}$  with dividends after time  $t$  is

$$V' e^{\delta_{\mathbf{v}}} = v_1 X'_1 e^{\delta_1} + \cdots + v_n X'_n e^{\delta_n},$$

so,

$$\delta_{\mathbf{v}} = \log(\pi'_1 e^{\delta_1} + \cdots + \pi'_n e^{\delta_n}),$$

where the portfolio weights  $\pi'_i$  satisfy

$$\pi'_i = v_i X'_i / V',$$

for  $i = 1, \dots, n$ . It follows that the total return of  $\mathbf{v}$  over the time period  $t$  is

$$\log(V'/V) + \delta_{\mathbf{v}},$$

and that the relative return of  $\mathbf{v}$  with respect to the market is

$$\log(V'/V) - \log(M'/M) + (\delta_{\mathbf{v}} - \delta).$$

Hence, the decomposition (3.4) now becomes

$$\log(V'/V) - \log(M'/M) = \log(\mathbf{S}(\mu')/\mathbf{S}(\mu)) + \log \mathbf{R}(\mu') + (\delta_{\mathbf{v}} - \delta). \quad (3.6)$$

This decomposition is completely analogous to the decomposition (2.4) for individual stocks. We shall now apply (3.6) to an actual portfolio of stocks.

## 4 Application: value stocks

In this section we apply the results of the previous section to factor the relative returns of value stocks. For our purposes, a value stock is one that is included in the S&P/Barra Value Index, and we consider the returns on this index relative to those of the the S&P 500 Index, which contains the S&P/Barra Value Index as a subindex. Since the S&P 500 Index is not a market portfolio, we must first extend our results to cover the relative returns of one portfolio versus another. (If we were to use the S&P 500 as the market, this would introduce the problem of *leakage*, which is discussed in Fernholz (2001).)

Let  $\mathbf{v}$  and  $\mathbf{w}$  be portfolios in the market  $\mathcal{M}$  with values  $V$  and  $W$ , respectively, and suppose that after time  $t$  these values have evolved from  $V$  to  $V'$  and  $W$  to  $W'$ , in both cases with dividends reinvested. We wish to decompose the relative return of  $\mathbf{v}$  with respect to  $\mathbf{w}$  in a manner similar to (3.4). Now, since both portfolios are in the same market, (3.4) applies to each of them. Hence, the relative return of  $\mathbf{v}$  with respect to  $\mathbf{w}$  satisfies

$$\log(V'/V) - \log(W'/W) = \log(\mathbf{S}_{\mathbf{v}}(\mu')/\mathbf{S}_{\mathbf{w}}(\mu')) + \log(\mathbf{R}_{\mathbf{v}}(\mu')/\mathbf{R}_{\mathbf{w}}(\mu')) + (\delta_{\mathbf{v}} - \delta_{\mathbf{w}}), \quad (4.1)$$

where in each case the subscripts for  $\mathbf{S}$ ,  $\mathbf{R}$ , and  $\delta$  indicate the portfolio that they represent.

We use equation (4.1) to factor the monthly returns of the S&P/Barra Value Index,  $\mathbf{v}$ , relative to the S&P 500 Index,  $\mathbf{w}$ , over the period from 1975, the first year for which S&P/Barra Value Index data are available, to 1998. About half of the capitalization of the S&P 500 is contained in the S&P/Barra Value Index, which is composed of those stocks with lower price-to-book ratios (see Standard & Poor's (1999)). Figure 1 displays the ratio of the weighted average capitalization of the S&P/Barra Value Index to that of the S&P 500 Index. (The weighted average capitalization of a portfolio is calculated as the sum over all the stocks in the portfolio of the portfolio weight of each stock times its total capitalization.) As can be seen from the chart, this ratio is usually less than one, and may be decreasing over time.

We consider the two indices to be portfolios contained in a market  $\mathcal{M}$  chosen from the monthly equity data base of the Center for Research in Securities Prices (CRSP) at the University of Chicago. We truncated the CRSP data base to eliminate ADRs, REITs, closed-end funds, as well as those stocks that remained under .005% market capitalization during their entire price history. After this truncation, there were about 7000 stocks in  $\mathcal{M}$ . The cumulative values of the four terms of the decomposition in (4.1) are presented in Figures 2 through 5.

In Figure 2 we see that that over the first three years of the time series the S&P/Barra Value Index returned almost 20% more than the S&P 500. However, after 1977 the Value Index underperformed the whole S&P 500 slightly, so whether or not value stocks may exhibit superior performance over the long term is not clear (see Black (1986) and Loughran (1997)). Let us now consider the three components of these relative returns.

The distributional component in Figure 3 behaved much like the relative return: over the first three years it moved almost 10% in favor of the Value Index, and then was essentially flat for the



rest of the period with a strong decline at the end. Over some periods, the relative return on value stocks appears to be positively correlated with the relative return of small stocks: during the first few years of the time series small stocks significantly outperformed large stocks in the aftermath of the “nifty-fifty” era, while during the last few years large stocks have prevailed (see Fernholz and Garvy (1999)). Moreover, Figure 1 indicates that such a correlation is likely, especially towards the end of the time series. Indeed, during the late 1990s, the distributional component is the dominant factor affecting the relative returns of the value stocks.

Figures 4 and 5 show that except for the first three years, the rank and dividend components essentially nullified each other. The value stocks paid higher dividends than the growth stocks as one would expect, but after 1977 the dividend payments were offset by loss of rank in the capital distribution. The behavior of the rank component over the first three years made a significant contribution to the value stocks’ outperformance over that period. For whatever reason, during the first three years of the simulation, all three components favored the value stocks.

Certain details in these charts may be significant. The sharp uptrend from 1975 to 1977 in the relative return shown in Figure 2 was the result of simultaneous uptrends in the distributional component in Figure 3 and the rank component in Figure 4. Likewise, the dip in 1980 and the double-pronged dip in 1990 to 1993 appear in all three of these time series. The rank-based Kendall’s  $\tau$  is a non-parametric test for positive correlation (see Hájek and Šidák (1967)), and the result of this test on the time series in Figures 3 and 4 indicates that the series are positively correlated with probability greater than .9999. This positive correlation may be evidence that both these factors are responding simultaneously to some other variable. It would seem natural here to reverse the causality and define a *value factor* that affected both the rank and distribution time series. Moreover, this would be consistent with the results of Fama and French (1992,1993,1995,1996).

## 5 Conclusions

We have shown that the relative return of a portfolio with respect to the market can be factored into three components. One component measures the contribution to the return of change in the capital distribution of the market, another measures the contribution of changes in rank among the stocks in the market, and the third measures the contribution of dividend payments. The factorization is of the nature of an accounting identity rather than the result of statistical factor analysis. An application of the factorization to the S&P/Barra Value Index indicates that the distributional factor has been an important factor affecting the relative returns on value stocks, particularly in the late 1990s.

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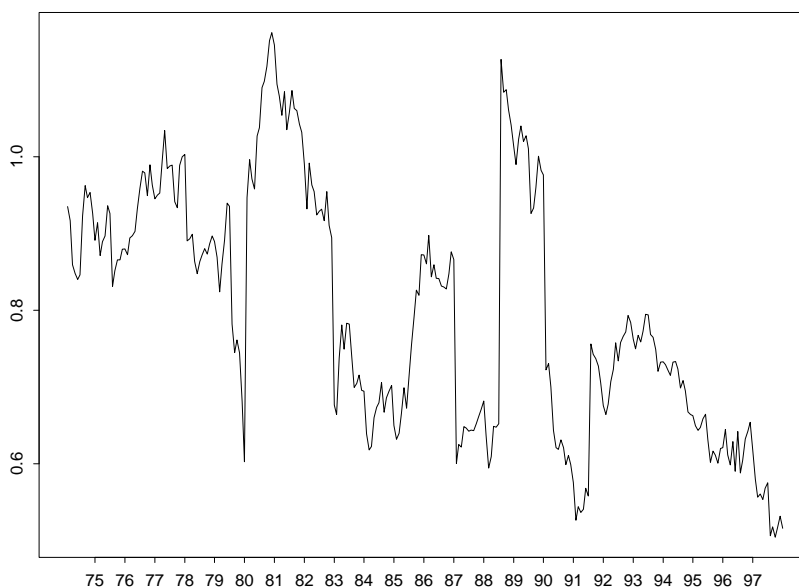


Figure 1: Ratio of weighted average capitalization

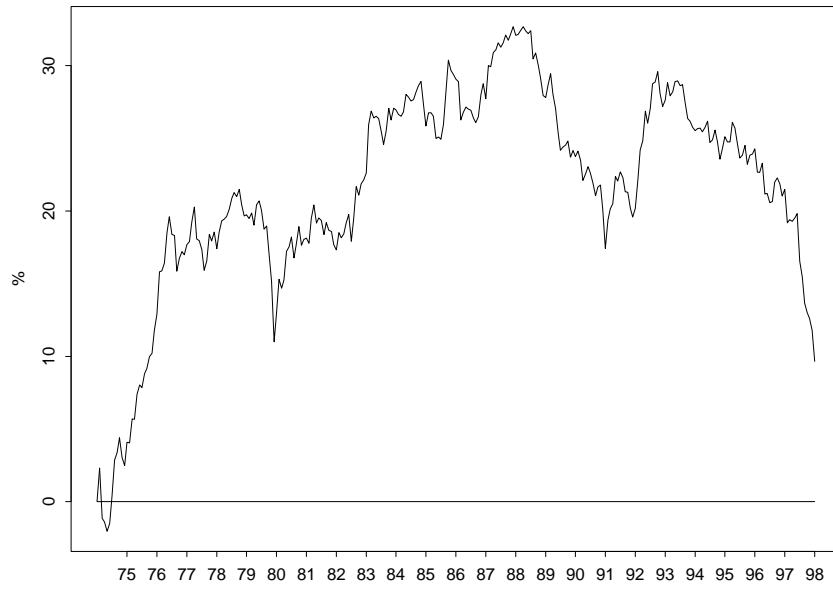


Figure 2: Relative return of Barra Value with respect to S&P 500

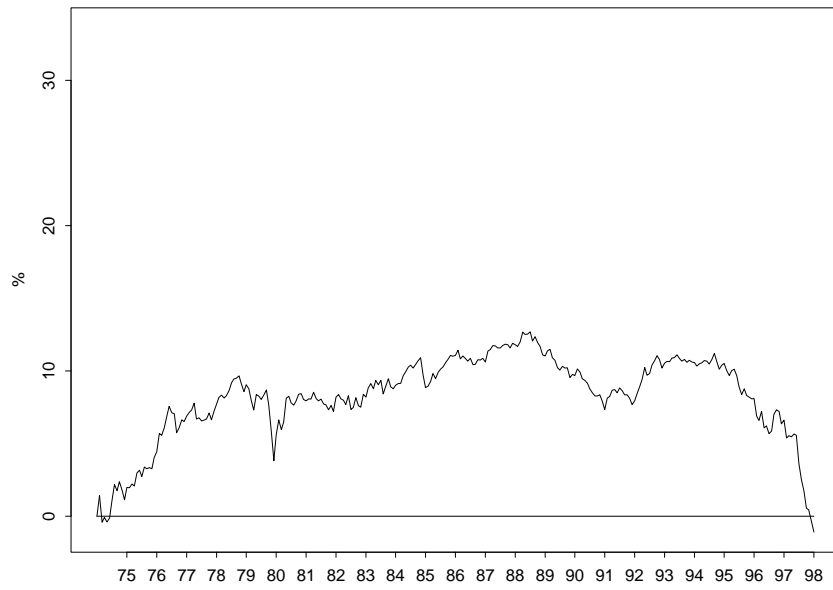


Figure 3: Distributional component of the relative return



Figure 4: Rank component of the relative return

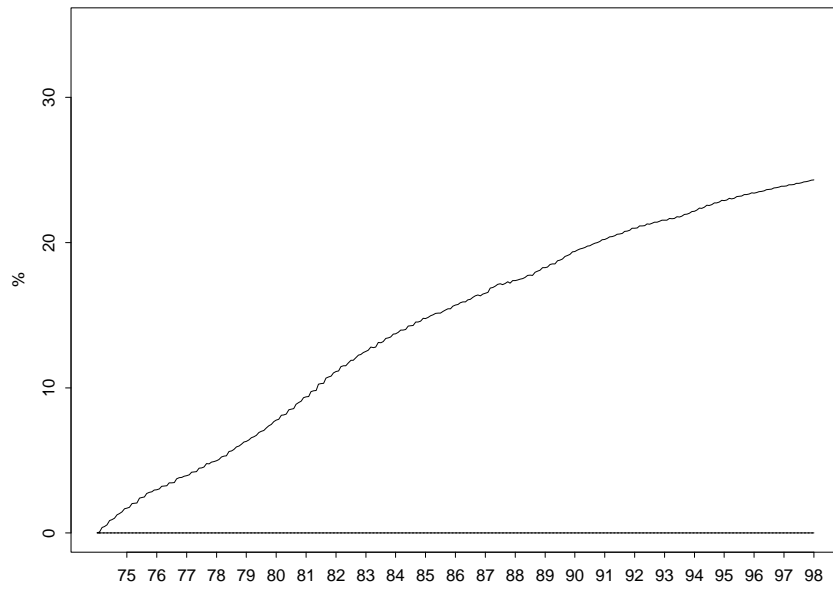


Figure 5: Dividend component of the relative return