LOCAL TIMES OF RANKED CONTINUOUS SEMIMARTINGALES

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Abstract

Given a finite collection of continuous semimartingales, we derive a semimartingale decomposition of the corresponding ranked (order-statistics) processes. We apply the decomposition to extend the theory of equity portfolios generated by ranked market weights to the case where the stock values admit triple points.

Key Words and Phrases: Portfolios, portfolio generating functions, continuous semimartingales, order statistics, local times.

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1 Introduction

Some recent developments in mathematical finance have led to the necessity of understanding the dynamics of the kth-ranked amongst n given semimartingales, at all levels $k = 1, \ldots, n$. For example, $k = 1$ and $k = n$ correspond to the maximum and minimum process of the collection, respectively. Particular applications include the theory of portfolios generated by ranked weights, given in Chapter 4 of R. Fernholz's monograph [2], and the Atlas and first-order models studied in [2] and [1]. In the former case, it has hitherto been assumed that no triple points exist, i.e., that no three stock values ever coincide, almost surely. In Problem 4.1.13 of [2], Fernholz poses the question of extending the theory to include triple

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points (and higher orders of incidence). Inspired by this, we develop some general formulas for rank processes of continuous semimartingales in Section 2, then apply them to the theory of portfolio generating functions in Section 3.

2 Decomposition of Ranked Continuous Semimartingales

Given continuous semimartingales $X_1, \ldots, X_n$, we define the *kth rank process* $X_{(k)}(\cdot)$ *of* $X_1, \ldots, X_n$ by

$$X_{(k)}(\cdot) := \max_{1 \le i_1 < \cdots < i_k \le n} \min\{X_{i_1}(\cdot), \ldots, X_{i_k}(\cdot)\}.$$

In particular, for any $t > 0$ we have

$$\max_{1 \le i \le n} X_i(t) \;=\; X_{(1)}(t) \;\ge\; X_{(2)}(t) \;\ge\; \ldots \;\ge\; X_{(n-1)}(t) \;\ge\; X_{(n)}(t) \;=\; \min_{1 \le i \le n} X_i(t)$$

so that at any given time, the values of the rank processes represent the values of the original processes arranged in descending order (i.e., the reverse order-statistics). We wish to show that the rank processes are continuous semimartingales and also to find their semimartingale decomposition. In order to do this, we will need to consider local times of certain continuous semimartingales. In particular, for any continuous semimartingale Z, we denote by $L_t(Z)$ the local time accumulated **at the origin** by $Z(\cdot)$ up to time t:

$$L_t(Z) := \frac{1}{2}\left(|Z(t)| - |Z(0)| - \int_0^t \operatorname{sgn}(Z(s))\, dZ(s)\right).$$

We now recall the following result:

Theorem 2.1 (Yan [5]) *For continuous semimartingales X and Y, we have*

$$L_t(X \vee Y) + L_t(X \wedge Y) = L_t(X) + L_t(Y) \qquad \textit{for all } t > 0.$$

This result can be extended to the case of three or more continuous semimartingales:

Theorem 2.2 *For continuous semimartingales $X_1, \ldots, X_n$, we have*

$$\sum_{k=1}^{n} L_t(X_{(k)}) = \sum_{i=1}^{n} L_t(X_i) \qquad \textit{for all } t > 0.$$

Proof: We proceed by induction. The case $n = 1$ is trivial, and the case $n = 2$ is precisely Theorem 2.1. Now assume that result holds for some n. Given continuous semimartingales

$X_1, \ldots, X_n, X_{n+1}$, we define $X_{(k)}$, $k = 1, \ldots, n$ as above and also set

$$X_{[k]}(\cdot) := \max_{1 \le i_1 < \cdots < i_k \le n+1} \min\{X_{i_1}(\cdot), \ldots, X_{i_k}(\cdot)\}.$$

The process $X_{[k]}(\cdot)$ is the kth-ranked process with respect to **all** $n+1$ **semimartingales** $X_1, \ldots, X_{n+1}$. It will also be convenient to set $X_{(0)}(\cdot) :\equiv \infty$. We now claim that

$$L_t(X_{(k-1)} \wedge X_{n+1}) + L_t(X_{(k)}) = L_t(X_{[k]}) + L_t(X_{(k)} \wedge X_{n+1}) \tag{2.1}$$

for all $k = 1, \ldots, n$ and $t > 0$. Suppose first that $k > 1$. By Theorem 2.1, we have

$$L_t(X_{(k-1)} \wedge X_{n+1}) + L_t(X_{(k)}) = L_t((X_{(k-1)} \wedge X_{n+1}) \vee X_{(k)}) + L_t((X_{(k-1)} \wedge X_{n+1}) \wedge X_{(k)}).$$

Since $X_{(k-1)}(t) \ge X_{(k)}(t)$ for all $t > 0$, the second term on the right-hand side of the above equation is simply $L_t(X_{(k)} \wedge X_{n+1})$. On the other hand, we have

$$\big((X_{(k-1)} \wedge X_{n+1}) \vee X_{(k)}\big)(t) = \begin{cases} X_{(k-1)}(t) & \text{if } X_{n+1}(t) \ge X_{(k-1)}(t) \ge X_{(k)}(t) \\ X_{n+1}(t) & \text{if } X_{(k-1)}(t) \ge X_{n+1}(t) \ge X_{(k)}(t) \\ X_{(k)}(t) & \text{if } X_{(k-1)}(t) \ge X_{(k)}(t) \ge X_{n+1}(t). \end{cases}$$

In each case it can be checked that $\big((X_{(k-1)} \wedge X_{n+1}) \vee X_{(k)}\big)(t)$ is the kth largest of the numbers $X_1(t), \ldots, X_{n+1}(t)$; that is, $\big((X_{(k-1)} \wedge X_{n+1}) \vee X_{(k)}\big)(\cdot) \equiv X_{[k]}(\cdot)$. Equation (2.1) now follows in the case where $k = 2, \ldots, n$. If $k = 1$, then (2.1) reduces to

$$L_t(X_{n+1}) + L_t(X_{(1)}) = L_t(X_{[1]}) + L_t(X_1 \wedge X_{n+1})$$

since $X_{(0)}(\cdot) \equiv \infty$. This equation follows from Theorem 2.1 and the simple observation that $(X_{(1)} \vee X_{n+1})(\cdot) \equiv X_{[1]}(\cdot)$. Finally, by the induction hypothesis, (2.1) and the facts that $(X_{(n)} \wedge X_{n+1})(\cdot) \equiv X_{[n+1]}(\cdot)$ and $(X_{(0)} \wedge X_{n+1})(\cdot) \equiv X_{n+1}(\cdot)$, we have

$$\begin{aligned} \sum_{i=1}^{n+1} L_t(X_i) &= \sum_{i=1}^{n} L_t(X_i) + L_t(X_{n+1}) = \sum_{k=1}^{n} L_t(X_{(k)}) + L_t(X_{n+1}) \\ &= \sum_{k=1}^{n} L_t(X_{[k]}) + \sum_{k=1}^{n} \big(L_t(X_{(k)} \wedge X_{n+1}) - L_t(X_{(k-1)} \wedge X_{n+1})\big) + L_t(X_{n+1}) \\ &= \sum_{k=1}^{n} L_t(X_{[k]}) + L_t(X_{(n)} \wedge X_{n+1}) - L_t(X_{(0)} \wedge X_{n+1}) + L_t(X_{n+1}) = \sum_{k=1}^{n+1} L_t(X_{[k]}) \end{aligned}$$

for all $t > 0$; the desired result follows by induction. $\square$

As in Definition 4.1.5 in [2], an *absolutely continuous semimartingale* X is a continuous semimartingale with decomposition $X(\cdot) = X(0) + M(\cdot) + V(\cdot)$ (where $M(\cdot)$ is a local martingale and $V(\cdot)$ is of bounded variation) such that the random signed measures dV and $d\langle X\rangle$ are both almost surely absolutely continuous with respect to Lebesgue measure. We now proceed under the following two assumptions:

1. $X_1, \ldots, X_n$ are absolutely continuous semimartingales; and
2. for all $i \neq j$, the set $\{t : X_i(t) = X_j(t)\}$ has Lebesgue measure zero, almost surely.

(Note that the second assumption appears as the first half of Definition 4.1.2 in [2].) Set

$$U = \{u.(\cdot) : [0,\infty) \times \{1,\ldots,n\} \to \{1,\ldots,n\} \,\big|\, u.(k) \text{ is predictable and } X_{(k)}(t) = X_{u_t(k)}(t) \text{ for all } t > 0, k = 1,\ldots,n\}. \quad (2.2)$$

Under the above assumptions, we have:

Proposition 2.1 *For fixed k, i and $t > 0$,*

$$\int_0^t 1_{\{u_s(k)=i\}} dX_i(s) \quad (2.3)$$

is independent of the choice of $u \in U$.

Proof: We define

$$S_t(k) := \{i : X_i(t) = X_{(k)}(t)\} \qquad \text{and} \qquad N_t(k) := |S_t(k)|\,; \quad (2.4)$$

in particular, $N_t(k)$ is the number of processes that are at (equal) rank k at time t. Then if $u, v \in U$, we have

$$\{s : u_s(k) \neq v_s(k)\} \subseteq \{s : N_s(k) > 1\} \subseteq \bigcup_{i \neq j} \{s : X_i(s) = X_j(s)\}.$$

It follows that $\{s : u_s(k) \neq v_s(k)\}$ has Lebesgue measure 0, so by our first assumption, the quantity in (2.3) is indeed independent of the choice of $u \in U$. □

We now use this observation to show that $\{X_{(k)}\}$ are continuous semimartingales and to provide their semimartingale decomposition.

Theorem 2.3 *If $X_1, \ldots, X_n$ are absolutely continuous semimartingales such that $\{t : X_i(t) =$*

$X_j(t)\}$ *is Lebesgue-null for all* $i \neq j$*, then*

$$dX_{(k)}(t) = \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dX_i(t) + \sum_{j=k+1}^{n} (N_k(t))^{-1} dL_t(X_{(k)} - X_{(j)}) - \sum_{j=1}^{k-1} (N_k(t))^{-1} dL_t(X_{(j)} - X_{(k)}) \quad (2.5)$$

for all $t > 0$ *a.s., where* u *is in the set* U *of (2.2) and* $N_k(\cdot)$ *is given in (2.4) above.*

Remark: the full semimartingale decompositions of $\{dX_{(k)}(\cdot)\}$ may be obtained from the semimartingale decompositions of $\{dX_i(\cdot)\}$.

Proof: Let

$$J := \{(j_1, \ldots, j_n) \,|\, j_i \in \{1, \ldots, i\} \text{ for all } i = 1, \ldots, n\}.$$

For any $\mathbf{j} \in J$, we define a member $u^{\mathbf{j}}$ of the set U by

$$u_t^{\mathbf{j}}(k) := \mathbf{j}_{N_t(k)}^{\text{th}} \text{ largest element of } S_t(k).$$

In other words, suppose that at time t precisely m of the processes $X_1, \ldots, X_n$ have rank k. If these processes are $X_{i_1}, \ldots, X_{i_m}$, then $u_t^{\mathbf{j}}(k)$ is simply the $\mathbf{j}_m$th largest among the indices $i_1, \ldots, i_m$. In the case where $\mathbf{j} = (1, 1, \ldots, 1)$, $u_t^{\mathbf{j}}(k)$ picks out the lowest index among all the processes of rank k at time t, regardless of how many such processes there are; that is, $u_t^{\mathbf{j}} \equiv p_t$ for the random permutation p_t defined in (4.1.26) of [2].

For any $u \in U$, we have

$$X_{(k)}(t) - X_{(k)}(0) = \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}} dX_i(s) + \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}} d(X_{(k)}(s) - X_i(s))$$

since $\{1_{\{u_s(k)=i\}}\}_{i=1}^n$ is a partition of unity. Replacing u by $u^{\mathbf{j}}$ for $\mathbf{j} \in J$, then summing over all such $\mathbf{j}$, we get

$$n!(X_{(k)}(t) - X_{(k)}(0)) = n! \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}} dX_i(s) + \sum_{i=1}^{n} \int_0^t \sum_{\mathbf{j} \in J} 1_{\{u_s^{\mathbf{j}}(k)=i\}} d(X_{(k)}(s) - X_i(s)) \quad (2.6)$$

for any $u \in U$. Here we have used the fact that $|J| = n!$ and Proposition 2.1. We now claim that

$$\sum_{\mathbf{j} \in J} 1_{\{u_s^{\mathbf{j}}(k)=i\}} = \frac{n!}{N_s(k)} 1_{\{X_{(k)}(s)=X_i(s)\}} \quad (2.7)$$

for all $i, k = 1, \ldots, n$, $s > 0$. Indeed, if $m = N_s(k)$ and $\mathbf{j}$ is expressed as $(j_1, \ldots, j_n)$, then

$$\sum_{\mathbf{j} \in J} 1_{\{u_s^{\mathbf{j}}(k)=i\}} = \sum_{j_1=1}^{1} \sum_{j_2=1}^{2}{}' \ldots \sum_{j_n=1}^{n} \left(\sum_{j_m=1}^{m} 1_{\{u_s^{\mathbf{j}}(k)=i\}} \right)$$

where the prime denotes omission of the mth sum. By definition,

$$\sum_{j_m=1}^{m} 1_{\{u_s^{(j_1,\ldots,j_n)}(k)=i\}} = 1_{\{X_{(k)}(s)=X_i(s)\}}$$

for any fixed $j_1, \ldots, j_{m-1}, j_{m+1}, \ldots, j_n$, so (2.7) follows. Using it in (2.6), we get

$$\begin{aligned} n!(X_{(k)}(t) - X_{(k)}(0)) = n! \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}} dX_i(s) \\ + n! \sum_{i=1}^{n} \int_0^t (N_s(k))^{-1} 1_{\{X_{(k)}(s)=X_i(s)\}} d(X_{(k)}(s) - X_i(s)). \end{aligned} \tag{2.8}$$

Using the formula

$$\int_0^t dL_s(Z) = \int_0^t 1_{\{Z(s)=0\}}\, dZ(s), \tag{2.9}$$

which is valid for continuous nonnegative semimartingales Z, we can write

$$\begin{aligned} &\sum_{i=1}^{n} \int_0^t (N_s(k))^{-1} 1_{\{X_{(k)}(s)=X_i(s)\}} d(X_{(k)}(s) - X_i(s)) \\ &\quad = \sum_{i=1}^{n} \int_0^t (N_s(k))^{-1} 1_{\{X_{(k)}(s)=X_i(s)\}} d((X_{(k)}(s) - X_i(s))^+) \\ &\qquad - \sum_{i=1}^{n} \int_0^t (N_s(k))^{-1} 1_{\{X_{(k)}(s)=X_i(s)\}} d((X_{(k)}(s) - X_i(s))^-) \\ &\quad = \sum_{i=1}^{n} \int_0^t (N_s(k))^{-1} dL_s((X_{(k)} - X_i)^+) - \sum_{i=1}^{n} \int_0^t (N_s(k))^{-1} dL_s((X_{(k)} - X_i)^-). \end{aligned}$$

This implies that (2.8) may be rewritten as

$$\begin{aligned} X_{(k)}(t) - X_{(k)}(0) = \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}} dX_i(s) + \sum_{i=1}^{n} \int_0^t (N_s(k))^{-1} dL_s((X_{(k)} - X_i)^+) \\ - \sum_{i=1}^{n} \int_0^t (N_s(k))^{-1} dL_s((X_{(k)} - X_i)^-). \end{aligned} \tag{2.10}$$

Noting that

$$(X_{(k)} - X_{(j)})^+ \equiv \begin{cases} X_{(k)} - X_{(j)}, & \text{if } j > k \\ 0, & \text{if } j \le k \end{cases}$$

and that

$$(X_{(k)} - X_{(j)})^- \equiv \begin{cases} X_{(j)} - X_{(k)}, & \text{if } j < k \\ 0, & \text{if } j \ge k, \end{cases}$$

the result follows by applying Theorem 2.2 to the last two sums on the right-hand side of (2.10). □

Corollary 2.1 *Suppose that $X_1, \ldots, X_n$ satisfy the hypotheses of Theorem 2.3, and in addition, for all $i < j < k$ we have $\{t : X_i(t) = X_j(t) = X_k(t)\} = \emptyset$, almost surely. Then*

$$dX_{(k)}(t) = \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dX_i(t) + \frac{1}{2} dL_t(X_{(k)} - X_{(k+1)}) - \frac{1}{2} dL_t(X_{(k-1)} - X_{(k)}). \tag{2.11}$$

Indeed, under the assumption of no triple points, we must have $N_t(k) = 1$ or 2 for all t. When $N_t(k) = 1$, the associated local times vanish, whereas when $N_t(k) = 2$, only the two local times appearing in (2.11) make a nonzero contribution. Note that in the case where $u_t(\cdot)$ is the random permutation $p_t(\cdot)$ defined in (4.1.26) of [2], Corollary 2.1 reduces to Proposition 4.1.11 in [2].

3 Application to Portfolio Generating Functions

In this section, we generalize Theorem 4.2.1 in [2] to the case where three or more stock capitalizations may be equal at a given time t (with positive probability). Let us briefly recall the setting of that theorem. As in Definition 1.1.1 of [2], we work in a market $\mathcal{M}$ consisting of stocks $X_1, \ldots, X_n$ which satisfy the stochastic differential equations

$$d \log X_i(t) = \gamma_i(t)\, dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t).$$

Here $(W_1, \ldots, W_n)$ is a Brownian motion, γ_i is measurable, adapted and satisfies the growth condition $\int_0^T |\gamma_i(t)|\, dt < \infty$ for all $T \ge 0$ and $i = 1, \ldots, n$, a.s., and for all such i, with $\Xi_i(t) := \xi_{i1}^2(t) + \cdots + \xi_{in}^2(t)$, we have $\int_0^T \Xi_i(t)\, dt < \infty$ for $T \ge 0$, a.s.; $\lim_{t\to\infty} t^{-1}\Xi_i(t) \log\log t = 0$, a.s.; and $\Xi_i(t) > 0$ for all $t > 0$, a.s. With

$$\Delta^n := \{x = (x_1, \ldots, x_n) \in I\!R^n : x_i \ge 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} x_i = 1\},$$

a *portfolio* in $\mathcal{M}$ is defined to be any measurable, adapted Δ^n-valued process. If $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_n(\cdot))$ is a portfolio, then $\pi_i(t)$ represents the proportion of the value of the stock X_i in the portfolio. We denote the total value of the portfolio at time t by $Z_\pi(t)$. If we set $\mu_i(t) = X_i(t)/\sum_{j=1}^n X_j(t)$, then the resulting portfolio μ is called the *market portfolio*, and its value is given by $Z_\mu(t) = C\sum_{j=1}^n X_j(t)$ for some constant C depending on the initial stock values and portfolio value $Z_\mu(0)$. The components $\mu_1, \ldots, \mu_n$ are referred to as the *market weights*.

We will also need to look at some covariance matrices. In particular, set $\sigma_{ij}(\cdot) := \sum_{\nu=1}^n \xi_{i\nu}(\cdot)\xi_{j\nu}(\cdot)$; the matrix σ is the *covariance matrix process* of the market. The *covariance process relative to the market* is defined by

$$\tau_{ij}(\cdot) := \sigma_{ij}(\cdot) - \sum_{k=1}^n \mu_k\sigma_{ik}(\cdot) - \sum_{k=1}^n \mu_k\sigma_{kj}(\cdot) + \sum_{k,l=1}^n \mu_k\mu_l\sigma_{kl}(\cdot).$$

We intend to apply the theory of Section 2 in the case where the continuous semimartingales $X_i(\cdot)$ of that section are precisely the stock values in a market $\mathcal{M}$. Note that the incidence $X_i(t) = X_j(t)$ corresponds to the incidence $\mu_i(t) = \mu_j(t)$, and that the rankings of μ_i correspond to the rankings of X_i. In this context, the ranked market weights are denoted $\mu_{(k)}(\cdot)$, for $k = 1 \ldots, n$. Let us now fix $u \in U$. The *ranked covariance process relative to the market* is defined by

$$\tau_{(ij)}(t) = \tau_{u_t(i)u_t(j)}(t)$$

for all $t > 0$ and $i, j = 1, \ldots, n$.

Finally, let us restrict our attention to a finite time horizon $[0, T]$. We say that a positive C^2 function $\mathbf{S}$ defined on a neighborhood of Δ^n *generates* the portfolio π if there esists a measurable process of bounded variation Θ, called the *drift process corrseponding to* $\mathbf{S}$, satisfying

$$\log(Z_\pi(t)/Z_\mu(t)) = \log \mathbf{S}(\mu(t)) + \Theta(t),$$

for all t in $[0, T]$, a.s. Armed with this plethora of definitions, we are now ready to state the following:

Theorem 3.4 *Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$ such that for each $i < j$, the set $\{X_i(t) = X_j(t)\}$ has Lebesgue measure 0. Suppose that u is an element of the set U defined in (2.2) above, and that $\mathbf{S}$ and S are positive C^2 functions defined on a neighborhood of Δ^n such that for any $x = (x_1, \ldots, x_n)$ in that neighborhood,*

$$\mathbf{S}(x_1, \ldots, x_n) = S(x_{(1)}, \ldots, x_{(n)}),$$

and for $i = 1, \ldots, n$, $x_i D_i \log S(x)$ is bounded for $x \in \Delta^n$. (Here D_i denotes differentiation

with respect to the ith variable.) Then $\mathbf{S}$ *generates the portfolio* π *such that for* $k = 1, \ldots, n$,

$$\pi_{u_t(k)}(t) = \left(D_k \log S(\mu_{(\cdot)}(t)) + 1 - \sum_{j=1}^{n} \mu_{(j)}(t) D_j \log S(\mu_{(\cdot)}(t)) \right) \mu_{(k)}(t),$$

for all $t \in [0, T]$, *a.s., with a drift process* Θ *that satisfies*

$$\begin{aligned} d\Theta(t) = \frac{-1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^{n} D_{ij} S(\mu_{(\cdot)}(t)) \mu_{(i)}(t) \mu_{(j)}(t) \tau_{(ij)}(t)\, dt \\ - \sum_{k=1}^{n} \pi_{u_t(k)}(t) (N_t(k))^{-1} \sum_{j=k+1}^{n} dL_t (\log \mu_{(k)} - \log \mu_{(j)}) \\ + \sum_{k=1}^{n} \pi_{u_t(k)}(t) (N_t(k))^{-1} \sum_{j=1}^{k-1} dL_t (\log \mu_{(j)} - \log \mu_{(k)}). \end{aligned}$$

The proof of this result is identical to that of the proof of Theorem 4.2.1 in [2], except that the appeal to Corollary 4.1.12 in that reference should be replaced by a similar appeal to Theorem 2.3 above. We note that $X_1, \ldots, X_n$ are absolutely continuous semimartingales by virtue of Lemma 4.1.7 in [2]. Also observe that the relative log return process for the generated portfolio π is given by

$$\begin{aligned} d \log(Z_\pi(t)/Z_\mu(t)) = \sum_{i=1}^{n} D_i \log S(\mu_{(\cdot)}(t)) d\mu_{(i)}(t) \\ - \frac{1}{2} \sum_{i,j=1}^{n} D_i \log S(\mu_{(\cdot)}(t)) D_j \log S(\mu_{(\cdot)}(t)) \mu_{(i)}(t) \mu_{(j)}(t) \tau_{(ij)}(t)\, dt \\ - \sum_{k=1}^{n} \pi_{u_t(k)}(t) (N_t(k))^{-1} \sum_{j=k+1}^{n} dL_t (\log \mu_{(k)} - \log \mu_{(j)}) \\ + \sum_{k=1}^{n} \pi_{u_t(k)}(t) (N_t(k))^{-1} \sum_{j=1}^{k-1} dL_t (\log \mu_{(j)} - \log \mu_{(k)}). \end{aligned}$$

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