
Planar Diffusions with Rank-Based Characteristics: Transition Probabilities, Time Reversal, Maximality and Perturbed Tanaka equations

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Abstract For given nonnegative constants g, h, ρ, σ with $\rho^2 + \sigma^2 = 1$ and $g + h > 0$, we construct a diffusion process $(X_1(\cdot), X_2(\cdot))$ with values in the plane and infinitesimal generator

$$\mathcal{L} = \mathbf{1}_{\{x_1 > x_2\}} \left(\frac{\rho^2}{2} \frac{\partial^2}{\partial x_1^2} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x_2^2} - h \frac{\partial}{\partial x_1} + g \frac{\partial}{\partial x_2} \right) + \mathbf{1}_{\{x_1 \leq x_2\}} \left(\frac{\sigma^2}{2} \frac{\partial^2}{\partial x_1^2} + \frac{\rho^2}{2} \frac{\partial^2}{\partial x_2^2} + g \frac{\partial}{\partial x_1} - h \frac{\partial}{\partial x_2} \right). \quad (0.1)$$

We compute the transition probabilities of this process, discuss its realization in terms of appropriate systems of stochastic differential equations, study its dynamics under a time reversal, and note that these involve singularly continuous components governed by local time. Crucial in our analysis are properties of Brownian and semimartingale local time; properties of the *generalized perturbed Tanaka equation*

$$dZ(t) = f(Z(t)) dM(t) + dN(t), \quad Z(0) = \xi$$

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driven by suitable continuous, orthogonal semimartingales $M(\cdot)$ and $N(\cdot)$ and with $f(\cdot)$ of bounded variation, which we study here in detail; and those of a one-dimensional diffusion $Y(\cdot)$ with *bang-bang* drift $dY(t) = -\lambda \operatorname{sgn}(Y(t)) dt + dW(t)$, $Y(0) = y$ driven by a standard Brownian motion $W(\cdot)$.

We also show that the planar diffusion $(X_1(\cdot), X_2(\cdot))$ can be represented in terms of this process $Y(\cdot)$, its local time $L^Y(\cdot)$ at the origin, and an independent standard Brownian motion $Q(\cdot)$, in a form which can be construed as a two-dimensional analogue of the stochastic equation satisfied by the so-called *skew Brownian motion*.

Keywords Diffusion · local time · bang-bang drift · Lévy characterization of Brownian motion · Tanaka formulae · weak and strong solutions · skew representation · skew Brownian motion · modified and perturbed Tanaka equations · time reversal

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1 INTRODUCTION

For given nonnegative constants g, h, ρ, σ with $\rho^2 + \sigma^2 > 0$ and $g + h > 0$, and for a given vector $(x_1, x_2) \in \mathbb{R}^2$, we shall consider the question of constructing a two-dimensional diffusion process $(X_1(\cdot), X_2(\cdot))$ with dynamics

$$dX_1(t) = (g\mathbf{1}_{\{X_1(t) \leq X_2(t)\}} - h\mathbf{1}_{\{X_1(t) > X_2(t)\}}) dt + (\rho\mathbf{1}_{\{X_1(t) > X_2(t)\}} + \sigma\mathbf{1}_{\{X_1(t) \leq X_2(t)\}}) dB_1(t), \quad (1.1)$$

$$dX_2(t) = (g\mathbf{1}_{\{X_1(t) > X_2(t)\}} - h\mathbf{1}_{\{X_1(t) \leq X_2(t)\}}) dt + (\rho\mathbf{1}_{\{X_1(t) \leq X_2(t)\}} + \sigma\mathbf{1}_{\{X_1(t) > X_2(t)\}}) dB_2(t), \quad (1.2)$$

initial condition $(X_1(0), X_2(0)) = (x_1, x_2)$, and $B_1(\cdot), B_2(\cdot)$ two independent, standard Brownian motions. For simplicity, we shall use throughout the normalization

$$\rho^2 + \sigma^2 = 1 \quad (1.3)$$

and refer to the case $\rho\sigma = 0$ as “degenerate”.

Speaking informally and a bit imprecisely for the moment about the system of (1.1)-(1.2), imagine you run two Brownian-like particles on the real line. At any given time, you assign positive drift g and diffusion σ to the laggard; and you assign negative drift $-h$ and diffusion ρ to the leader. What is the probabilistic structure of the resulting two-dimensional diffusion process? Can it be realized as the solution of a system of stochastic differential equations other than (1.1), (1.2)? What are its transition probabilities? How does it look like, when time is reversed?

It has been known for some time now, at least for the non-degenerate case (cf. STROOCK & VARADHAN (1979), pages 193-194; BASS & PARDOUX (1987); or KRYLOV (2004), page 45) that a unique probability measure μ can be constructed on the canonical filtered measurable space $(\mathfrak{W}, \mathcal{G})$, $\mathbf{G} = \{\mathcal{G}(t)\}_{0 \leq t < \infty}$ of continuous functions $\mathfrak{w} : [0, \infty) \rightarrow \mathbb{R}^2$ endowed with the topology of uniform convergence on

compact intervals, such that the process $f(\mathbf{w}(t)) - \int_0^t \mathcal{L}f(\mathbf{w}(s)) ds$, $0 \leq t < \infty$ is a local martingale under $\boldsymbol{\mu}$ for every function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ of class \mathcal{C}^2 (here \mathcal{L} is the second-order partial differential operator in (0.1), the infinitesimal generator of the resulting diffusion). Our goal in this paper is to describe this probability measure $\boldsymbol{\mu}$ as explicitly as possible; to study its behavior under time-reversal; and to understand the solvability of systems of stochastic differential equations, such as (1.1), (1.2) above, which correspond to this martingale problem and help realize its solution.

We shall show in sections 2, 3 and 4 that the system of stochastic differential equations (1.1), (1.2) has a solution which is unique in the sense of the probability distribution (thus the above martingale problem is indeed well-posed). This solution is shown to be strong in section 5; it is characterized in terms of a one-dimensional diffusion process $Y(\cdot)$, which has “bang-bang” drift with intensity $\lambda = g + h > 0$ and is driven by yet another standard Brownian motion process $W(\cdot)$, namely

$$dY(t) = -\lambda \operatorname{sgn}(Y(t))dt + dW(t). \quad (1.4)$$

Here and in what follows we shall use the convention for the signum function

$$\operatorname{sgn}(y) := \mathbf{1}_{(0,\infty)}(y) - \mathbf{1}_{(-\infty,0]}(y), \quad y \in \mathbb{R}. \quad (1.5)$$

The one-dimensional diffusion of (1.4) was studied in some detail by KARATZAS & SHREVE (1984), who found its transition probabilities as in equations (6.3), (6.4) below. We shall use this analysis to compute, in section 6, the transition probabilities of the two-dimensional process $(X_1(\cdot), X_2(\cdot))$. As in that earlier paper, a crucial rôle will be played here again by the local time

$$L^Y(t) := \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \int_0^t \mathbf{1}_{\{-\varepsilon < Y(s) < \varepsilon\}} ds \quad (1.6)$$

accumulated at the origin during the interval $[0, t]$ by the diffusion $Y(\cdot)$ of (1.4). In terms of this diffusion, its local time (1.6), and an independent standard Brownian motion $Q(\cdot)$, the unique-in-distribution weak solution of the system in (1.1), (1.2) will be shown to admit the *skew representation* of (2.22), (2.23) below.

The diffusion process of (1.4) is also instrumental in a time-reversal analysis we carry out in section 7, where the dynamics of time-reversed versions of the processes $X_1(\cdot)$, $X_2(\cdot)$ are derived in the spirit of HAUSSMANN & PARDOUX (1986) and MEYER (1994). We were quite surprised, at first, that these reverse-time dynamics should feature terms involving singularly continuous components such as local times, as indeed they do. To the best of our knowledge, this is the first instance where such a structure is observed in the context of a “purely forward” system of stochastic differential equations such as (1.1)-(1.2) that does not involve reflection; see Remark 7.3 in this regard. We also study the forward and backward dynamics of the ranks $R_1(\cdot) = \max(X_1(\cdot), X_2(\cdot))$, $R_2(\cdot) = \min(X_1(\cdot), X_2(\cdot))$ in this two-dimensional diffusion, in sections 4 and 7, respectively.

The planar diffusion process with infinitesimal generator (0.1) has local covariance matrix

$$\mathcal{A}(x_1, x_2) = \begin{pmatrix} \rho^2 \mathbf{1}_{\{x_1 > x_2\}} + \sigma^2 \mathbf{1}_{\{x_1 \leq x_2\}} & 0 \\ 0 & \rho^2 \mathbf{1}_{\{x_1 \leq x_2\}} + \sigma^2 \mathbf{1}_{\{x_1 > x_2\}} \end{pmatrix}. \quad (1.7)$$

There is a continuum of real square roots for this matrix, of the form $\Sigma(x_1, x_2) = \Sigma_+ \mathbf{1}_{\{x_1 > x_2\}} + \Sigma_- \mathbf{1}_{\{x_1 \leq x_2\}}$ with

$$\Sigma_+ := \begin{pmatrix} \rho \cos \varphi & -\rho \sin \varphi \\ \varepsilon \sigma \sin \varphi & \varepsilon \sigma \cos \varphi \end{pmatrix}, \quad \Sigma_- := \begin{pmatrix} \sigma \cos \vartheta & -\sigma \sin \vartheta \\ \rho \delta \sin \vartheta & \rho \delta \cos \vartheta \end{pmatrix}, \quad (1.8)$$

parametrized by $\varepsilon = \pm 1$, $\delta = \pm 1$, $0 \leq \varphi, \vartheta \leq 2\pi$. All such configurations lead to systems of stochastic differential equations that admit a (unique in distribution) weak solution. We show in subsection 5.1 that *those solutions that correspond to configurations with*

$$(\sigma^2 \varepsilon - \rho^2 \delta) \sin(\vartheta - \varphi) + \rho \sigma (1 + \varepsilon \delta) \cos(\vartheta - \varphi) = -1 \quad (1.9)$$

are not strong; see the system (2.14), (2.15) for an example. Whereas *all other configurations lead to strongly solvable systems*; one such system appears in (1.1), (1.2), and another one in (2.12), (2.13).

Crucial in this analysis of strength and weakness is the following recent result by PROKAJ (2011) on the pathwise uniqueness of the ‘‘perturbed Tanaka equation’’

$$Y(t) = y + \int_0^t \operatorname{sgn}(Y(s)) dM(s) + N(t), \quad 0 \leq t < \infty. \quad (1.10)$$

Theorem 1.1 (PROKAJ (2011)). *Suppose that $M(\cdot), N(\cdot)$ are continuous local martingales with $M(0) = N(0) = 0$ and quadratic and cross-variations that satisfy the conditions of orthogonality and domination*

$$\langle M, N \rangle(t) = 0, \quad \langle M \rangle(t) = \int_0^t q(s) d\langle N \rangle(s); \quad 0 \leq t < \infty, \quad (1.11)$$

respectively, for some progressively measurable process $q(\cdot)$ with values in a compact interval $[0, c]$. Under these assumptions, pathwise uniqueness holds for the perturbed Tanaka equation (1.10).

In section 8 we shall use the local time techniques introduced by PERKINS (1982) and further developed by LE GALL (1983), to provide a simple proof of a considerably more general result of this type, Theorem 8.1 (see also Proposition 8.1), in which the signum function is replaced in (1.10) by an arbitrary function of finite variation, and $M(\cdot), N(\cdot)$ by continuous semimartingales such that (1.11) is satisfied.

Multidimensional processes of the type (1.1), (1.2) were introduced by FERNHOLZ (2002), and their ergodic behavior was studied by BANNER et al (2005), PAL & PITMAN (2008), ICHIBA et al (2011), among others. Here we focus on the two-dimensional case, and concentrate on the precise probabilistic structure of the resulting diffusions governed by stochastic differential equations such as (1.1), (1.2).

2 ANALYSIS

Let us assume that the system of stochastic differential equations (1.1), (1.2) has a weak solution: To wit, that there exists a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbf{F} = \{\mathfrak{F}(t)\}_{0 \leq t < \infty}$ and on it two pairs $(B_1(\cdot), B_2(\cdot))$ and $(X_1(\cdot), X_2(\cdot))$ of continuous, \mathbf{F} -adapted processes, such that $B_1(\cdot)$ and $B_2(\cdot)$ are independent standard Brownian motions and (1.1), (1.2), $X_1(0) = x_1$ and $X_2(0) = x_2$ hold. We shall fix the nonnegative constants g, h, ρ, σ with $g + h > 0$, and impose the normalization (1.3).

We shall assume throughout the paper, and without further mention, that the filtrations we are dealing with are in their right-continuous versions and have been augmented by sets of \mathbb{P} -measure zero. We shall also use the convention

$$\mathbf{F}^\Xi = \{\mathfrak{F}^\Xi(t)\}_{0 \leq t < \infty}, \quad \mathfrak{F}^\Xi(t) := \sigma(\Xi(s), 0 \leq s \leq t)$$

for the \mathbb{P} -augmentation of the filtration generated by a given process $\Xi : [0, \infty) \times \Omega \rightarrow \mathbb{R}^d$ with values in some Euclidean space and RCLL paths.

With such a setup, and with the notation

$$\lambda = g + h, \quad \nu = g - h, \quad y = x_1 - x_2, \quad z = x_1 + x_2, \quad (2.1)$$

we note that the difference

$$Y(t) := X_1(t) - X_2(t), \quad 0 \leq t < \infty \quad (2.2)$$

satisfies the integral version

$$Y(t) = y - \lambda \int_0^t \operatorname{sgn}(Y(s)) \, ds + W(t), \quad 0 \leq t < \infty \quad (2.3)$$

of the equation (1.4). The equation (2.3) is driven by the process

$$W(t) := \rho W_1(t) + \sigma W_2(t), \quad 0 \leq t < \infty, \quad (2.4)$$

where we have set

$$W_1(t) := \int_0^t \mathbf{1}_{\{Y(s) > 0\}} \, dB_1(s) - \int_0^t \mathbf{1}_{\{Y(s) \leq 0\}} \, dB_2(s), \quad (2.5)$$

$$W_2(t) := \int_0^t \mathbf{1}_{\{Y(s) \leq 0\}} \, dB_1(s) - \int_0^t \mathbf{1}_{\{Y(s) > 0\}} \, dB_2(s). \quad (2.6)$$

It is also seen from (1.1) and (1.2) that the sum of the two processes $X_1(\cdot), X_2(\cdot)$ is of the form

$$X_1(t) + X_2(t) = z + \nu t + V(t), \quad V(t) := \rho V_1(t) + \sigma V_2(t), \quad (2.7)$$

where, by analogy with (2.5), (2.6) we have set

$$V_1(t) := \int_0^t \mathbf{1}_{\{Y(s) > 0\}} \, dB_1(s) + \int_0^t \mathbf{1}_{\{Y(s) \leq 0\}} \, dB_2(s), \quad (2.8)$$

$$V_2(t) := \int_0^t \mathbf{1}_{\{Y(s) \leq 0\}} \, dB_1(s) + \int_0^t \mathbf{1}_{\{Y(s) > 0\}} \, dB_2(s). \quad (2.9)$$

• The processes $W_1(\cdot)$, $W_2(\cdot)$, $V_1(\cdot)$, $V_2(\cdot)$ are continuous \mathbf{F} -martingales with quadratic variations $\langle W_1 \rangle(t) = \langle W_2 \rangle(t) = \langle V_1 \rangle(t) = \langle V_2 \rangle(t) = t$, i.e., Brownian motions by the P. LÉVY theorem (e.g., KARATZAS & SHREVE (1991), p. 157); also $\langle V_1, V_2 \rangle(\cdot) = \langle W_1, W_2 \rangle(\cdot) = \langle V_1, W_2 \rangle(\cdot) = \langle W_1, V_2 \rangle(\cdot) = 0$.

The pairs $(W_1(\cdot), W_2(\cdot))$ and $(V_1(\cdot), V_2(\cdot))$, as well as $(V_1(\cdot), W_2(\cdot))$ and $(W_1(\cdot), V_2(\cdot))$, are thus two-dimensional Brownian motions, by the P. LÉVY theorem. In conjunction with (1.3) this implies, in particular, that $W(\cdot)$ and $V(\cdot)$ in (2.4), (2.7) are standard, one-dimensional Brownian motions.

• On the other hand, we see from (2.5), (2.8) and with the notation of (1.5) the intertwinelements

$$V_1(t) = \int_0^t \operatorname{sgn}(Y(s)) dW_1(s), \quad W_1(t) = \int_0^t \operatorname{sgn}(Y(s)) dV_1(s); \quad (2.10)$$

and from (2.6), (2.9) we obtain the intertwinelements

$$V_2(t) = - \int_0^t \operatorname{sgn}(Y(s)) dW_2(s), \quad W_2(t) = - \int_0^t \operatorname{sgn}(Y(s)) dV_2(s). \quad (2.11)$$

• The equation (2.3) admits a pathwise unique, strong solution; in particular, uniqueness in the sense of the probability distribution holds as well (cf. Proposition 5.5.17, page 341 and the YAMADA-WATANABE Proposition 5.3.20, page 309 in KARATZAS & SHREVE (1991)), and we have the identity $\mathbf{F}^Y \equiv \mathbf{F}^W$.

2.1 TWO AUXILIARY SYSTEMS

The relations $X_1(t) + X_2(t) = x_1 + x_2 + (g - h)t + \rho V_1(t) + \sigma V_2(t)$ from (2.7), and

$$\begin{aligned} X_1(t) - X_2(t) = \\ x_1 - x_2 + (g + h) \int_0^t (\mathbf{1}_{\{X_1(s) \leq X_2(s)\}} - \mathbf{1}_{\{X_1(s) > X_2(s)\}}) ds + \rho W_1(t) + \sigma W_2(t) \end{aligned}$$

from (1.4), (2.2), (2.4) lead, in conjunction with the intertwinelements of (2.10) and (2.11), to the system of stochastic integral equations

$$\begin{aligned} X_1(t) = x_1 + \int_0^t (g \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} - h \mathbf{1}_{\{X_1(s) > X_2(s)\}}) ds \\ + \rho \int_0^t \mathbf{1}_{\{X_1(s) > X_2(s)\}} dW_1(s) + \sigma \int_0^t \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} dW_2(s), \quad (2.12) \end{aligned}$$

$$\begin{aligned} X_2(t) = x_2 + \int_0^t (g \mathbf{1}_{\{X_1(s) > X_2(s)\}} - h \mathbf{1}_{\{X_1(s) \leq X_2(s)\}}) ds \\ - \rho \int_0^t \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} dW_1(s) - \sigma \int_0^t \mathbf{1}_{\{X_1(s) > X_2(s)\}} dW_2(s) \quad (2.13) \end{aligned}$$

driven by the planar Brownian motion $(W_1(\cdot), W_2(\cdot))$ of (2.5), (2.6).

Furthermore, we observe that the intertwining of (2.10), (2.11) allow us to recast this system as driven by the planar Brownian motion $(V_1(\cdot), V_2(\cdot))$ of (2.8), (2.9), namely

$$\begin{aligned} X_1(t) &= x_1 + \int_0^t (g \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} - h \mathbf{1}_{\{X_1(s) > X_2(s)\}}) ds \\ &\quad + \rho \int_0^t \mathbf{1}_{\{X_1(s) > X_2(s)\}} dV_1(s) + \sigma \int_0^t \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} dV_2(s), \end{aligned} \quad (2.14)$$

$$\begin{aligned} X_2(t) &= x_2 + \int_0^t (g \mathbf{1}_{\{X_1(s) > X_2(s)\}} - h \mathbf{1}_{\{X_1(s) \leq X_2(s)\}}) ds \\ &\quad + \rho \int_0^t \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} dV_1(s) + \sigma \int_0^t \mathbf{1}_{\{X_1(s) > X_2(s)\}} dV_2(s). \end{aligned} \quad (2.15)$$

It is quite clear, though perhaps worth noting, that the systems of stochastic equations (1.1)-(1.2), as well as (2.12)-(2.13) and (2.14)-(2.15), give rise and correspond to the same martingale problem – namely, the one with infinitesimal generator (0.1).

2.2 SKEW AND INTEGRAL REPRESENTATIONS

By analogy with (2.4), (2.7) let us introduce the standard Brownian motions

$$W^b(\cdot) := \rho W_1(\cdot) - \sigma W_2(\cdot), \quad V^b(\cdot) := \rho V_1(\cdot) - \sigma V_2(\cdot) \quad (2.16)$$

and note the new intertwining

$$V(t) = \int_0^t \operatorname{sgn}(Y(s)) dW^b(s), \quad V^b(t) = \int_0^t \operatorname{sgn}(Y(s)) dW(s). \quad (2.17)$$

It will be convenient to cast the Brownian motion $W^b(\cdot)$ of (2.16) in the decomposition

$$W^b(\cdot) = \gamma W(\cdot) + \delta U^b(\cdot), \quad \text{where } \gamma := \rho^2 - \sigma^2, \quad \delta := \sqrt{1 - \gamma^2} = 2\rho\sigma, \quad (2.18)$$

in terms of the independent Brownian motions

$$U^b(\cdot) := \sigma W_1(\cdot) - \rho W_2(\cdot) \quad \text{and} \quad W(\cdot) = \rho W_1(\cdot) + \sigma W_2(\cdot) \quad (2.19)$$

as in (2.4). With this setup, the Brownian motion $V(\cdot)$ defined in (2.7) takes the form $V(t) = \int_0^t \operatorname{sgn}(Y(s)) [\gamma dW(s) + \delta dU^b(s)]$, equivalently

$$\begin{aligned} V(t) &= \gamma \int_0^t \operatorname{sgn}(Y(s)) [dY(s) + \lambda \operatorname{sgn}(Y(s)) ds] + \delta Q(t) \\ &= \gamma (|Y(t)| - |y| + \lambda t - 2L^Y(t)) + \delta Q(t), \quad 0 \leq t < \infty \end{aligned} \quad (2.20)$$

thanks to (2.17), (2.18), (1.4) and the TANAKA formulas (e.g., KARATZAS & SHREVE (1991), page 220). Here

$$Q(t) := \int_0^t \operatorname{sgn}(Y(s)) dU^b(s) = \sigma V_1(t) + \rho V_2(t), \quad 0 \leq t < \infty \quad (2.21)$$

is yet another Brownian motion with $\langle Q, W \rangle = \langle U^b, W \rangle = 0$. Thus, $Q(\cdot)$ is independent of the Brownian motion $W(\cdot)$ and of the diffusion process $Y(\cdot)$ that $W(\cdot)$ engenders via (2.3).

• We can combine now the last expression in (2.20) with $X_1(t) - X_2(t) = Y(t)$ of (2.2) and with $X_1(t) + X_2(t) = x_1 + x_2 + vt + V(t)$ of (2.7), and arrive at the *skew representations*

$$X_1(t) = x_1 + \mu t + \rho^2(Y^+(t) - y^+) - \sigma^2(Y^-(t) - y^-) - \gamma L^Y(t) + \rho \sigma Q(t), \quad (2.22)$$

$$X_2(t) = x_2 + \mu t - \sigma^2(Y^+(t) - y^+) + \rho^2(Y^-(t) - y^-) - \gamma L^Y(t) + \rho \sigma Q(t) \quad (2.23)$$

for the components of the two-dimensional diffusion $(X_1(\cdot), X_2(\cdot))$ of the system (1.1), (1.2); we have set

$$\mu := \frac{1}{2}(v + \lambda \gamma) = g\rho^2 - h\sigma^2. \quad (2.24)$$

These formulas involve the positive and negative parts of the current value of the one-dimensional diffusion process $Y(\cdot)$ in (1.4), the current value of its local time $L^Y(\cdot)$ at the origin, and the current value of the *independent* Brownian motion $Q(\cdot)$ of (2.21). Two cases stand out.

(A) In the *Equal Variance* (Isotropic) case $\rho = \sigma = 1/\sqrt{2}$, the local times disappear from the expressions (2.22), (2.23), which then take the very simple form

$$X_1(t) = x_1 + \frac{1}{2}(vt + Y(t) - y + Q(t)), \quad X_2(t) = x_2 + \frac{1}{2}(vt - Y(t) + y + Q(t)). \quad (2.25)$$

(B) In the *Degenerate* case $\rho \sigma = 0$ the independent Brownian motion $Q(\cdot)$ disappears from these expressions; for instance, with $\sigma = 0$ and $\rho = 1$, they become

$$X_1(t) = x_1 - y^+ + gt + Y^+(t) - L^Y(t), \quad X_2(t) = x_2 - y^- + gt + Y^-(t) - L^Y(t). \quad (2.26)$$

2.3 UNIQUENESS IN DISTRIBUTION

The analysis of this section shows that, given *any* weak solution of the system of stochastic equations (1.1), (1.2), its vector process $(X_1(\cdot), X_2(\cdot))$ can be cast in the form (2.22)–(2.23). Here the diffusion process $Y(\cdot)$ is the pathwise unique, strong solution of the stochastic integral equation

$$Y(t) = y - \lambda \int_0^t \operatorname{sgn}(Y(s)) \, ds + W(t), \quad 0 \leq t < \infty$$

as in (2.3) and with the notation of (1.5), driven by the Brownian motion $W(\cdot)$ of (2.4); whereas the Brownian motion $Q(\cdot)$ is independent of $W(\cdot)$, thus also of $Y(\cdot)$. In other words, the joint distribution of the pair $(Y(\cdot), Q(\cdot))$ is determined uniquely – and thus, from (2.22)–(2.23), so is the joint distribution of the vector process $(X_1(\cdot), X_2(\cdot))$.

To put it a bit more succinctly: *uniqueness in distribution holds for the system of equations (1.1), (1.2), as well as for the systems of equations (2.12), (2.13) and (2.14), (2.15).*

3 SYNTHESIS

Let us begin now to reverse the steps of the preceding analysis. We start with a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbf{F} = \{\mathfrak{F}(t)\}_{0 \leq t < \infty}$ rich enough to support two independent, standard Brownian motion $W_1(\cdot)$ and $W_2(\cdot)$; and without sacrificing generality, we shall assume $\mathbf{F} \equiv \mathbf{F}^{(W_1, W_2)}$, i.e., that the filtration is generated by this planar Brownian motion.

With given nonnegative constants g, h, ρ, σ that satisfy (1.3) and $g + h > 0$, with a given vector $(x_1, x_2) \in \mathbb{R}^2$, and with the notation of (2.1), we construct the pairs of independent Brownian motions

$$W(\cdot) := \rho W_1(\cdot) + \sigma W_2(\cdot), \quad U^b(\cdot) := \sigma W_1(\cdot) - \rho W_2(\cdot) \quad (3.1)$$

and

$$U(\cdot) := \sigma W_1(\cdot) + \rho W_2(\cdot), \quad W^b(\cdot) := \rho W_1(\cdot) - \sigma W_2(\cdot) \quad (3.2)$$

in accordance with (2.19), (2.16). Clearly, $\mathbf{F}^{(W_1, W_2)} \equiv \mathbf{F}^{(W, U^b)} \equiv \mathbf{F}^{(U, W^b)}$.

We construct also the pathwise unique, strong solution $Y(\cdot)$ of the stochastic equation (2.3) driven by the Brownian motion $W(\cdot)$ in (3.1). This is a strong MARKOV and FELLER process, whose transition probabilities can be computed explicitly; see (6.3)-(6.5) below.

With the process $Y(\cdot)$ thus in place, we introduce the continuous, \mathbf{F} -adapted processes

$$V_1(t) = \int_0^t \operatorname{sgn}(Y(s)) dW_1(s), \quad V_2(t) = - \int_0^t \operatorname{sgn}(Y(s)) dW_2(s) \quad (3.3)$$

in accordance with (2.10) and (2.11). These are martingales with $\langle V_1 \rangle(t) = \langle V_2 \rangle(t) = t$ and $\langle V_1, V_2 \rangle(t) = 0$ for all $0 \leq t < \infty$, thus independent Brownian motions in their own right. We construct from them, and by analogy with (3.1) and (3.2), two additional pairs of independent, standard Brownian motions, namely

$$V(\cdot) := \rho V_1(\cdot) + \sigma V_2(\cdot), \quad Q^b(\cdot) := \sigma V_1(\cdot) - \rho V_2(\cdot) \quad (3.4)$$

and

$$Q(\cdot) := \sigma V_1(\cdot) + \rho V_2(\cdot), \quad V^b(\cdot) := \rho V_1(\cdot) - \sigma V_2(\cdot). \quad (3.5)$$

We note the intertwining (2.17), (2.21) and $Q^b(\cdot) = \int_0^\cdot \operatorname{sgn}(Y(t)) dU(t)$, as well as the filtration identities $\mathbf{F}^{(V_1, V_2)} \equiv \mathbf{F}^{(V, Q^b)} \equiv \mathbf{F}^{(Q, V^b)}$.

Finally, we introduce the continuous, \mathbf{F} -adapted processes

$$X_1(t) := x_1 + \int_0^t (g \mathbf{1}_{\{Y(s) \leq 0\}} - h \mathbf{1}_{\{Y(s) > 0\}}) ds + M_1(t) \quad (3.6)$$

$$X_2(t) := x_2 + \int_0^t (g \mathbf{1}_{\{Y(s) > 0\}} - h \mathbf{1}_{\{Y(s) \leq 0\}}) ds + M_2(t) \quad (3.7)$$

for $0 \leq t < \infty$, where we set

$$M_1(t) := \int_0^t \left(\rho \mathbf{1}_{\{Y(s) > 0\}} dW_1(s) + \sigma \mathbf{1}_{\{Y(s) \leq 0\}} dW_2(s) \right), \quad (3.8)$$

$$M_2(t) := \int_0^t \left(-\rho \mathbf{1}_{\{Y(s) \leq 0\}} dW_1(s) - \sigma \mathbf{1}_{\{Y(s) > 0\}} dW_2(s) \right), \quad (3.9)$$

by analogy with the equations of (2.12), (2.13). We have for these processes

$$X_1(t) - X_2(t) = Y(t), \quad X_1(t) + X_2(t) = x_1 + x_2 + vt + V(t) \quad (3.10)$$

in accordance with (2.2), (2.7), and note that $M_1(\cdot)$, $M_2(\cdot)$ are continuous \mathbf{F} -martingales, with $\langle M_1, M_2 \rangle(\cdot) \equiv 0$ and quadratic variations

$$\begin{aligned} \langle M_1 \rangle(t) &= \int_0^t \left(\rho^2 \mathbf{1}_{\{Y(s) > 0\}} + \sigma^2 \mathbf{1}_{\{Y(s) \leq 0\}} \right) ds, \\ \langle M_2 \rangle(t) &= \int_0^t \left(\rho^2 \mathbf{1}_{\{Y(s) \leq 0\}} + \sigma^2 \mathbf{1}_{\{Y(s) > 0\}} \right) ds. \end{aligned}$$

There exist then independent Brownian motions $B_1(\cdot)$, $B_2(\cdot)$ on our filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbf{F} = \{\mathfrak{F}(t)\}_{0 \leq t < \infty}$, so the continuous martingales of (3.8), (3.9) are cast in their DOOB representations

$$\begin{aligned} M_1(t) &= \int_0^t \left(\rho \mathbf{1}_{\{Y(s) > 0\}} + \sigma \mathbf{1}_{\{Y(s) \leq 0\}} \right) dB_1(s), \\ M_2(t) &= \int_0^t \left(\rho \mathbf{1}_{\{Y(s) \leq 0\}} + \sigma \mathbf{1}_{\{Y(s) > 0\}} \right) dB_2(s) \end{aligned} \quad (3.11)$$

for $0 \leq t < \infty$; for instance, we can take the Brownian motions

$$B_1(t) = \int_0^t \left(\mathbf{1}_{\{Y(s) > 0\}} dW_1(s) + \mathbf{1}_{\{Y(s) \leq 0\}} dW_2(s) \right), \quad (3.12)$$

$$B_2(t) = - \int_0^t \left(\mathbf{1}_{\{Y(s) \leq 0\}} dW_1(s) + \mathbf{1}_{\{Y(s) > 0\}} dW_2(s) \right) \quad (3.13)$$

that one gets by disentangling $(B_1(\cdot), B_2(\cdot))$ from $(W_1(\cdot), W_2(\cdot))$ in (2.5), (2.6).

3.1 TAKING STOCK

To recapitulate: we have constructed a weak solution for the system of stochastic differential equations (1.1), (1.2), as is seen clearly from (3.6), (3.7) and (3.11); and as we stressed in subsection 2.3, this solution is unique in distribution. We remarked in the Introduction that this is in accordance with general results of STROOCK & VARADHAN (1979), pages 193-194; BASS & PARDOUX (1987); or KRYLOV (2004), page 45.

It develops from (3.6)–(3.9) that we have also constructed a weak solution to the system (2.12), (2.13), once again unique in distribution. On the other hand, we can express the martingales of (3.8), (3.9) as

$$\begin{aligned} M_1(t) &= \int_0^t \left(\rho \mathbf{1}_{\{Y(s)>0\}} dV_1(s) + \sigma \mathbf{1}_{\{Y(s)\leq 0\}} dV_2(s) \right), \\ M_2(t) &= \int_0^t \left(\rho \mathbf{1}_{\{Y(s)\leq 0\}} dV_1(s) + \sigma \mathbf{1}_{\{Y(s)>0\}} dV_2(s) \right), \end{aligned}$$

in terms of the independent, standard Brownian motions of (3.3). Back into (3.6) and (3.7), these expressions show that we have also constructed a weak solution for the system of stochastic equations (2.14), (2.15), once again unique in the sense of the probability distribution.

In the next two sections we shall discuss in detail properties of strength/weakness for the solutions to these systems of equations, namely, (2.14)–(2.15), (2.12)–(2.13) and (1.1)–(1.2). For the moment, let us remark that the Brownian motion $V(\cdot) = \rho V_1(\cdot) + \sigma V_2(\cdot)$ determines the *sum* of $X_1(\cdot), X_2(\cdot)$ as in (2.7) or (3.10); and that the Brownian motion $W(\cdot) = \rho W_1(\cdot) + \sigma W_2(\cdot)$ determines the *difference* $Y(\cdot) = X_1(\cdot) - X_2(\cdot)$ via the solution of the stochastic equation (2.3).

3.2 A TWO-DIMENSIONAL ANALOGUE OF THE SKEW BROWNIAN MOTION

The equations of (2.22)–(2.23) can be cast in the form

$$X_1(t) = x_1 + \mu t + \rho^2 \left((X_1(t) - X_2(t))^+ - (x_1 - x_2)^+ \right) \quad (3.14)$$

$$- \sigma^2 \left((X_1(t) - X_2(t))^- - (x_1 - x_2)^- \right) - (\rho^2 - \sigma^2) L^{X_1 - X_2}(t) + \rho \sigma Q(t),$$

$$X_2(t) = x_2 + \mu t - \sigma^2 \left((X_1(t) - X_2(t))^+ - (x_1 - x_2)^+ \right) \quad (3.15)$$

$$+ \rho^2 \left((X_1(t) - X_2(t))^- - (x_1 - x_2)^- \right) - (\rho^2 - \sigma^2) L^{X_1 - X_2}(t) + \rho \sigma Q(t),$$

namely, as a system of stochastic differential equations involving the local time $L^{X_1 - X_2}(\cdot)$ at the origin of the difference $X_1(\cdot) - X_2(\cdot)$. We have constructed a solution to this system, subject to the requirement that the difference $X_1(\cdot) - X_2(\cdot)$ be independent of the driving Brownian motion $Q(\cdot)$.

The system (3.14), (3.15) can be construed as a two-dimensional analogue of the stochastic equation derived by HARRISON & SHEPP (1981) for the WALSH (1978) skew Brownian motion; see the recent survey by LEJAY (2006), and the references cited there, for the various constructions and properties of this process.

4 RANKS AND FILTRATIONS

Let us look now at the ranked versions

$$R_1(\cdot) = \max(X_1(\cdot), X_2(\cdot)), \quad R_2(\cdot) = \min(X_1(\cdot), X_2(\cdot)) \quad (4.1)$$

of the components of the diffusion process constructed in the previous section. We have $R_1(t) + R_2(t) = X_1(t) + X_2(t) = x_1 + x_2 + \nu t + V(t)$ from (3.10), as well as

$$\begin{aligned} R_1(t) - R_2(t) &= |X_1(t) - X_2(t)| = |Y(t)| = |y| + \int_0^t \operatorname{sgn}(Y(s)) dY(s) + 2L^Y(t) \\ &= |x_1 - x_2| - \lambda t + V^b(t) + 2L^Y(t), \quad 0 \leq t < \infty \end{aligned} \quad (4.2)$$

from the TANAKA formulas and (2.3), (2.17); and from the theory of the SKOROKHOD reflection problem (e.g., KARATZAS & SHREVE (1991), page 210), we note

$$2L^Y(t) = \max_{0 \leq s \leq t} \left(-(|y| + V^b(s) - \lambda s) \right)^+, \quad 0 \leq t < \infty. \quad (4.3)$$

We also note from (4.2), (4.3) and (1.6) the filtration relations

$$\mathfrak{F}^{V^b}(t) = \mathfrak{F}^{|Y|}(t) \subsetneq \mathfrak{F}^Y(t), \quad 0 < t < \infty; \quad (4.4)$$

the strict inclusion comes from the fact that $Y(\cdot)$ has a zero on $[0, t]$ with positive probability, so that the random variable $\operatorname{sgn}(Y(t))$ is not $\mathfrak{F}^{|Y|}(t)$ -measurable. Alternatively, one can note from (6.2) below that both $\mathbb{P}(Y(t) > 0 | \mathfrak{F}^{|Y|}(t))$ and $\mathbb{P}(Y(t) < 0 | \mathfrak{F}^{|Y|}(t))$ are non-trivial.

With $r_1 = \max(x_1, x_2)$, $r_2 = \min(x_1, x_2)$ we deduce from these equations and (3.4), (3.5) the dynamics for the ranks

$$R_1(t) = r_1 - ht + \rho V_1(t) + L^Y(t), \quad 0 \leq t < \infty \quad (4.5)$$

$$R_2(t) = r_2 + gt + \sigma V_2(t) - L^Y(t), \quad 0 \leq t < \infty. \quad (4.6)$$

Equations (4.5), (4.6) identify the processes $V_1(\cdot)$ and $V_2(\cdot)$ of (3.3) as the independent Brownian motions associated with individual *ranks*; such an interpretation is also possible using the equations of (2.14), (2.15). On the other hand, the independent, standard Brownian motions $B_1(\cdot)$, $B_2(\cdot)$ of (3.12) and (3.13), are those associated with the “*names*” (indices, identities) of the individual particles.

4.1 THE DEGENERATE CASE

We embark now on a detailed study of the solutions to the systems of equations (1.1), (1.2), as well as of those of (2.12), (2.13) and (2.14), (2.15). In this effort it is instructive to focus first on the degenerate case $\rho \sigma = 0$, which has some interesting features of its own.

Proposition 4.1. *In the degenerate case with $\sigma = 0$, thus $\rho = 1$ in light of (1.3), we have the relations*

$$\mathfrak{F}^{(R_1, R_2)}(t) = \mathfrak{F}^V(t) = \mathfrak{F}^{X_1 + X_2}(t) \subsetneq \mathfrak{F}^{(X_1, X_2)}(t) = \mathfrak{F}^Y(t) = \mathfrak{F}^{X_1 - X_2}(t) = \mathfrak{F}^W(t) \quad (4.7)$$

for every $0 < t < \infty$, where the inclusion is strict.

Proof. With $\rho = 1$, $\sigma = 0$ we have $V_1(\cdot) = V(\cdot) = V^b(\cdot)$ in (3.4), (3.5); the first of the claims in (4.7) follows from (4.3) and (4.5)-(4.6), which read now

$$R_1(t) = r_1 - ht + V(t) + L^Y(t), \quad R_2(t) = r_2 + gt - L^Y(t), \quad 0 \leq t < \infty. \quad (4.8)$$

The second claim is immediate from (3.10).

The third (strict inclusion) is fairly obvious from (2.26). Indeed, the equations of (2.26) show

$$X_1(t) + X_2(t) = z - |y| + 2gt + |Y(t)| - 2L^Y(t),$$

thus $\mathfrak{F}^{X_1+X_2}(t) \subseteq \mathfrak{F}^{|Y|}(t) \subseteq \mathfrak{F}^Y(t)$ for $0 \leq t < \infty$; and, as we remarked in (4.4), the second inclusion is strict for $0 < t < \infty$.

The fourth and fifth relations in (4.7) follow from the equations in (2.26) and $X_1(\cdot) - X_2(\cdot) = Y(\cdot)$; whereas the sixth relation is a consequence of the strong solvability of the stochastic equation (2.3). \square

The strictness of the inclusion in $\mathfrak{F}^{(R_1, R_2)}(t) = \mathfrak{F}^V(t) \subsetneq \mathfrak{F}^W(t) = \mathfrak{F}^{(X_1, X_2)}(t)$ for $0 < t < \infty$ reflects the fact that some information is inevitably lost when one passes from the “names” $(X_1(\cdot), X_2(\cdot))$ to the “ranks” $(R_1(\cdot), R_2(\cdot))$. In the case $\rho = 1$, $\sigma = 0$ that we are studying here, the equations (2.12), (2.13) read

$$X_1(t) = x_1 + \int_0^t (g \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} - h \mathbf{1}_{\{X_1(s) > X_2(s)\}}) ds + \int_0^t \mathbf{1}_{\{X_1(s) > X_2(s)\}} dW(s), \quad (4.9)$$

$$X_2(t) = x_2 + \int_0^t (g \mathbf{1}_{\{X_1(s) > X_2(s)\}} - h \mathbf{1}_{\{X_1(s) \leq X_2(s)\}}) ds - \int_0^t \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} dW(s) \quad (4.10)$$

with $W(\cdot) \equiv W_1(\cdot)$ in accordance with (2.4). The construction (synthesis) of section 3 shows that the solution constructed there is *strong*, that is, adapted to the filtration \mathbf{F}^W generated by the Brownian motion $W(\cdot)$ that drives this system. Repeating the analysis of section 2, we see that *any* solution $(X_1(\cdot), X_2(\cdot))$ of the system (4.9), (4.10) can be cast in the form (2.26), where $Y(\cdot)$ is the pathwise unique, strong solution of the diffusion equation (2.3) with $\mathbf{F}^Y \equiv \mathbf{F}^W$ and $L^Y(\cdot)$ is the local time in (1.6). In particular, any such solution of the system (4.9), (4.10) is a non-anticipative functional (given as in (2.26)) of its driving Brownian motion $W(\cdot)$. Thus, pathwise uniqueness holds for the system of equations (4.9), (4.10).

On the other hand, and always in the case $\rho = 1$, $\sigma = 0$, we can cast the equations (2.14), (2.15) as

$$X_1(t) = x_1 + \int_0^t (g \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} - h \mathbf{1}_{\{X_1(s) > X_2(s)\}}) ds + \int_0^t \mathbf{1}_{\{X_1(s) > X_2(s)\}} dV(s), \quad (4.11)$$

$$X_2(t) = x_2 + \int_0^t (g \mathbf{1}_{\{X_1(s) > X_2(s)\}} - h \mathbf{1}_{\{X_1(s) \leq X_2(s)\}}) ds + \int_0^t \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} dV(s) \quad (4.12)$$

now driven by the Brownian motion $V(\cdot) \equiv V_1(\cdot)$ in accordance with (3.4). We know from Proposition 4.1 that this system does *not* admit a strong solution.

Collecting the results of this section and of the preceding one, we see that we have established the following theorem; this can be seen as a two-dimensional analogue of [BARLOW \(1988\)](#).

Theorem 4.1. *The system of stochastic differential equations (1.1), (1.2) with $\rho \sigma = 0$ has a weak solution which is unique in the sense of the probability distribution. The same is true for each of the systems of equations (4.9), (4.10) and (4.11), (4.12).*

On the other hand, the system of stochastic differential equations (4.9), (4.10) admits a strong solution, which is therefore pathwise unique; whereas the system (4.11), (4.12) does not admit a strong solution.

We have deduced here pathwise uniqueness from the “obverse YAMADA & WATANABE” results of [ENGELBERT \(1991\)](#) and [CHERNÝĚ \(2001\)](#).

The following Figure is a Simulation of the Diffusion with $\rho = 1$, $\sigma = 0$ and $g = h = 1$.

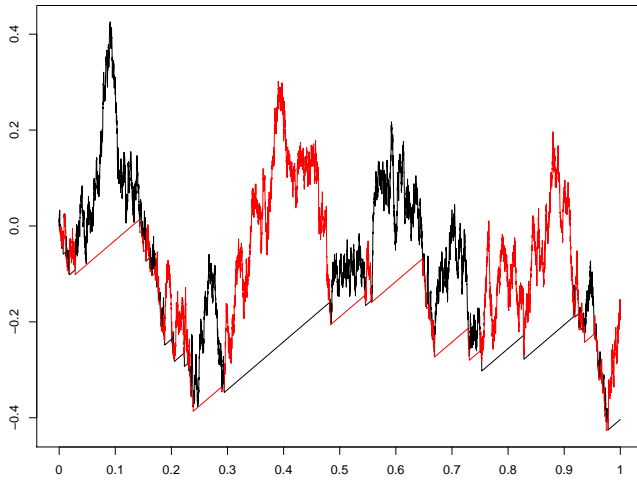


Fig. 1 Simulated processes; Black = $X_1(\cdot)$, Red = $X_2(\cdot)$.

4.2 THE NON-DEGENERATE CASE

Let us move now on to the non-degenerate case $\rho \sigma > 0$. In contrast to the situation we encountered in [Proposition 4.1](#), now both pairs $(X_1(\cdot), X_2(\cdot))$ (of positions by “name”) and $(R_1(\cdot), R_2(\cdot))$ (of positions by “rank”) generate two-dimensional Brownian filtrations, as our next result shows.

Proposition 4.2. *In the non-degenerate case $\rho \sigma > 0$, we have for every $0 < t < \infty$ the filtration relations*

$$\begin{aligned} \mathfrak{F}^{(V_1, V_2)}(t) &= \mathfrak{F}^{(R_1, R_2)}(t) = \mathfrak{F}^{(|Y|, V)}(t) = \mathfrak{F}^{(|Y|, Q)}(t) \subsetneq \mathfrak{F}^{(X_1, X_2)}(t) = \mathfrak{F}^{(W_1, W_2)}(t) \\ &= \mathfrak{F}^{(Y, V)}(t) = \mathfrak{F}^{(W, U^b)}(t) = \mathfrak{F}^{(Y, U^b)}(t) = \mathfrak{F}^{(W, Q)}(t) = \mathfrak{F}^{(Y, Q)}(t), \end{aligned} \quad (4.13)$$

where the inclusion is strict.

Proof. We have clearly $\mathfrak{F}^{L^Y}(t) \subseteq \mathfrak{F}^{V^b}(t) \subseteq \mathfrak{F}^{(V_1, V_2)}(t)$ by virtue of (4.3), (3.4); back into the equations of (4.5), (4.6), this implies $\mathfrak{F}^{(R_1, R_2)}(t) \subseteq \mathfrak{F}^{(V_1, V_2)}(t)$. For the reverse inclusion, we note that $\mathfrak{F}^{L^Y}(t) \subseteq \mathfrak{F}^{|Y|}(t) \subseteq \mathfrak{F}^{(R_1, R_2)}(t)$ holds, thanks to (1.6) and $|Y(\cdot)| = R_1(\cdot) - R_2(\cdot)$. Back into (4.5), (4.6) this gives $\mathfrak{F}^{(V_1, V_2)}(t) \subseteq \mathfrak{F}^{(R_1, R_2)}(t)$, and the first equality in (4.13) is proved.

In conjunction with (2.7) and (2.20), these considerations also give $\mathfrak{F}^{(R_1, R_2)}(t) = \mathfrak{F}^{(|Y|, X_1 + X_2)}(t) = \mathfrak{F}^{(|Y|, V)}(t) = \mathfrak{F}^{(|Y|, Q)}(t)$ for every $0 \leq t < \infty$, justifying the second and third equalities in (4.13).

For the fourth equality in (4.13), we have noted already the inclusion $\mathfrak{F}^{(X_1, X_2)}(t) \subseteq \mathfrak{F}^{(W_1, W_2)}(t)$ from the construction of (3.6)-(3.9). In order to argue the reverse inclusion, let us note that (3.8), (3.9) imply

$$\begin{aligned} W_1(t) &= (1/\rho) \int_0^t (\mathbf{1}_{\{X_1(s) > X_2(s)\}} dM_1(s) - \mathbf{1}_{\{X_1(s) \leq X_2(s)\}} dM_2(s)), \\ W_2(t) &= (1/\sigma) \int_0^t (\mathbf{1}_{\{X_1(s) \leq X_2(s)\}} dM_1(s) - \mathbf{1}_{\{X_1(s) > X_2(s)\}} dM_2(s)). \end{aligned}$$

On the strength of (3.6) and (3.7), the $\mathbf{F}^{(W_1, W_2)}$ -martingales $M_1(\cdot)$ and $M_2(\cdot)$ are also martingales of the filtration $\mathbf{F}^{(X_1, X_2)}$ generated by the state process $(X_1(\cdot), X_2(\cdot))$, so we deduce $\mathbf{F}^{(W_1, W_2)} \subseteq \mathbf{F}^{(X_1, X_2)}$ and the fourth equality in (4.13) follows.

For the fifth equality we note that $\sigma(X_1(t), X_2(t)) = \sigma(Y(t), V(t))$ holds for every $t \geq 0$, courtesy of (3.10), and gives $\mathfrak{F}^{(X_1, X_2)}(t) = \mathfrak{F}^{(Y, V)}(t)$. The sixth equality in (4.13) is a consequence of (2.19), which gives actually the stronger statement $\sigma(W_1(t), W_2(t)) = \sigma(W(t), U^b(t))$ for every $t \geq 0$.

The seventh and eighth equalities in (4.13) are straightforward consequences of $\mathbf{F}^W = \mathbf{F}^Y$ and of the equation (2.21), along with its ‘‘twin’’ $U^b(\cdot) = \int_0^\cdot \text{sgn}(Y(t)) dQ(t)$. The ninth equality is now obvious.

Finally, the strictness of the inclusion in (4.13) follows now from the strictness of the inclusion in

$$\mathfrak{F}^{(R_1, R_2)}(t) = \mathfrak{F}^{(|Y|, Q)}(t) \subsetneq \mathfrak{F}^{(Y, Q)}(t) = \mathfrak{F}^{(X_1, X_2)}(t), \quad 0 < t < \infty,$$

itself a consequence of the strictness of the inclusion in (4.4) and of the independence of the processes $Y(\cdot)$, $Q(\cdot)$. \square

Arguing as in the previous subsection and using Proposition 4.2, we obtain the following analogue of Theorem 4.1 for the non-degenerate case.

Theorem 4.2. *The system of stochastic differential equations (1.1), (1.2) with $\rho \sigma > 0$ has a weak solution which is unique in the sense of the probability distribution. The same is true for each of the systems of equations (2.12), (2.13) and (2.14), (2.15).*

On the other hand, the system of stochastic differential equations (2.12), (2.13) admits a strong solution, which is therefore pathwise unique; whereas the system (2.14), (2.15) admits no strong solution.

The existence and uniqueness-in-distribution of a weak solution to (1.1), (1.2) in the non-degenerate case $\rho \sigma > 0$ follow also from the general, multidimensional results of BASS & PARDOUX (1987). The approach here is more direct and concrete, capitalizing on the special two-dimensional nature of our setting; the BASS-PARDOUX results cannot, however, be applied to the degenerate case $\rho \sigma = 0$, so it does not seem possible to obtain even the first part of Theorem 4.1 from them.

We shall see in Theorem 5.1 of the next section that the system of equations (1.1), (1.2) admits a strong solution; this too goes beyond the BASS & PARDOUX (1987) results. As far as we know it is an open issue, whether such strength might obtain in their setting as well (to wit, where \mathbb{R}^d , $d \geq 2$ is partitioned into disjoint polyhedral chambers, diffusion characteristics are constant in each chamber, and strong non-degeneracy prevails).

4.3 DISCUSSION

Theorem 4.2 shows that the system of (2.14), (2.15) with $\rho \sigma > 0$ can be thought of as a *genuinely two-dimensional TANAKA example*: a system of stochastic differential equations that admits a weak solution which is unique in the sense of the probability distribution, but no strong solution.

Theorem 4.2 might usefully be compared also with the strictly one-dimensional results by BARLOW (1988) and NAKAO (1972), as well as with Theorem 2 in VERE-TENNIKOV (1982); see also BARLOW (1982). BARLOW (1988) shows, in particular, that with any given constants $x \in \mathbb{R}$, $\rho > 0$ and $\sigma > 0$, the one-dimensional stochastic differential equation

$$X(t) = x + \int_0^t \tau(X(s)) dB(s), \quad 0 \leq t < \infty,$$

which has a pathwise unique, strong solution for $\tau(\cdot)$ given as $\tau_1(y) := \rho \mathbf{1}_{(0,\infty)}(y) + \sigma \mathbf{1}_{(-\infty,0]}(y)$ by virtue of the NAKAO (1972) results, admits only a weak solution (and no strong solution) for $\tau(\cdot)$ equal to

$$\tau_2(y) = \rho \mathbf{1}_{(0,\infty)}(y) - \sigma \mathbf{1}_{(-\infty,0]}(y).$$

In both these cases the variance structure is the same, namely $\tau_1^2(y) = \tau_2^2(y) = \rho^2 \mathbf{1}_{(0,\infty)}(y) + \sigma^2 \mathbf{1}_{(-\infty,0]}(y)$.

A rather similar situation obtains in Theorem 4.2 (and with obvious minor changes, in Theorem 4.1 as well). To wit, both systems of equations (2.12), (2.13) and (2.14), (2.15) have the same covariance structure, namely $\mathcal{A}(x_1, x_2)$ in (1.7), as the original system (1.1), (1.2); whereas the diffusion matrices for (2.12), (2.13) and (2.14), (2.15) are given respectively by

$$\Sigma_W(x_1, x_2) = \begin{pmatrix} \rho \mathbf{1}_{\{x_1 > x_2\}} & \sigma \mathbf{1}_{\{x_1 \leq x_2\}} \\ -\rho \mathbf{1}_{\{x_1 \leq x_2\}} & -\sigma \mathbf{1}_{\{x_1 > x_2\}} \end{pmatrix}, \quad \Sigma_V(x_1, x_2) = \begin{pmatrix} \rho \mathbf{1}_{\{x_1 > x_2\}} & \sigma \mathbf{1}_{\{x_1 \leq x_2\}} \\ \rho \mathbf{1}_{\{x_1 \leq x_2\}} & \sigma \mathbf{1}_{\{x_1 > x_2\}} \end{pmatrix}. \quad (4.14)$$

These two matrices differ only by changes of signs in the second row; yet the system (2.12), (2.13) of stochastic differential equations induced by the first of them is

strongly solvable, whereas the system (2.14), (2.15) induced by the second diffusion matrix is solvable only weakly. To paraphrase VERETENNIKOV (1982), page 448, in this latter case the diffusion generated by the system (2.14), (2.15) “cannot leave the diagonal $\{x_1 = x_2\}$ strongly”.

We shall see in the next section that the original system (1.1), (1.2), with diffusion matrix of the form

$$\Sigma_B(x_1, x_2) = \begin{pmatrix} \rho \mathbf{1}_{\{x_1 > x_2\}} + \sigma \mathbf{1}_{\{x_1 \leq x_2\}} & 0 \\ 0 & \rho \mathbf{1}_{\{x_1 \leq x_2\}} + \sigma \mathbf{1}_{\{x_1 > x_2\}} \end{pmatrix}, \quad (4.15)$$

is also strongly solvable. A thorough discussion of weak and strong solutions, for all possible real square roots of the covariance matrix $\mathcal{A}(x_1, x_2)$ in (1.7), appears in subsection 5.1.

5 STRENGTH, RESTORED

In conjunction with the identity $\mathfrak{F}^{(X_1, X_2)}(t) = \mathfrak{F}^{(W_1, W_2)}(t)$ from (4.13), the expressions in (3.12), (3.13) imply

$$\mathfrak{F}^{(B_1, B_2)}(t) \subseteq \mathfrak{F}^{(X_1, X_2)}(t), \quad \forall 0 \leq t < \infty \quad (5.1)$$

in the non-degenerate case $\rho \sigma > 0$. It is an interesting question to settle, whether the reverse inclusion might also hold in (5.1), thus implying the strength of the solution to the system (1.1), (1.2) constructed in section 3.

We do have such strength in the isotropic case $\rho^2 = \sigma^2 = 1/2$ of equal variances, as has been well known since VERETENNIKOV (1979, 1980, 1982). To obtain this property from first principles in our context, let us observe from (2.4)-(2.6) the representation $W(t) = (B_1(t) - B_2(t))/\sqrt{2}$ in the isotropic case, thus also the inclusion

$$\mathfrak{F}^Y(t) = \mathfrak{F}^W(t) \subseteq \mathfrak{F}^{(B_1, B_2)}(t) \quad (5.2)$$

for each $t \geq 0$; back into (2.5), (2.6) and in conjunction with (4.13), this gives $\mathfrak{F}^{(X_1, X_2)}(t) = \mathfrak{F}^{(W_1, W_2)}(t) \subseteq \mathfrak{F}^{(B_1, B_2)}(t)$, showing that (5.1) holds with equality in the isotropic case.

More generally, the inclusion in (5.2) will follow whenever one is able to show that the *modified TANAKA equation*

$$Y(t) = y - \lambda \int_0^t \operatorname{sgn}(Y(s)) ds + \int_0^t \left(\rho \mathbf{1}_{\{Y(s) > 0\}} + \sigma \mathbf{1}_{\{Y(s) \leq 0\}} \right) dB_1(s) - \int_0^t \left(\rho \mathbf{1}_{\{Y(s) \leq 0\}} + \sigma \mathbf{1}_{\{Y(s) > 0\}} \right) dB_2(s), \quad 0 \leq t < \infty, \quad (5.3)$$

a consequence of (2.3)-(2.6), has a strong (that is, $\mathbf{F}^{(B_1, B_2)}$ -adapted) solution; then *this strong solvability implies the inclusion*

$$\mathfrak{F}^{(X_1, X_2)}(t) \subseteq \mathfrak{F}^{(B_1, B_2)}(t), \quad \forall 0 \leq t < \infty, \quad (5.4)$$

that is, the strong solvability of the system (1.1), (1.2). Indeed, as Johannes RUF suggests, the strong solvability of (5.3) implies the inclusion $\mathfrak{F}^{(Y, B_1, B_2)}(t) \subseteq \mathfrak{F}^{(B_1, B_2)}(t)$, and (5.4) follows then directly from the equations (1.1), (1.2).

Theorem 5.1. *The system of stochastic differential equations (1.1), (1.2) admits a pathwise unique, strong solution; in particular, the inclusion (5.4) holds.*

Proof. We have seen already that the weak solution of (1.1), (1.2) constructed in section 3 is actually strong in the isotropic case $\rho^2 = \sigma^2 = 1/2$ of equal variances; so we focus on the case $\rho \neq \sigma$.

From the results of YAMADA & WATANABE (e.g., p. 310 in KARATZAS & SHREVE (1991)) and their “obverse” counterparts due to ENGELBERT (1991) and CHERNYĀ (2001) we know that, in the presence of uniqueness in distribution, strong existence and pathwise uniqueness are equivalent; and in this case every solution is strong. We have already seen that the solution of (1.1), (1.2) is unique in distribution, so we need only show that there is a strong solution; and by the argument before the statement of the Theorem, it is enough to prove that the solution of (5.3) is pathwise unique. Furthermore, it suffices to argue such pathwise uniqueness on an arbitrary but fixed time-horizon $[0, T]$ of finite length $T \in (0, \infty)$.

A further reduction is that we need to prove such pathwise uniqueness only as regards the “driftless” modified TANAKA equation

$$Y(t) = y + \int_0^t \left(\rho \mathbf{1}_{\{Y(s) > 0\}} + \sigma \mathbf{1}_{\{Y(s) \leq 0\}} \right) d\boldsymbol{\beta}_1(s) - \int_0^t \left(\rho \mathbf{1}_{\{Y(s) \leq 0\}} + \sigma \mathbf{1}_{\{Y(s) > 0\}} \right) d\boldsymbol{\beta}_2(s) \quad (5.5)$$

for $0 \leq t \leq T$, with $\boldsymbol{\beta}_1(\cdot)$ and $\boldsymbol{\beta}_2(\cdot)$ independent, standard Brownian motions under a suitable equivalent probability measure \mathbb{Q} ; because then a CAMERON-MARTIN-GIRSANOV change of measure brings us back to (5.3), with

$$B_1(t) = \boldsymbol{\beta}_1(t) + \frac{\lambda t}{\rho - \sigma}, \quad B_2(t) = \boldsymbol{\beta}_2(t) + \frac{\lambda t}{\rho - \sigma}, \quad 0 \leq t \leq T$$

independent Brownian motions under the original probability measure \mathbb{P} . Let us observe also, that uniqueness in distribution holds for the equation (5.5): every solution is a standard Brownian motion starting at $Y(0) = y$, under the auxiliary, equivalent probability measure \mathbb{Q} .

Introducing the independent, standard \mathbb{Q} -Brownian motions

$$\boldsymbol{\beta}(\cdot) := \frac{\boldsymbol{\beta}_1(\cdot) + \boldsymbol{\beta}_2(\cdot)}{\sqrt{2}}, \quad \boldsymbol{\vartheta}(\cdot) := \frac{\boldsymbol{\beta}_1(\cdot) - \boldsymbol{\beta}_2(\cdot)}{\sqrt{2}},$$

we can write (5.5) in the equivalent form

$$Y(t) = y + \frac{\rho - \sigma}{\sqrt{2}} \int_0^t \operatorname{sgn}(Y(s)) d\boldsymbol{\beta}(s) - \frac{\rho + \sigma}{\sqrt{2}} \boldsymbol{\vartheta}(t).$$

Pathwise uniqueness for this equation follows now directly from Theorem 1.1. \square

5.1 THE ALGEBRA AND GEOMETRY OF STRENGTH

The matrices of (4.14), (4.15) are square roots of the covariance matrix $\mathcal{A}(x_1, x_2)$ in (1.7). The general real square root of this covariance matrix is of the form

$$\begin{aligned} \mathcal{A}^{1/2}(x_1, x_2) = \Sigma(x_1, x_2) = & \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \mathbf{1}_{\{x_1 > x_2\}} \\ & + \begin{pmatrix} \sigma & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix} \mathbf{1}_{\{x_1 \leq x_2\}}, \end{aligned} \quad (5.6)$$

where $\varepsilon \in \{-1, 1\}$, $\delta \in \{-1, 1\}$, $0 \leq \varphi, \vartheta \leq 2\pi$, or equivalently

$$\mathcal{A}^{1/2}(x_1, x_2) = \Sigma(x_1, x_2) = \Sigma_+ \mathbf{1}_{\{x_1 > x_2\}} + \Sigma_- \mathbf{1}_{\{x_1 \leq x_2\}} \quad (5.7)$$

in the notation of (1.8). We also write the drift term of the generator (0.1) in vector form

$$\mathbf{G}(x_1, x_2) = \begin{pmatrix} g \\ -h \end{pmatrix} \mathbf{1}_{\{x_1 \leq x_2\}} + \begin{pmatrix} -h \\ g \end{pmatrix} \mathbf{1}_{\{x_1 > x_2\}}$$

Each choice in (5.6) (or equivalently, (5.7)) leads to a system of stochastic differential equations with rank-based characteristics

$$d\mathbf{X}(t) = \mathcal{A}^{1/2}(\mathbf{X}(t)) d\mathbf{U}(t) + \mathbf{G}(\mathbf{X}(t)) dt, \quad (5.8)$$

driven by a two-dimensional Brownian motion $\mathbf{U}(\cdot) = (U_1(\cdot), U_2(\cdot))'$. This system admits a weak solution, which is unique in the sense of the probability distribution.

Theorem 5.2. *With the notation of (1.8) and $\mathbf{e}_1 = (1, 0)'$, $\mathbf{e}_2 = (0, 1)'$, the (unique in distribution) weak solution of the system of stochastic differential equations (5.8) fails to be strong, if and only if*

$$(\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_- = -(\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_+. \quad (5.9)$$

Proof. We note that the equation satisfied by the difference $Y(\cdot) = X_1(\cdot) - X_2(\cdot)$ is

$$\begin{aligned} dY(t) &= \frac{(\mathbf{e}_1 - \mathbf{e}_2)'}{2} \left((\Sigma_+ + \Sigma_-) + \operatorname{sgn}(Y(t))(\Sigma_+ - \Sigma_-) \right) d\mathbf{U}(t) - \lambda \operatorname{sgn}(Y(t)) dt \\ &= -\lambda \operatorname{sgn}(Y(t)) dt + \left((\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_+ \mathbf{1}_{\{Y(t) > 0\}} + (\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_- \mathbf{1}_{\{Y(t) \leq 0\}} \right) d\mathbf{U}(t) \\ &= \operatorname{sgn}(Y(t)) dM(t) + dN(t), \end{aligned} \quad (5.10)$$

where we have set

$$M(t) := \frac{(\mathbf{e}_1 - \mathbf{e}_2)'}{2} (\Sigma_+ - \Sigma_-) \mathbf{U}(t) - \lambda t \quad \text{and} \quad N(t) := \frac{(\mathbf{e}_1 - \mathbf{e}_2)'}{2} (\Sigma_+ + \Sigma_-) \mathbf{U}(t).$$

Elementary calculation shows that $\langle M, N \rangle(\cdot) = 0$, since $(\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_{\pm}$ are two unit vectors on the plane.

We observe next that, because the indicators $\mathbf{1}_{\{X_1(t) \geq X_2(t)\}}$ and $\mathbf{1}_{\{X_1(t) < X_2(t)\}}$ are functions of the difference $Y(t) = X_1(t) - X_2(t)$, the σ -algebra $\mathfrak{F}^{(X_1, X_2)}(t)$ is contained in the σ -algebra $\mathfrak{F}^{(U_1, U_2, Y)}(t)$, for every $t \in [0, \infty)$, by construction as in (5.8).

When the equation (5.10) for $Y(\cdot)$ is solvable strongly with respect to the planar Brownian motion $\mathbf{U}(\cdot) = (U_1(\cdot), U_2(\cdot))'$, we have the filtration comparisons

$$\mathfrak{F}^{X_1-X_2}(t) = \mathfrak{F}^Y(t) \subseteq \mathfrak{F}^{(U_1, U_2)}(t), \quad \text{thus} \quad \mathfrak{F}^{(X_1, X_2)}(t) \subseteq \mathfrak{F}^{(U_1, U_2, Y)}(t) = \mathfrak{F}^{(U_1, U_2)}(t) :$$

the system (5.8) is then strongly solvable. When the equation (5.10) for $Y(\cdot)$ admits no strong solution with respect to $\mathbf{U}(\cdot)$, the system (5.8) admits no strong solution. There are now three possibilities.

1. When $(\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_+ = (\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_-$, the second line of the equation (5.10) for $Y(\cdot)$ simplifies to (2.3) which, as we have already remarked in subsection 2.3, has a pathwise unique, strong solution; so the system (5.8) is then strongly solvable.
2. If both $(\mathbf{e}_1 - \mathbf{e}_2)'(\Sigma_+ \pm \Sigma_-)$ are non-zero vectors, we can apply the CAMERON-MARTIN-GIRSANOV theorem as in the proof of Theorem 5.1. To wit, we can restrict ourselves to a finite time horizon $[0, T]$ and define a suitable equivalent probability measure \mathbb{Q} under which $N(\cdot)$ and $M(\cdot)$ are continuous, strongly orthogonal martingales. In this case the quadratic variations $\langle M \rangle(\cdot)$ and $\langle N \rangle(\cdot)$ are proportional, so the domination condition of Theorem 1.1 also holds. Then $Y(\cdot)$ is adapted to the filtration of $(M(\cdot), N(\cdot))$, and this proves that the solution to (5.8) is strong in this case as well.
3. Finally, it is possible that $(\mathbf{e}_1 - \mathbf{e}_2)'(\Sigma_+ + \Sigma_-)$ is zero; this condition is formulated in the statement as (5.9). Then the perturbation $N(\cdot)$ vanishes in (5.10), which becomes then an ordinary *drifted TANAKA equation*

$$dY(t) = \text{sgn}(Y(t)) dM(t), \quad M(t) = (\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_+ \mathbf{U}(t) - \lambda t \quad (5.11)$$

driven by the Brownian motion $M(\cdot)$ with negative drift. From the TANAKA formula, the theory of the SKOROKHOD reflection problem, and (4.4), we note

$$M(t) = |Y(t)| - |y| - 2L^Y(t), \quad \text{thus} \quad \mathfrak{F}^M(t) = \mathfrak{F}^{|Y|}(t) \subsetneq \mathfrak{F}^Y(t), \quad 0 < t < \infty.$$

We consider also the independent, one-dimensional standard Brownian motions

$$U_+(\cdot) := (\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_+ \mathbf{U}(\cdot), \quad \mathcal{Z}(\cdot) := \mathbf{v}' \mathbf{U}(\cdot),$$

where the unit vectors $\Sigma_+'(\mathbf{e}_1 - \mathbf{e}_2)$ and \mathbf{v} are orthogonal, and note $\mathfrak{F}^{\mathbf{U}}(t) = \mathfrak{F}^{U_+}(t) \vee \mathfrak{F}^{\mathcal{Z}}(t) = \mathfrak{F}^M(t) \vee \mathfrak{F}^{\mathcal{Z}}(t)$, $0 \leq t < \infty$.

The equation (5.11) has no strong solution; that is, $\mathfrak{F}^Y(t) \subseteq \mathfrak{F}^{\mathbf{U}}(t)$ cannot possibly hold for $0 < t < \infty$. For if it did, then the process

$$W(\cdot) := Y(\cdot) - y + \lambda \int_0^\cdot \text{sgn}(Y(t)) dt,$$

which generates exactly the same filtration as the process $Y(\cdot)$, would be a standard one-dimensional Brownian motion and independent of $\mathcal{Z}(\cdot)$, yet adapted to the filtration generated by $\mathbf{U}(\cdot)$; but this is impossible, since $\mathfrak{F}^M(t) = \mathfrak{F}^{|Y|}(t) \subsetneq \mathfrak{F}^Y(t) = \mathfrak{F}^W(t) \perp \mathfrak{F}^{\mathcal{Z}}(t)$ for $0 < t < \infty$ would lead then to

$$\mathfrak{F}^{\mathbf{U}}(t) = \mathfrak{F}^{U_+}(t) \vee \mathfrak{F}^{\mathcal{Z}}(t) = \mathfrak{F}^M(t) \vee \mathfrak{F}^{\mathcal{Z}}(t) \subsetneq \mathfrak{F}^W(t) \vee \mathfrak{F}^{\mathcal{Z}}(t) \subseteq \mathfrak{F}^{\mathbf{U}}(t).$$

To summarize: the solution to the system (5.8) fails to be strong if, and only if, (5.9) holds. \square

Remark 5.1. Under the condition (5.9), the diffusion vectors $\mathbf{s}_+ := (\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_+$ (for the half-plane $\{x_1 > x_2\}$) and $\mathbf{s}_- := (\mathbf{e}_1 - \mathbf{e}_2)' \Sigma_-$ (for the half-plane $\{x_1 \leq x_2\}$) in the equation (5.10) point in *exactly opposite directions*. As a result, it becomes impossible for the planar diffusion $\mathbf{X}(\cdot) = (X_1(\cdot), X_2(\cdot))$ of (5.8) to “escape strongly from the diagonal $\{x_1 = x_2\}$ ”.

Let us also note that \mathbf{s}_\pm are unit vectors, so the condition (5.9) holds if and only if their scalar product is -1 . Working out this relation using the representation given in (1.8), we obtain the condition (1.9).

Remark 5.2. Since $(\sigma^2 \varepsilon - \rho^2 \delta)^2 + \rho^2 \sigma^2 (1 + \varepsilon \delta)^2 = 1$, the condition of (1.9) is equivalent to

$$\pi + \vartheta = \varphi + \psi \quad (5.12)$$

(modulo 2π), which involves only the angles ϑ and φ of (5.6), as well as the angle $\psi \in (-\pi, \pi]$ determined from $\cos \psi = \rho \sigma (1 + \varepsilon \delta)$, $\sin \psi = \sigma^2 \varepsilon - \rho^2 \delta$.

Example 5.1. The system of (1.1), (1.2), with diffusion matrix Σ_B in (4.15), corresponds to $\varepsilon = \delta = 1$, $\varphi = \vartheta = 0$ and thus to ψ that has to satisfy $\cos \psi = 2\rho\sigma \geq 0$, $\sin \psi = \sigma^2 - \rho^2$; there is no way that (5.12) can hold, so the system (5.8) is strongly solvable for all choices of ρ , σ as in (1.3).

Similarly, the system of (2.14), (2.15) with diffusion matrix Σ_W in (4.14) corresponds to $\varepsilon = -1$, $\delta = 1$, $\varphi = 0$, $\vartheta = -\pi/2$, and thus $\psi = -\pi/2$ as well. Once again there is no way for (5.12) to hold, so the system (5.8) is always strongly solvable.

Whereas the system of (2.12), (2.13) with diffusion matrix Σ_V in (4.14) corresponds to $\varepsilon = 1$, $\delta = -1$, $\varphi = 0$, $\vartheta = -\pi/2$, so $\psi = \pi/2$ for all choices of ρ , σ as in (1.3). In this case (5.12) always holds, and the system (5.8) is never strongly solvable.

Example 5.2. The matrix $\mathcal{A}(x_1, x_2)$ in (1.7) has a total of 64 square roots of the form $\Sigma_1 \mathbf{1}_{\{x_1 > x_2\}} + \Sigma_2 \mathbf{1}_{\{x_1 \leq x_2\}}$ with

$$\Sigma_1 \in \left\{ \begin{pmatrix} \pm\rho & 0 \\ 0 & \pm\sigma \end{pmatrix}, \begin{pmatrix} 0 & \pm\rho \\ \pm\sigma & 0 \end{pmatrix} \right\} \quad \text{and} \quad \Sigma_2 \in \left\{ \begin{pmatrix} \pm\sigma & 0 \\ 0 & \pm\rho \end{pmatrix}, \begin{pmatrix} 0 & \pm\sigma \\ \pm\rho & 0 \end{pmatrix} \right\}.$$

Among these, 48 lead to strongly solvable systems in the isotropic ($\rho = \sigma = 1/\sqrt{2}$) or degenerate ($\rho\sigma = 0$) cases, whereas 56 choices lead to strongly solvable systems in all other cases.

6 JOINT DISTRIBUTIONS

The representations (2.22), (2.23) involve the triple $(Y^+(t), Y^-(t), L^Y(t))$ as well as the independent random variable $Q(t)$, where $Y(\cdot)$ is the diffusion process in (2.3), $L^Y(\cdot)$ the local time of this process at the origin as in (1.6), and $Q(\cdot)$ an independent, standard Brownian motion. Thus, in order to compute the joint distribution of $(X_1(t), X_2(t))$ via (2.22), (2.23), we need first to find that of $(Y^+(t), Y^-(t), L^Y(t))$.

In order to do this, we consider the “reference probability measure” \mathbb{P}_* , under which the process $Y(\cdot)$ becomes standard Brownian motion. According to (2.3) and the GIRSANOV theorem (e.g., KARATZAS & SHREVE (1991), section 3.5), we have for every $t \in [0, \infty)$ the RADON-NIKODÝM derivative

$$\begin{aligned} \frac{d\mathbb{P}}{d\mathbb{P}_*} \Big|_{\mathfrak{F}^{Y(t)}} &= \exp \left\{ -\lambda \int_0^t \operatorname{sgn}(Y(s)) dY(s) - \frac{\lambda^2}{2} \int_0^t \operatorname{sgn}^2(Y(s)) ds \right\} \\ &= \exp \left\{ \lambda \left(|y| - |Y(t)| + 2L^Y(t) \right) - \frac{\lambda^2}{2} t \right\}, \end{aligned} \quad (6.1)$$

thanks to the TANAKA formulas once again. Thus, for any Borel subsets A, B of $[0, \infty)$ and Θ of \mathbb{R} , and with \mathbb{E}_* denoting expectation with respect to the reference probability measure, we have

$$\begin{aligned} \mathbb{P}(Y^\pm(t) \in A, Y^\mp(t) = 0, 2L^Y(t) \in B, Q(t) \in \Theta) &= \exp \left\{ \lambda |y| - \frac{\lambda^2}{2} t \right\} \times \\ \mathbb{E}_* \left[\exp \left\{ \lambda (2L^Y(t) - Y^\pm(t)) \right\} \mathbf{1}_{\{Y^\pm(t) \in A, Y^\mp(t) = 0, 2L^Y(t) \in B\}} \right] &\int_{\Theta} \frac{\exp \left\{ -\frac{\vartheta^2}{2t} \right\}}{\sqrt{2\pi t}} d\vartheta. \end{aligned} \quad (6.2)$$

Remark 6.1. We have the classical result $\mathbb{P}_*(\operatorname{Leb}\{0 \leq t \leq T : Y(t) = 0\} > 0) = 0$ for the Lebesgue measure of the Brownian zero-set, thus $\mathbb{P}(\operatorname{Leb}\{0 \leq t \leq T : Y(t) = 0\} > 0) = 0$ holds from (6.1) for every $T \in (0, \infty)$. We deduce $\mathbb{P}(\operatorname{Leb}\{0 \leq t < \infty : X_1(t) = X_2(t)\} = 0) = 1$.

Let us also recall that the transition probability density function $\mathbb{P}(Y(t) \in d\xi | Y(0) = y) = p_t(y, \xi) d\xi$ for the one-dimensional “bang-bang” diffusion process $Y(\cdot)$ of (2.3) is given by

$$p_t(y, \xi) = \frac{1}{\sqrt{2\pi t}} \left(\exp \left\{ -\frac{(y - \xi - \lambda t)^2}{2t} \right\} + \lambda e^{-2\lambda \xi} \int_{y+\xi}^{\infty} e^{-(u-\lambda t)^2/(2t)} du \right) \quad (6.3)$$

for $\xi > 0$, and by

$$p_t(y, \xi) = \frac{1}{\sqrt{2\pi t}} \left(\exp \left\{ 2\lambda y - \frac{(y - \xi + \lambda t)^2}{2t} \right\} + \lambda e^{2\lambda \xi} \int_{y-\xi}^{\infty} e^{-(u-\lambda t)^2/(2t)} du \right) \quad (6.4)$$

for $\xi \leq 0$. In particular, with $y = 0$ the function

$$\begin{aligned} \xi \mapsto p_t(\xi) \equiv p_t(0, \xi) &= \\ \frac{1}{\sqrt{2\pi t}} \left(\exp \left\{ -\frac{(|\xi| + \lambda t)^2}{2t} \right\} + \lambda e^{-2\lambda |\xi|} \int_{|\xi|}^{\infty} e^{-(u-\lambda t)^2/(2t)} du \right) \end{aligned} \quad (6.5)$$

is evenly symmetric about the origin. Similar formulas hold for $y < 0$; for the details of these computations, see KARATZAS & SHREVE (1984).

6.1 THE ISOTROPIC CASE WITH $y = x_1 - x_2 \geq 0$

This equal variance case $\rho = \sigma = 1/\sqrt{2}$ affords the most straightforward computation: from the representations of (2.25) and the independence of $Y(t)$ and $Q(t)$, we obtain

$$\mathbb{P}(X_1(t) \in d\xi_1, X_2(t) \in d\xi_2) = \frac{\mathfrak{p}_t(y, \xi_1 - \xi_2)}{2\sqrt{2\pi t}} \exp\left\{-\frac{(\xi_1 + \xi_2 - z - vt)^2}{2t}\right\} d\xi_1 d\xi_2 \quad (6.6)$$

for $(\xi_1, \xi_2) \in \mathbb{R}^2$ in the notation of (2.1), (6.3) and (6.5). The resulting transition probability density function

$$\mathfrak{P}_t(\xi_1, \xi_2) = \frac{\mathfrak{p}_t(y, \xi_1 - \xi_2)}{2\sqrt{2\pi t}} \exp\left\{-\frac{(\xi_1 + \xi_2 - z - vt)^2}{2t}\right\}$$

is continuous and strictly positive on all of \mathbb{R}^2 , and of class \mathcal{C}^∞ on $\mathbb{R}^2 \setminus \{(\xi_1, \xi_2) : \xi_1 = \xi_2\}$.

6.2 THE DEGENERATE CASE WITH $y = x_1 - x_2 \geq 0$

Let us focus now on the degenerate case with $\sigma = 0$, thus $\rho = 1$, and with $x_1 \geq x_2$ as in Figure 1. From formulae (6.5.9)-(6.5.11), page 440 in KARATZAS & SHREVE (1991), we have the joint probability distribution computations

$$\mathbb{P}_*(Y(t) \in da, 2L^Y(t) \in db) = \frac{(|a| + b + y)}{\sqrt{2\pi t^3}} \cdot \exp\left\{-\frac{(|a| + b + y)^2}{2t}\right\} da db, \quad a \in \mathbb{R}, b > 0 \quad (6.7)$$

as well as

$$\mathbb{P}_*(Y(t) \in da, 2L^Y(t) = 0) = \frac{1}{\sqrt{2\pi t}} \left(\exp\left\{-\frac{(a-y)^2}{2t}\right\} - \exp\left\{-\frac{(a+y)^2}{2t}\right\} \right) da, \quad a > 0, \quad (6.8)$$

which are based on the theory of the so-called ‘‘elastic Brownian motion’’. Substituting into (6.2) with $\Theta = \mathbb{R}$ we obtain from these expressions

$$\begin{aligned} & \mathbb{P}[Y^+(t) \in da, Y^-(t) = 0, 2L^Y(t) \in db] = \\ & \mathbb{P}[Y^-(t) \in da, Y^+(t) = 0, 2L^Y(t) \in db] = \\ & = \exp\left\{\lambda(y+b-a) - \frac{\lambda^2}{2}t\right\} \cdot \frac{(a+b+y)}{\sqrt{2\pi t^3}} \cdot \exp\left\{-\frac{(a+b+y)^2}{2t}\right\} da db \\ & = e^{-2\lambda a} \cdot \frac{(a+b+y)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(a+b+y-\lambda t)^2}{2t}\right\} da db, \quad a > 0, b > 0. \quad (6.9) \end{aligned}$$

In a similar fashion, we obtain

$$\begin{aligned} & \mathbb{P}(Y^\pm(t) \in da, Y^\mp(t) = 0, 2L^Y(t) = 0) = \\ & = \exp\left\{\lambda(y-a) - \frac{\lambda^2}{2}t\right\} \cdot \frac{1}{\sqrt{2\pi t}} \left(\exp\left\{-\frac{(a-y)^2}{2t}\right\} - \exp\left\{-\frac{(a+y)^2}{2t}\right\}\right) da \\ & = \frac{1}{\sqrt{2\pi t}} \left(\exp\left\{-\frac{(a-y+\lambda t)^2}{2t}\right\} - e^{-2\lambda a} \exp\left\{-\frac{(a+y-\lambda t)^2}{2t}\right\}\right) da \quad (6.10) \end{aligned}$$

for $a > 0$. This expression vanishes, as it should, for $y = 0$: in this case the process $Y(\cdot)$ starts accumulating local time at the origin straightaway, that is, $\mathbb{P}(L^Y(t) > 0) = 1$ holds for every $t \in (0, \infty)$.

• We set out to compute the joint distribution of the random vector $(X_1(t), X_2(t))$ for given, fixed $t > 0$. From Remark 6.1 and either (2.26) or (4.8), it is clear that this distribution is supported in the planar region $\mathfrak{B}_1 \cup \mathfrak{B}_2$, the union of the two blunt (135°-) wedges

$$\mathfrak{B}_1 := \{(\xi_1, \xi_2) : \xi_1 > \xi_2, \xi_2 \leq x_2 + gt\}, \quad \mathfrak{B}_2 := \{(\xi_1, \xi_2) : \xi_1 < \xi_2, \xi_1 < x_2 + gt\}.$$

• Let us consider the wedge \mathfrak{B}_1 first. With given real numbers ξ_1, ξ_2 that satisfy $\xi_1 > \xi_2, \xi_2 \leq x_2 + gt$, setting

$$\mathfrak{C} := \{(a, b) \in (0, \infty)^2 : x_2 + gt + a - (b/2) \geq \xi_1, x_2 + gt - (b/2) \leq \xi_2\},$$

and with the help of the expressions (2.26) and (6.9), we obtain

$$\begin{aligned} & \mathbb{P}(X_1(t) \geq \xi_1, X_2(t) \leq \xi_2) = \\ & \int_{\mathfrak{C}} \int_{\mathfrak{C}} e^{-2\lambda a} \cdot \frac{(a+b+y)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(a+b+y-\lambda t)^2}{2t}\right\} da db = \\ & \int_{2(x_2+gt-\xi_2)}^{\infty} \left(\int_{\frac{b}{2}+\xi_1-x_2-gt}^{\infty} e^{-2\lambda a} \cdot \frac{(a+b+y)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(a+b+y-\lambda t)^2}{2t}\right\} da\right) db. \end{aligned}$$

We differentiate this expression, first with respect to ξ_2 , then with respect to ξ_1 ; recalling the notation of (2.1), we obtain

$$\begin{aligned} & \mathbb{P}(X_1(t) \in d\xi_1, X_2(t) \in d\xi_2) = \\ & 2e^{-2\lambda(\xi_1-\xi_2)} \cdot \frac{\xi_1 - 3\xi_2 + z + 2gt}{\sqrt{2\pi t^3}} \cdot \exp\left\{-\frac{1}{2t}(\xi_1 - 3\xi_2 + z + vt)^2\right\} d\xi_1 d\xi_2; \\ & \xi_1 > \xi_2, \quad \xi_2 < x_2 + gt. \quad (6.11) \end{aligned}$$

On the other hand, with $\xi_1 > \xi_2 = x_2 + gt$ there is no accumulation of local time at the origin over the interval $[0, t]$, so the expressions of (2.26) and (6.10) give

$$\begin{aligned} & \mathbb{P}(X_1(t) \in d\xi_1, X_2(t) = \xi_2) = \frac{1}{\sqrt{2\pi t}} \cdot \left(\exp\left\{-\frac{(\xi_1 - x_1 + ht)^2}{2t}\right\}\right. \\ & \left. - e^{-2\lambda(\xi_1-\xi_2)} \cdot \exp\left\{-\frac{(\xi_1 - 2\xi_2 + x_1 - ht)^2}{2t}\right\}\right) d\xi_1; \quad \xi_1 > \xi_2 = x_2 + gt. \end{aligned} \quad (6.12)$$

• Next, we consider the wedge \mathfrak{B}_2 ; equivalently, we work on the event $\{X_1(t) < X_2(t)\}$, on which the initial order $x_1 \geq x_2$ stands reversed at time t and $L^Y(t) > 0$ holds a.e. In particular, the joint distribution of the random vector $(X_1(t), X_2(t))$ assigns zero mass to the region $\{(\xi_1, \xi_2) : \xi_2 > \xi_1 = x_2 + gt\}$, as we have already observed.

With given real numbers ξ_1, ξ_2 that satisfy $\xi_1 < \xi_2, \xi_1 < x_2 + gt$, denoting

$$\mathfrak{D} := \left\{ (a, b) \in (0, \infty)^2 : x_2 + gt - (b/2) \leq \xi_1, x_2 + gt + a - (b/2) \geq \xi_2 \right\},$$

and with the help of the expressions (2.26) and (6.9), we obtain then

$$\begin{aligned} \mathbb{P}(X_1(t) \leq \xi_1, X_2(t) \geq \xi_2) &= \\ &= \int \int_{\mathfrak{D}} e^{-2\lambda a} \cdot \frac{(a+b+y)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(a+b+y-\lambda t)^2}{2t}\right\} da db = \\ &= \int_{2(x_2+gt-\xi_1)}^{\infty} \left(\int_{\frac{b}{2}+\xi_2-x_2-gt}^{\infty} e^{-2\lambda a} \cdot \frac{(a+b+y)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(a+b+y-\lambda t)^2}{2t}\right\} da \right) db. \end{aligned}$$

Differentiating this expression, first with respect to ξ_1 and then with respect to ξ_2 , we obtain the following expression for the probability density function:

$$\begin{aligned} \mathbb{P}(X_1(t) \in d\xi_1, X_2(t) \in d\xi_2) &= \\ 2e^{-2\lambda(\xi_2-\xi_1)} \cdot \frac{\xi_2 - 3\xi_1 + z + 2gt}{\sqrt{2\pi t^3}} \cdot \exp\left\{-\frac{1}{2t}(\xi_2 - 3\xi_1 + z + vt)^2\right\} d\xi_1 d\xi_2; \\ &\quad \xi_1 < \xi_2, \quad \xi_1 < x_2 + gt. \quad (6.13) \end{aligned}$$

This is the same as the expression on the right-hand side of (6.11), except ξ_1, ξ_2 have now traded places.

The following Figure plots the joint Probability Density Function of $(X_1(t), X_2(t))$.

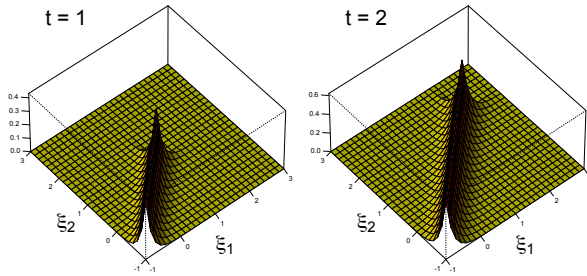


Fig. 2 Joint density: $g = h = 1, \sigma = 0, \rho = 1, x_1 = x_2 = 0, t = 1$ (left), $t = 2$ (right).

Remark 6.2. The joint distribution for the ranks $R_1(t) = \max(X_1(\cdot), X_2(t))$, $R_2(t) = \min(X_1(t), X_2(t))$ is supported by the planar region $\{(\rho_1, \rho_2) : \rho_1 > \rho_2, \rho_2 \leq x_2 + gt\}$; from (6.11)-(6.13), or directly from $R_2(t) = r_2 + gt - L^Y(t)$ and $R_1(t) = r_2 + gt + |Y(t)| - L^Y(t)$ and (6.7)-(6.8), it is computed as

$$\begin{aligned} & \mathbb{P}(R_1(t) \in d\rho_1, R_2(t) \in d\rho_2) = \\ & \frac{4(2gt + z + \rho_1 - 3\rho_2)}{\sqrt{2\pi t^3}} \cdot \exp\left\{-2\lambda(\rho_1 - \rho_2) - \frac{1}{2t}(z + vt + \rho_1 - 3\rho_2)^2\right\} d\rho_1 d\rho_2; \\ & \rho_2 < x_2 + gt, \rho_1 > \rho_2 \quad (6.14) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}(R_1(t) \in d\rho_1, R_2(t) = \rho_2) = \\ & \frac{1}{\sqrt{2\pi t}} \cdot \left(\exp\left\{-\frac{(\rho_1 - x_2 - y + ht)^2}{2t}\right\} - e^{-2\lambda(\rho_1 - \rho_2)} \cdot \exp\left\{-\frac{(\rho_1 - 2\rho_2 + x_1 - ht)^2}{2t}\right\} \right) d\xi_1; \\ & \rho_1 > \rho_2 = x_2 + gt. \quad (6.15) \end{aligned}$$

Remark 6.3. Once again, the probabilities in (6.12) and (6.15) vanish for $y = 0$, i.e., when the two particles start at the same point. In this case the distribution of $(X_1(t), X_2(t))$ is absolutely continuous with respect to Lebesgue measure on the plane, with probability density function $\mathfrak{P}_t(\xi_1, \xi_2)$ given by (6.11) on $\xi_1 > \xi_2$, $\xi_2 < x_2 + gt$ (in the wedge \mathfrak{B}_1), and by (6.13) on $\xi_1 < \xi_2$, $\xi_1 < x_2 + gt$ (in the wedge \mathfrak{B}_2).

Even in this case, though, there is a discontinuity along the front $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \wedge \xi_2 = x + gt\}$ (cf. Figure 2), of size

$$\frac{2|\xi_1 - \xi_2|}{\sqrt{2\pi t^3}} \exp\left\{-2\lambda|\xi_1 - \xi_2| - \frac{1}{2t}(|\xi_1 - \xi_2| - \lambda t)^2\right\}.$$

6.3 THE NON-DEGENERATE CASE WITH UNEQUAL VARIANCES

More generally, and by virtue of (4.2), (6.9) and (6.10), the quadrivariate joint distribution of $(Y^+(t), Y^-(t), L^Y(t), Q(t))$ is given as

$$\mathbb{P}[Y^\pm(t) \in da, Y^\mp(t) = 0, 2L^Y(t) \in db, Q(t) \in d\vartheta] = f_1(a, b, \vartheta) da db d\vartheta \quad (6.16)$$

for $a > 0$, $b > 0$, $\vartheta \in \mathbb{R}$, and

$$\mathbb{P}[Y^\pm(t) \in da, Y^\mp(t) = 0, 2L^Y(t) = 0, Q(t) \in d\vartheta] = f_2(a, \vartheta) da d\vartheta \quad (6.17)$$

for $a > 0$, $\vartheta \in \mathbb{R}$, with

$$f_1(a, b, \vartheta) = e^{-2\lambda a} \cdot \frac{(a+b+y)}{2\pi t^2} \exp\left\{-\frac{\vartheta^2 + (a+b+y - \lambda t)^2}{2t}\right\}, \quad (6.18)$$

$$f_2(a, \vartheta) = \frac{e^{-\vartheta^2/(2t)}}{2\pi t} \left(\exp \left\{ -\frac{(a-y+\lambda t)^2}{2t} \right\} - \exp \left\{ -2\lambda a - \frac{(a+y-\lambda t)^2}{2t} \right\} \right). \quad (6.19)$$

In conjunction with the skew representations of (2.22) and (2.23), now written in the form

$$X_1(t) = x_1 + \mu t + \Psi_1(t), \quad X_2(t) = x_2 + \mu t + \Psi_2(t)$$

with

$$\Psi_1(t) := \rho^2 (Y^+(t) - y^+) - \sigma^2 (Y^-(t) - y^-) - \gamma L^Y(t) + \rho \sigma Q(t), \quad (6.20)$$

$$\Psi_2(t) := -\sigma^2 (Y^+(t) - y^+) + \rho^2 (Y^-(t) - y^-) - \gamma L^Y(t) + \rho \sigma Q(t), \quad (6.21)$$

it clearly suffices to compute the joint distribution of $(\Psi_1(t), \Psi_2(t))$. This is facilitated by the observation that the system of (6.20), (6.21) can be “inverted”, in the sense

$$Y^+(t) - y^+ = \frac{1}{\gamma} \left(\rho^2 \Psi_1(t) + \sigma^2 \Psi_2(t) \right) + L^Y(t) - \frac{\rho \sigma}{\gamma} Q(t),$$

$$Y^-(t) - y^- = \frac{1}{\gamma} \left(\sigma^2 \Psi_1(t) + \rho^2 \Psi_2(t) \right) + L^Y(t) - \frac{\rho \sigma}{\gamma} Q(t).$$

To proceed further, the cases $y \geq 0$, $y < 0$ and $\gamma > 0$, $\gamma < 0$ have to be considered separately. We shall discuss briefly only the case $y = x_1 - x_2 \geq 0$, $\gamma > 0$ (i.e., $\rho > \sigma > 0$) In this case, the joint probability density function

$$\mathbb{P}(\Psi_1(t) \in d\psi_1, \Psi_2(t) \in d\psi_2) = \mathfrak{P}_t(\psi_1, \psi_2) d\psi_1 d\psi_2$$

of $(\Psi_1(t), \Psi_2(t))$ in (6.20), (6.21) is given by

$$\begin{aligned} \gamma \mathfrak{P}_t(\psi_1, \psi_2) &= \\ &= \int_{-\infty}^{\vartheta(\psi_1, \psi_2)} \left\{ \int_0^\infty f_1(b - \mathfrak{b}^*(\psi_1, \psi_2, \vartheta)) db + f_2(-\mathfrak{b}^*(\psi_1, \psi_2, \vartheta)) \right\} d\vartheta \\ &\quad + \int_{\vartheta(\psi_1, \psi_2)}^\infty \left(\int_{\mathfrak{b}^*(\psi_1, \psi_2, \vartheta)}^\infty f_1(b - \mathfrak{b}^*(\psi_1, \psi_2, \vartheta)) db \right) d\vartheta, \quad (6.22) \end{aligned}$$

where we have set

$$\vartheta(\psi_1, \psi_2) := \frac{\psi_1 + \psi_2 + \gamma y}{2\rho\sigma}, \quad \mathfrak{b}^*(\psi_1, \psi_2, \vartheta) := \frac{2\rho\sigma}{\gamma} (\vartheta - \vartheta(\psi_1, \psi_2)).$$

After some calculations, (6.22) reduces to

$$\begin{aligned} \mathfrak{P}_t(\psi_1, \psi_2) &= \frac{1}{\sqrt{2\pi t}} \left[(1 + \Phi(A_+(\psi_1, \psi_2))) \exp \left\{ -\frac{1}{2t} (\psi_1 + \psi_2 + \lambda \gamma t)^2 \right\} \right. \\ &\quad \left. - e^{2\lambda \gamma^2 y} \Phi(A_-(\psi_1, \psi_2)) \exp \left\{ -\frac{1}{2t} (\psi_1 + \psi_2 + 2\gamma y + \lambda \gamma t)^2 \right\} \right], \quad (6.23) \end{aligned}$$

where $\Phi(\cdot) := (1/\sqrt{2\pi}) \int_{-\infty}^{\cdot} e^{-z^2/2} dz$ is the cumulative standard normal distribution function and

$$\sqrt{t} A_\pm(\psi_1, \psi_2) := \frac{\gamma}{2\rho\sigma} (\psi_1 + \psi_2 + \gamma y) \pm 2\rho\sigma (y \mp \lambda t), \quad (\psi_1, \psi_2) \in \mathbb{R}^2.$$

7 TIME REVERSAL

Let us consider now the time-reversed versions (“free” and “anchored”, respectively)

$$\widehat{Y}(t) := Y(T-t), \quad \widetilde{W}(t) := W(T-t) - W(T), \quad 0 \leq t \leq T, \quad (7.1)$$

over a given time horizon $[0, T]$ of finite length, of the diffusion process $Y(\cdot)$ in (2.3) and of the Brownian motion $W(\cdot)$ which drives that equation. Both of the processes introduced in (7.1) are adapted to the *backwards filtration* $\widehat{\mathbb{F}} = \{\widehat{\mathcal{F}}(t)\}_{0 \leq t \leq T}$ defined as

$$\widehat{\mathcal{F}}(t) := \sigma(Y(T)) \vee \mathfrak{F}^{\widetilde{W}}(t), \quad \mathfrak{F}^{\widetilde{W}}(t) := \sigma(\widetilde{W}(\theta); 0 \leq \theta \leq t). \quad (7.2)$$

Note that the process $\widetilde{W}(\cdot)$ is Brownian motion with respect to the filtration $\mathbf{F}^{\widetilde{W}} = \{\mathfrak{F}^{\widetilde{W}}(t)\}_{0 \leq t \leq T}$; this is because $\widetilde{W}(\cdot)$ has continuous paths, and is easily checked to be an $\mathbf{F}^{\widetilde{W}}$ -martingale with the same quadratic variation as Brownian motion. However, $\widetilde{W}(\cdot)$ is only a semimartingale with respect to the larger backwards filtration $\widehat{\mathbb{F}}$; its semimartingale decomposition is provided by the fact that *the process*

$$W^\#(t) := \widetilde{W}(t) - \int_0^t \mathfrak{q}(T-s, \widehat{Y}(s)) ds = W(T-t) - W(T) - \int_{T-t}^T \mathfrak{q}(s, Y(s)) ds \quad (7.3)$$

for $0 \leq t \leq T$, is a standard $\widehat{\mathbb{F}}$ -Brownian motion. Here

$$\mathfrak{q}(\tau, \xi) := \frac{\partial}{\partial \xi} \log p_\tau(y, \xi), \quad (\tau, \xi) \in (0, \infty) \times \mathbb{R} \quad (7.4)$$

is the logarithmic derivative of the transition probability density function in (6.3), (6.4).

Remark 7.1. This result is proved as in MEYER (1994), who notes also the following corollary: the (once-more-time-reversed) process

$$\begin{aligned} \boldsymbol{\eta}(t) &:= W^\#(T-t) - W^\#(T) = Y(t) + \int_0^t \left(\mathfrak{q}(s, Y(s)) + \lambda \operatorname{sgn}(Y(s)) \right) ds \\ &= W(t) + \int_0^t \mathfrak{q}(s, Y(s)) ds = \int_0^t \left(\operatorname{sgn}(Y(s)) dV^b(s) + \mathfrak{q}(s, Y(s)) ds \right), \quad 0 \leq t \leq T \end{aligned} \quad (7.5)$$

is also Brownian motion *with respect to its own filtration*, and is adapted to the filtration $\mathbf{F}^Y \equiv \mathbf{F}^W$ (though not to the filtration \mathbf{F}^{V^b} ; recall the filtration identity (4.4) in this regard).

The process $\boldsymbol{\eta}(\cdot)$ of (7.5) is independent of the random variable $Y(T)$. In fact, for every given $t \in [0, T)$, the σ -algebra $\mathfrak{F}^{\boldsymbol{\eta}}(t) = \sigma(\boldsymbol{\eta}(s), 0 \leq s \leq t)$ is independent of the σ -algebra $\sigma(Y(\theta), t \leq \theta \leq T)$; in particular, of the random variable $Y(t)$. This shows that the inclusion $\mathfrak{F}^{\boldsymbol{\eta}}(t) \subset \mathfrak{F}^Y(t)$, quite obvious from (7.5), is strict; or, put another way, that the stochastic integral equation

$$Y(t) = y - \int_0^t \left(\lambda \operatorname{sgn}(Y(s)) + \mathfrak{q}(s, Y(s)) \right) ds + \boldsymbol{\eta}(t), \quad 0 \leq t \leq T \quad (7.6)$$

cannot possibly have a strong solution. \square

A result by [HAUSSMANN & PARDOUX \(1986\)](#) (see also the expository papers by [PARDOUX \(1986\)](#) and [MEYER \(1994\)](#), among many other works on this subject) is that, with respect to the backwards filtration $\widehat{\mathbb{F}} = \{\widehat{\mathcal{F}}(t)\}_{0 \leq t \leq T}$, the time-reversed process $\widehat{Y}(\cdot)$ in (7.1) is not just a semimartingale but also a *diffusion process* driven by the $\widehat{\mathbb{F}}$ -Brownian motion in (7.3), of the form

$$\widehat{Y}(t) = \widehat{Y}(0) + \int_0^t \widehat{\mathbf{b}}(T-s, \widehat{Y}(s)) ds + W^\#(t), \quad 0 \leq t \leq T. \quad (7.7)$$

Here, the new (backward) drift function $\widehat{\mathbf{b}}(\cdot, \cdot)$ is given in the notation of (7.4) by the *generalized NELSON equation*

$$\widehat{\mathbf{b}}(\tau, \xi) = \lambda \operatorname{sgn}(\xi) + \mathfrak{q}(\tau, \xi), \quad (\tau, \xi) \in (0, T] \times \mathbb{R}. \quad (7.8)$$

Remark 7.2. In the special case $y = 0$, the function of (7.4) takes the form

$$\begin{aligned} \mathfrak{q}(\tau, \xi) &= \left(\varphi^{(\lambda)}(\tau, -\xi) + \lambda e^{2\lambda\xi} \int_{-\xi}^{\infty} \varphi^{(\lambda)}(\tau, -u) du \right)^{-1} \\ &\cdot \left(\left(2\lambda - \frac{\xi}{\tau} \right) \varphi^{(\lambda)}(\tau, -\xi) + 2\lambda^2 e^{2\lambda\xi} \int_{-\xi}^{\infty} \varphi^{(\lambda)}(\tau, -u) du \right), \quad \xi \leq 0, \end{aligned} \quad (7.9)$$

and $\mathfrak{q}(\tau, \xi) = -\mathfrak{q}(\tau, -\xi)$ for $\xi > 0$, $\tau > 0$ with

$$\varphi^{(\lambda)}(\tau, \xi) := \frac{1}{\sqrt{2\pi\tau}} \exp \left\{ -\frac{(\xi + \lambda\tau)^2}{2\tau} \right\}. \quad (7.10)$$

In particular, we obtain from the equations (7.9)-(7.10) the explicit expression

$$\begin{aligned} \widehat{\mathbf{b}}(\tau, \xi) &= \lambda \operatorname{sgn}(\xi) - \left(\varphi^{(\lambda)}(\tau, |\xi|) + \lambda e^{-2\lambda|\xi|} \int_{|\xi|}^{\infty} \varphi^{(\lambda)}(\tau, -u) du \right)^{-1} \\ &\cdot \left(\left(2\lambda + \frac{|\xi|}{\tau} \right) \varphi^{(\lambda)}(\tau, |\xi|) + 2\lambda^2 e^{-2\lambda|\xi|} \int_{|\xi|}^{\infty} \varphi^{(\lambda)}(\tau, -u) du \right), \quad \xi \in \mathbb{R}. \end{aligned} \quad (7.11)$$

The singularity at $\tau = 0$ of the drift in (7.11) is of the “bridge” type: it ensures that, as the time-to-go $\tau \downarrow 0$ decreases to zero, the backward diffusion $\widehat{Y}(\cdot)$ zooms into the prescribed terminal condition $\widehat{Y}(T) = 0$ (the initial condition $Y(0) = 0$ of the forward process), as it must. \square

Let us also note that the semimartingale local time $L^{\widehat{Y}}(\cdot)$ of the backward diffusion $\widehat{Y}(\cdot)$, and the semimartingale local time $L^Y(\cdot)$ of the forward diffusion $Y(\cdot)$, are linked via

$$L^{\widehat{Y}}(t) = L^Y(T) - L^Y(T-t), \quad 0 \leq t \leq T. \quad (7.12)$$

7.1 A TIME REVERSAL FOR NAMES

We consider now the “free” time-reversals

$$\widehat{X}_1(t) := X_1(T-t), \quad \widehat{X}_2(t) := X_2(T-t), \quad 0 \leq t \leq T \quad (7.13)$$

of the components of the vector process $(X_1(\cdot), X_2(\cdot))$ constructed in section 3, as well as their “anchored” versions

$$\widetilde{X}_j(t) := \widehat{X}_j(t) - \widehat{X}_j(0) = X_j(T-t) - X_j(T); \quad 0 \leq t \leq T, \quad j = 1, 2. \quad (7.14)$$

By analogy with (7.2), we also look at the new backwards filtration $\widetilde{\mathbf{F}} = \{\widetilde{\mathfrak{F}}(t)\}_{0 \leq t \leq T}$ given by

$$\widetilde{\mathfrak{F}}(t) := \boldsymbol{\sigma}(Y(T)) \vee \mathfrak{F}^{(\widetilde{Q}, \widetilde{W})}(t), \quad \text{with} \quad \mathfrak{F}^{(\widetilde{Q}, \widetilde{W})}(t) = \boldsymbol{\sigma}(\widetilde{Q}(\theta), \widetilde{W}(\theta); 0 \leq \theta \leq t)$$

generated by the random variable $Y(T)$ and by the time-reversed versions

$$\widetilde{Q}(t) = Q(T-t) - Q(T), \quad \widetilde{W}(t) = W(T-t) - W(T), \quad 0 \leq t \leq T \quad (7.15)$$

of the independent Brownian motions $W(\cdot)$, $Q(\cdot)$ of (2.19), (2.21) in the manner of (7.1). In particular, $\widetilde{Q}(\cdot)$ is independent of both $\widetilde{W}(\cdot)$ and $Y(T)$, thus also of the Brownian motion $W^\#(\cdot)$ in (7.3).

Then the skew representations (2.22) and (2.23), along with the notation of (7.1) and the local time identity (7.12), imply that the “anchored time-reversals” of (7.14) are $\widetilde{\mathbf{F}}$ -adapted and are given by

$$\begin{aligned} \widetilde{X}_1(t) &= -\mu t + \rho^2 (\widehat{Y}^+(t) - \widehat{Y}^+(0)) - \sigma^2 (\widehat{Y}^-(t) - \widehat{Y}^-(0)) + \gamma L^{\widehat{Y}}(t) + \rho \sigma \widetilde{Q}(t) \\ &= -\mu t + \int_0^t (\rho^2 \mathbf{1}_{\{\widehat{Y}(s) > 0\}} + \sigma^2 \mathbf{1}_{\{\widehat{Y}(s) \leq 0\}}) d\widehat{Y}(s) + 2\gamma L^{\widehat{Y}}(t) + \rho \sigma \widetilde{Q}(t), \end{aligned} \quad (7.16)$$

and

$$\begin{aligned} \widetilde{X}_2(t) &= -\mu t - \sigma^2 (\widehat{Y}^+(t) - \widehat{Y}^+(0)) + \rho^2 (\widehat{Y}^-(t) - \widehat{Y}^-(0)) + \gamma L^{\widehat{Y}}(t) + \rho \sigma \widetilde{Q}(t) \\ &= -\mu t - \int_0^t (\rho^2 \mathbf{1}_{\{\widehat{Y}(s) \leq 0\}} + \sigma^2 \mathbf{1}_{\{\widehat{Y}(s) > 0\}}) d\widehat{Y}(s) + 2\gamma L^{\widehat{Y}}(t) + \rho \sigma \widetilde{Q}(t), \end{aligned} \quad (7.17)$$

respectively, thanks once again to the TANAKA formulas.

We recall now the “backwards dynamics” of (7.7) as well as the notation of (7.4), (7.8), and write these equations in the *time-reversed skew representation* form

$$\begin{aligned} \widetilde{X}_1(t) &= \widehat{X}_1(t) - \widehat{X}_1(0) = \\ &= \int_0^t \left(h \mathbf{1}_{\{\widehat{X}_1(s) > \widehat{X}_2(s)\}} - g \mathbf{1}_{\{\widehat{X}_1(s) \leq \widehat{X}_2(s)\}} \right) ds + (\rho^2 - \sigma^2) L^{|\widehat{X}_1 - \widehat{X}_2|}(t) + \rho \sigma \widetilde{Q}(t) + \\ &\int_0^t (\rho^2 \mathbf{1}_{\{\widehat{X}_1(s) > \widehat{X}_2(s)\}} + \sigma^2 \mathbf{1}_{\{\widehat{X}_1(s) \leq \widehat{X}_2(s)\}}) \left[q(T-s, \widehat{X}_1(s) - \widehat{X}_2(s)) ds + dW^\#(s) \right] \end{aligned} \quad (7.18)$$

and

$$\begin{aligned} \tilde{X}_2(t) &= \hat{X}_2(t) - \hat{X}_2(0) = \\ &= \int_0^t \left(h \mathbf{1}_{\{\hat{X}_1(t) \leq \hat{X}_2(t)\}} - g \mathbf{1}_{\{\hat{X}_1(t) > \hat{X}_2(t)\}} \right) dt + (\rho^2 - \sigma^2) L^{|\hat{X}_1 - \hat{X}_2|}(t) + \rho \sigma \tilde{Q}(t) + \\ &\int_0^t \left(\rho^2 \mathbf{1}_{\{\hat{X}_1(s) \leq \hat{X}_2(s)\}} + \sigma^2 \mathbf{1}_{\{\hat{X}_1(s) > \hat{X}_2(s)\}} \right) [\mathfrak{q}(T-s, \hat{X}_1(s) - \hat{X}_2(t)) ds + dW^\#(s)]. \end{aligned} \quad (7.19)$$

Remark 7.3. A somewhat interesting dichotomy emerges. In the case of equal variances ($\rho^2 = \sigma^2 = 1/2$) these anchored time-reversals are $\tilde{\mathbf{F}}$ -adapted IT $\hat{\mathcal{O}}$ processes: the bounded variation terms in their semimartingale decompositions are absolutely continuous with respect to LEBESGUE measure. In the case of unequal variances $\rho^2 \neq \sigma^2$, terms which are singular with respect to LEBESGUE measure appear, and are governed by local time.

To the best of our knowledge, this is the first time such a feature is observed in the context of time-reversal of a “purely forward” stochastic differential equation without reflection; its occurrence and significance need to be understood further. For a similar but different phenomenon, in the context of time reversal of Brownian motion reflected on an independent time-reversed Brownian motion, see SOUCALIUC & WERNER (2002) (as well as SOUCALIUC et al (2000), BURDZY & NUALART (2002)).

7.2 A TIME REVERSAL FOR RANKS

By analogy with (7.13), we introduce the time-reversed versions

$$\hat{R}_1(t) := R_1(T-t) = \max(\hat{X}_1(t), \hat{X}_2(t)), \quad \hat{R}_2(t) := R_2(T-t) = \min(\hat{X}_1(t), \hat{X}_2(t)) \quad (7.20)$$

of the ranked processes in (4.1) for $0 \leq t \leq T$. We have $\hat{R}_1(t) + \hat{R}_2(t) = \hat{X}_1(t) + \hat{X}_2(t)$ and $\hat{R}_1(t) - \hat{R}_2(t) = |\hat{X}_1(t) - \hat{X}_2(t)| = |\hat{Y}(t)|$, so the time-reversed skew representations of (7.16), (7.17) cast the “anchored” versions of the processes of (7.20) in the form

$$\tilde{R}_1(t) := \hat{R}_1(t) - \hat{R}_1(0) = -\mu t + \rho^2 (|\hat{Y}(t)| - |\hat{Y}(0)|) + \gamma L^{\hat{Y}}(t) + \rho \sigma \tilde{Q}(t), \quad (7.21)$$

$$\tilde{R}_2(t) := \hat{R}_2(t) - \hat{R}_2(0) = -\mu t - \sigma^2 (|\hat{Y}(t)| - |\hat{Y}(0)|) + \gamma L^{\hat{Y}}(t) + \rho \sigma \tilde{Q}(t). \quad (7.22)$$

In conjunction with the reverse-time dynamics of (7.7), (7.8), the TANAKA-MEYER formulas give now

$$|\hat{Y}(t)| - |\hat{Y}(0)| = \lambda t + \int_0^t \operatorname{sgn}(\hat{Y}(s)) \mathfrak{q}(T-s, \hat{Y}(s)) ds + V^\#(t) + 2L^{\hat{Y}}(t), \quad (7.23)$$

where

$$V^\#(t) := \int_0^t \operatorname{sgn}(\hat{Y}(s)) dW^\#(s), \quad 0 \leq t \leq T \quad (7.24)$$

is a Brownian motion of the backwards filtration $\tilde{\mathbf{F}}$, and is independent of the Brownian motion $\tilde{Q}(\cdot)$.

Substituting the expression of (7.23) back into (7.21) and (7.22), and recalling (2.1), (2.24), we obtain the dynamics

$$\hat{R}_1(t) - \hat{R}_1(0) = ht + \rho^2 \int_0^t \operatorname{sgn}(\hat{Y}(s)) \mathfrak{q}(T-s, \hat{Y}(s)) ds + \rho V_1^\#(t) + (4\rho^2 - 1) L^{\hat{Y}}(t), \quad (7.25)$$

$$\hat{R}_2(t) - \hat{R}_2(0) = -gt - \sigma^2 \int_0^t \operatorname{sgn}(\hat{Y}(s)) \mathfrak{q}(T-s, \hat{Y}(s)) ds + \sigma V_2^\#(t) - (4\sigma^2 - 1) L^{\hat{Y}}(t). \quad (7.26)$$

Here the processes

$$V_1^\#(\cdot) := \rho V^\#(\cdot) + \sigma \tilde{Q}(\cdot), \quad V_2^\#(\cdot) := \rho \tilde{Q}(\cdot) - \sigma V^\#(\cdot) \quad (7.27)$$

are independent, standard Brownian motions of the backwards filtration $\tilde{\mathbf{F}}$.

Comparing the equations of (7.25), (7.26) with those of (4.5), (4.6), we see that $V_1^\#(\cdot)$, $V_2^\#(\cdot)$ play in the context of time reversal the same rôles that the Brownian motions $V_1(\cdot)$, $V_2(\cdot)$ play on the forward context: to wit, that of independent, standard Brownian motions associated with individual *ranks*.

7.3 STEADY STATE

Finally, let us note that the diffusion process $Y(\cdot)$ of (2.3) has invariant distribution with double exponential probability density function

$$p(\xi) = \lambda e^{-2\lambda|\xi|}, \quad \xi \in \mathbb{R}.$$

For this function, the analogue $q(\xi) = (\partial/\partial\xi) \log p(\xi)$ of the logarithmic derivative in (7.4) becomes $q(\xi) = -2\lambda \operatorname{sgn}(\xi)$ in the notation of (6.1), and from the generalized NELSON equation (7.8) the drift in the backward equation (7.7) becomes

$$\hat{b}(\xi) = -\lambda \operatorname{sgn}(\xi).$$

This reflects the fact that the diffusion process $Y(\cdot)$ of (2.3) is strictly time-reversible when started at its invariant distribution, that is, the processes $Y(\cdot)$ and $\hat{Y}(\cdot)$ are then identically distributed; and then the Brownian motion in (7.5) takes the form $\boldsymbol{\eta}(t) = \int_0^t \operatorname{sgn}(Y(s)) (dV^b(s) - 2\lambda ds)$, $0 \leq t \leq T$.

In such a setting, the equations of (7.6) and (7.18), (7.19) continue to hold, now with $q(\cdot) = -2\lambda \operatorname{sgn}(\cdot)$; whereas the equations of (7.25), (7.26) assume the rather concrete form

$$\hat{R}_1(t) - \hat{R}_1(0) = \left(h - 2(g+h)\rho^2 \right) t + \rho V_1^\#(t) + \frac{4\rho^2 - 1}{2} L^{\hat{R}_1 - \hat{R}_2}(t),$$

$$\hat{R}_2(t) - \hat{R}_2(0) = \left(2(g+h)\sigma^2 - g \right) t + \sigma V_2^\#(t) - \frac{4\sigma^2 - 1}{2} L^{\hat{R}_1 - \hat{R}_2}(t),$$

where $L^{\hat{R}_1 - \hat{R}_2}(\cdot) = 2L^{\hat{Y}}(\cdot)$ is the local time accumulated at the origin by $\hat{R}_1(\cdot) - \hat{R}_2(\cdot) = |\hat{Y}(\cdot)|$.

8 A GENERALIZATION OF THE PERTURBED TANAKA EQUATION

We have the following generalization of Theorem 1.1, which partially answers a question posed by Professor Marc YOR. In what follows, we shall agree to denote by $\langle \mathcal{L} \rangle(\cdot)$ the quadratic variation $\langle \mathcal{V} \rangle(\cdot)$ of the continuous local martingale $\mathcal{V}(\cdot)$ in the decomposition of a continuous semimartingale $\mathcal{L}(\cdot) = \mathcal{L}(0) + \mathcal{V}(\cdot) + \mathcal{B}(\cdot)$, where $\mathcal{B}(\cdot)$ is a continuous, adapted process of finite variation on compact intervals.

Theorem 8.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function of finite variation, that is, $f = f_+ - f_-$ where $f_{\pm} : \mathbb{R} \rightarrow \mathbb{R}$ are increasing. On some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathfrak{F}(t)\}_{t \geq 0}$, consider two continuous local martingales $M(\cdot), N(\cdot)$ which satisfy the conditions of Theorem 1.1, and a continuous, adapted process $A(\cdot)$ with $A(0) = 0$ and finite total variation $\dot{A}(t)$ on compact intervals of the form $[0, t]$.*

Then pathwise uniqueness holds for the stochastic differential equation

$$Y(t) = y + \int_0^t f(Y(s)) dM(s) + A(t) + N(t), \quad 0 \leq t < \infty. \quad (8.1)$$

Proof. We shall prove the statement for $f = f_+ - f_-$ with bounded, increasing f_{\pm} . The general case will then follow by considering $f^{(n)} = f_+^{(n)} - f_-^{(n)}$, where $f_{\pm}^{(n)} = \min(n, \max(-n, f_{\pm}))$, $n \in \mathbb{N}$ are truncated versions of f . If pathwise uniqueness holds in the bounded case, then the solution is pathwise unique up to the stopping time $\sup \vartheta_n$, where $\vartheta_n = \inf\{t > 0 : |f_{\pm}(Y(t))| > n\}$. That is, if we have two solutions, $X(\cdot), Y(\cdot)$ to (8.1) on the same probability space, issued from the same initial value and driven by the processes $M(\cdot), N(\cdot)$ and $A(\cdot)$, then $X(t) = Y(t)$ holds for $t < \zeta(X) \wedge \zeta(Y)$, where $\zeta(X), \zeta(Y)$ denote the lifetimes of the processes $X(\cdot)$ and $Y(\cdot)$, respectively.

Suppose then that the processes $X(\cdot), Y(\cdot)$ both solve the equation (8.1), so their difference

$$D(t) := X(t) - Y(t) = \int_0^t (f(X(s)) - f(Y(s))) dM(s), \quad 0 \leq t < \infty \quad (8.2)$$

is a continuous local martingale with quadratic variation

$$\langle D \rangle(t) = \int_0^t (f(X(s)) - f(Y(s)))^2 d\langle M \rangle(s) \leq \|f\|_{TV}^2 \int_0^t |f(X(s)) - f(Y(s))| d\langle M \rangle(s). \quad (8.3)$$

Here $\|f\|_{TV}$ is the total variation of f over the real line, and we have used the elementary comparison $(f(x) - f(y))^2 \leq \|f\|_{TV} |f(x) - f(y)|$, $\forall (x, y) \in \mathbb{R}^2$. Following LE GALL (1983), we shall establish the estimate

$$\mathbb{E} \int_0^T \frac{d\langle D \rangle(t)}{D(t)} \mathbf{1}_{\{D(t) > 0\}} \leq c \cdot \|f\|_{TV}^2 \cdot \sup_{\substack{a \in \mathbb{R} \\ u \in [0, 1]}} \mathbb{E} \left(2L^{(u)}(T, a) \right) \quad (8.4)$$

where, for each $u \in [0, 1]$, the quantity $L^{(u)}(T, a) \equiv \Lambda^{Z^{(u)}}(T, a)$ is the local time accumulated at the site $a \in \mathbb{R}$ during the time interval $[0, T]$ by the continuous semi-

martingale

$$\begin{aligned} Z^{(u)}(\cdot) &:= (1-u)X(\cdot) + uY(\cdot) = \\ &y + \int_0^\cdot \left((1-u)f(X(t)) + uf(Y(t)) \right) dM(t) + A(\cdot) + N(\cdot). \end{aligned} \quad (8.5)$$

To this end, we introduce a sequence $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{C}^1(\mathbb{R})$ of continuous and continuously differentiable functions that converge to f pointwise, and are bounded in total variation norm, that is $\sup_{k \in \mathbb{N}} \|f_k\|_{TV} < \infty$. Since $\limsup_k \|f_k\|_{TV} \leq \|f\|_{TV}$ obviously holds, it is only possible if f is of bounded variation and in this case an approximating sequence is easily obtained, e.g., by mollifiers. We note the identity $f_k(X(t)) - f_k(Y(t)) = (X(t) - Y(t)) \int_0^1 f'_k(Z^{(u)}(t)) du$, as well as the comparison

$$\begin{aligned} \mathbb{E} \int_0^T \frac{|f_k(X(t)) - f_k(Y(t))|}{X(t) - Y(t)} \mathbf{1}_{\{X(t) - Y(t) > \delta\}} d\langle M \rangle(t) \\ \leq \int_0^1 \left(\mathbb{E} \int_0^T |f'_k(Z^{(u)}(t))| \mathbf{1}_{\{X(t) - Y(t) > \delta\}} d\langle M \rangle(t) \right) du \end{aligned}$$

for $\delta > 0$. We also note that the orthogonality of $M(\cdot)$ and $N(\cdot)$ implies $\langle Z^{(u)} \rangle(\cdot) \geq \langle N \rangle(\cdot)$ in conjunction with (8.5). Let us recall now the domination condition (1.11), which gives

$$\langle M \rangle(\cdot) \leq c \langle N \rangle(\cdot) \leq c \langle Z^{(u)} \rangle(\cdot) \leq C \langle N \rangle(\cdot)$$

for a suitable real constant $C > c$. We deduce from this

$$\begin{aligned} &\int_0^1 \left(\mathbb{E} \int_0^T |f'_k(Z^{(u)}(t))| \mathbf{1}_{\{D(t) > \delta\}} d\langle M \rangle(t) \right) du \\ &\leq c \int_0^1 \left(\mathbb{E} \int_0^T |f'_k(Z^{(u)}(t))| d\langle Z^{(u)} \rangle(t) \right) du \\ &= c \int_0^1 \left(\mathbb{E} \int_{\mathbb{R}} |f'_k(a)| 2L^{(u)}(T, a) da \right) du \\ &\leq c \cdot \sup_{\substack{a \in \mathbb{R} \\ u \in [0,1]}} \mathbb{E} \left(2L^{(u)}(T, a) \right) \cdot \int_{\mathbb{R}} |f'_k(a)| da \leq c \cdot \|f_k\|_{TV} \cdot \sup_{\substack{a \in \mathbb{R} \\ u \in [0,1]}} \mathbb{E} \left(2L^{(u)}(T, a) \right). \end{aligned}$$

Letting $k \uparrow \infty$ and then $\delta \downarrow 0$, these estimates give

$$\mathbb{E} \int_0^T \frac{|f(X(t)) - f(Y(t))|}{X(t) - Y(t)} \mathbf{1}_{\{X(t) - Y(t) > 0\}} d\langle M \rangle(t) \leq c \cdot \|f\|_{TV} \cdot \sup_{\substack{a \in \mathbb{R} \\ u \in [0,1]}} \mathbb{E} \left(2L^{(u)}(T, a) \right)$$

and, in conjunction with (8.3), the claim (8.4) as well.

Suppose now we can show

$$\sup_{\substack{a \in \mathbb{R} \\ u \in [0,1]}} \mathbb{E} \left(2L^{(u)}(T, a) \right) < \infty. \quad (8.6)$$

On the strength of (8.4), this then implies $\mathbb{E} \int_0^T (D(t))^{-1} \mathbf{1}_{\{D(t)>0\}} d\langle D \rangle(t) < \infty$ for each $T \in (0, \infty)$; and arguing as in LE GALL (1983), Lemma 1.0 (see also Exercise 3.7.12, pages 225-226 in KARATZAS & SHREVE (1991)), we deduce that the local time $L(\cdot) \equiv \Lambda^D(\cdot, 0)$, accumulated at the origin by the continuous local martingale $D(\cdot)$ of (8.2), is identically equal to zero. But then, by the TANAKA formula once again, we obtain that $|D(\cdot)|$ is a local martingale, thus also a (nonnegative) continuous supermartingale with $|D(0)| = 0$, and consequently $X(\cdot) - Y(\cdot) \equiv D(\cdot) \equiv 0$; that is, pathwise uniqueness holds.

• The property (8.6) is checked by standard methods: with the help of the TANAKA formula

$$|Z^{(u)}(T) - a| = |Z^{(u)}(0) - a| + \int_0^T \operatorname{sgn}(Z^{(u)}(t) - a) dZ^{(u)}(t) + 2L^{(u)}(T, a)$$

and the Itô isometry (e.g., KARATZAS & SHREVE (1991), p. 144) we get

$$\begin{aligned} \mathbb{E} \left(2L^{(u)}(T, a) \right) &\leq \mathbb{E} \left| Z^{(u)}(T) - Z^{(u)}(0) \right| + \mathbb{E}^{1/2} \left(\langle Z^{(u)} \rangle(T) \right) + \mathbb{E}(\check{A}(T)) \\ &\leq 2\mathbb{E}^{1/2} \left(\langle Z^{(u)} \rangle(T) \right) + 2\mathbb{E}(\check{A}(T)) \\ &\leq 2\sqrt{C} \mathbb{E}^{1/2}(\langle N \rangle(T)) + 2\mathbb{E}(\check{A}(T)). \end{aligned}$$

This last quantity does not depend on $a \in \mathbb{R}$ or $u \in [0, 1]$; if it is also finite, we are done.

If not, we deploy standard localization arguments: to wit, we consider the stopping times

$$\tau_m = \inf \{ t \geq 0 : \max(\check{A}(t), \langle N \rangle(t)) \geq m \},$$

and deduce $D(\cdot \wedge \tau_m) \equiv 0$ from the above analysis, for every $m \in \mathbb{N}$, \mathbb{P} -a.s. But then $\lim_{m \rightarrow \infty} \tau_m = \infty$ also holds \mathbb{P} -a.s., and this leads to $D(\cdot) \equiv 0$ once again. \square

Our next result covers cases discussed in Theorems 5.1 and 5.2. More importantly, it generalizes Theorem 8.1, by replacing in the equation (1.10) (respectively, in the equation (8.1)) both driving local martingales $M(\cdot)$ and $N(\cdot)$ by semimartingales. This generalization can be construed as an analogue of the results in ZVONKIN (1974) and VERETENNIKOV (1979). With bounded, measurable $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$, these authors show (for $m = 1$ and for general $m \in \mathbb{N}$, respectively) that, even in situations in which the ordinary differential equation $dY(t) = f(Y(t)) dt$ might not be solvable, the addition of a Brownian perturbation $W(\cdot)$ as in

$$dY(t) = f(Y(t)) dt + dW(t)$$

restores to the differential equation a pathwise unique, strong solution.

Proposition 8.1. *With the same assumptions and notation as in Theorem 8.1, and with $\Gamma(\cdot)$ a continuous, adapted process of finite first variation on compact intervals, pathwise uniqueness holds for the stochastic differential equation*

$$Y(t) = y + \int_0^t f(Y(s)) d(M(s) + \Gamma(s)) + A(t) + N(t), \quad 0 \leq t < \infty, \quad (8.7)$$

provided that either

- (i) the process $\Gamma(\cdot)$ is increasing, and the function $f(\cdot)$ is decreasing; or
(ii) the process $\Gamma(\cdot)$ is decreasing, and the function $f(\cdot)$ is increasing; or
(iii) the process $\Gamma(\cdot)$ is of the form

$$\Gamma(\cdot) = \int_0^\cdot \varphi(t) d\langle M \rangle(t)$$

for some progressively measurable process $\varphi(\cdot)$ which satisfies the integrability condition $\int_0^T |\varphi(t)|^2 d\langle M \rangle(t) < \infty$, $\forall T \in (0, \infty)$.

Proof. We show exactly as before that the continuous semimartingale $D(\cdot) = X(\cdot) - Y(\cdot)$ accumulates zero local time $L(\cdot) \equiv \Lambda^D(\cdot, 0) \equiv 0$ at the origin, so we have

$$|D(\cdot)| = \int_0^\cdot \eta(t) (dM(t) + d\Gamma(t)), \quad (8.8)$$

with

$$\eta(t) := \operatorname{sgn}(X(t) - Y(t)) (f(X(t)) - f(Y(t))).$$

Under either of the conditions (i) or (ii), the process $|D(\cdot)|$ is now a continuous local supermartingale thanks to the representation (8.8), thus a true supermartingale (by FATOU's lemma) since it is nonnegative. As we have $D(0) = 0$, we conclude $D(\cdot) \equiv 0$ just as before.

Under the condition (iii), we note first that the process $\varphi(\cdot)$ satisfies also the integrability condition $\int_0^T |\varphi(t)| d\langle M \rangle(t) < \infty$ for every $T \in (0, \infty)$, by the KUNITA-WATANABE inequality; cf. KARATZAS & SHREVE (1991), p. 142. We introduce the continuous local martingale

$$K(\cdot) := \int_0^\cdot \varphi(t) dM(t), \quad \text{note} \quad \Gamma(\cdot) = \langle M, K \rangle(\cdot), \quad (8.9)$$

and summon the “stochastic exponential”

$$\mathfrak{Z}(\cdot) := \exp \left\{ -K(\cdot) - \frac{1}{2} \langle K \rangle(\cdot) \right\},$$

the unique solution of the “simplest stochastic integral equation” (in the terminology of MCKEAN (1969)) $\mathfrak{Z}(\cdot) = 1 - \int_0^\cdot \mathfrak{Z}(t) dK(t)$.

We want to show $D(\cdot) \equiv 0$. Observe that $\Pi(\cdot) := \mathfrak{Z}(\cdot) |D(\cdot)|$ is a local martingale by the product rule of the stochastic calculus:

$$\begin{aligned} \Pi(\cdot) &= \int_0^\cdot \mathfrak{Z}(t) d|D(t)| + \int_0^\cdot |D(t)| d\mathfrak{Z}(t) - \int_0^\cdot \eta(t) \mathfrak{Z}(t) d\langle M, K \rangle(t) \\ &= \int_0^\cdot \mathfrak{Z}(t) \eta(t) dM(t) + \int_0^\cdot |D(t)| d\mathfrak{Z}(t). \end{aligned}$$

Here $M(\cdot)$ and $\mathfrak{Z}(\cdot)$ are continuous local martingales and the integrands are locally bounded, hence both terms on the right are continuous local martingales. Now, $\Pi(\cdot)$ is a nonnegative, continuous local martingale, starting from the origin at time zero, so it must stay at the origin at all times; and since $\mathfrak{Z}(\cdot)$ is strictly positive, this leads to the desired conclusion $D(\cdot) \equiv 0$. \square

8.1 MAXIMALITY

Let us place ourselves in the non-degenerate case $\rho\sigma > 0$ and in the constructive setup (synthesis) of section 3, where we work with the filtration $\mathbf{F}^{(W_1, W_2)}$ generated by the planar Brownian motion $(W_1(\cdot), W_2(\cdot))$.

In the terminology of [BROSSARD & LEURIDAN \(2008\)](#), we shall say that a one-dimensional Brownian motion $\beta(\cdot)$ in the filtration $\mathbf{F}^{(W_1, W_2)}$, is

- (i) *complementable* in $\mathbf{F}^{(W_1, W_2)}$, if there exists an independent one-dimensional Brownian motion $\eta(\cdot)$ in the filtration $\mathbf{F}^{(W_1, W_2)}$, and for which $\mathbf{F}^{(\beta, \eta)} = \mathbf{F}^{(W_1, W_2)}$;
- (ii) *maximal* in $\mathbf{F}^{(W_1, W_2)}$, if for any one-dimensional Brownian motion $\vartheta(\cdot)$ in the filtration $\mathbf{F}^{(W_1, W_2)}$ and for which $\mathfrak{F}^\beta(t) \subseteq \mathfrak{F}^\vartheta(t)$ holds for all $0 \leq t < \infty$, we have $\mathfrak{F}^\vartheta(t) = \mathfrak{F}^\beta(t)$, $0 \leq t < \infty$.

[BROSSARD & LEURIDAN \(2008\)](#) show that complementability implies maximality; it is still an open question whether the reverse is true.

In section 3 we start with the planar Brownian motion $(W_1(\cdot), W_2(\cdot))$ and construct 12 one-dimensional Brownian motions in the filtration $\mathbf{F}^{(W_1, W_2)}$. We shall examine these properties for each of them.

Let us start with the independent one-dimensional Brownian motions $B_1(\cdot)$ of (3.12) and $B_2(\cdot)$ of (3.13). In view of (5.1), (5.4), Theorem 5.1 and Proposition 4.2, we have then $\mathfrak{F}^{(B_1, B_2)}(t) = \mathfrak{F}^{(X_1, X_2)}(t) = \mathfrak{F}^{(W_1, W_2)}(t)$, $0 \leq t < \infty$. Thus, each of $B_1(\cdot)$ of (3.12) and $B_2(\cdot)$ is complementable (by the other one), therefore also maximal, in the filtration $\mathbf{F}^{(W_1, W_2)}$.

It follows also fairly directly from (3.1) and (3.2), that the independent Brownian motions $W(\cdot)$ and $U^b(\cdot)$ complement each other in $\mathbf{F}^{(W_1, W_2)}$; the same is true of $U(\cdot)$ and $W^b(\cdot)$. Thus, all these one-dimensional Brownian motions are maximal in $\mathbf{F}^{(W_1, W_2)}$.

On the other hand, let us consider the one-dimensional Brownian motion $V^b(\cdot)$ of (3.5). This process is adapted to the filtration \mathbf{F}^W generated by the Brownian motion $W(\cdot)$ of (3.1); this is a consequence of the representation $V^b(\cdot) = \int_0^\cdot \text{sgn}(Y(t)) dW(t)$ in (2.17), and of the strong solvability of the stochastic equation (1.4). But we also have $\mathfrak{F}^{V^b}(t) = \mathfrak{F}^{|Y|}(t) \subsetneq \mathfrak{F}^Y(t) = \mathfrak{F}^W(t)$, $0 < t < \infty$ from (4.4), so $V^b(\cdot)$ cannot possibly be maximal in the two-dimensional filtration $\mathbf{F}^{(W_1, W_2)}$.

As a corollary of these considerations, we observe also that a linear combination of independent Brownian motions which are maximal, such as $V^b(\cdot) = \rho V_1(\cdot) - \sigma V_2(\cdot)$ in (3.5), can fail to be maximal.

As for the Brownian motion $Q^b(\cdot)$ of (3.4), it can fail to be maximal. Indeed, it is clear from (3.4), (3.5) and the observations in the previous paragraph that in the isotropic case $\rho = \sigma$ this Brownian motion $Q^b(\cdot) \equiv V^b(\cdot)$ is not maximal in $\mathbf{F}^{(W_1, W_2)}$. It turns out that $Q^b(\cdot)$ is maximal in $\mathbf{F}^{(W_1, W_2)}$, however, if $\rho \neq \sigma$. To see this, recall the notation $\delta = 2\rho\sigma$ and $\gamma = \rho^2 - \sigma^2$ from (2.18), and observe

$$\begin{aligned} \delta \int_0^t \text{sgn}(Y(s)) dQ^b(s) + \gamma W^b(t) &= (\delta\sigma + \gamma\rho)W_1(t) + (\delta\rho - \gamma\sigma)W_2(t) = \\ &= \rho W_1(t) + \sigma W_2(t) = W(t) = Y(t) - y - \lambda \int_0^t \text{sgn}(Y(s)) ds; \end{aligned}$$

equivalently,

$$Y(t) = y + \int_0^t \operatorname{sgn}(Y(s)) d(\delta Q^b(s) + \lambda s) + \gamma W^b(t), \quad 0 \leq t < \infty.$$

Then, since we are assuming $\delta = 2\rho\sigma > 0$ and $\gamma = \rho^2 - \sigma^2 \neq 0$, it follows from Proposition 8.1 (iii) that pathwise uniqueness holds for the above equation; and since the equation admits a weak solution, this solution is actually strong. In particular, the process $Y(\cdot)$ is adapted to $\mathbf{F}^{(Q^b, W^b)}$, and we arrive at the filtration identities $\mathbf{F}^{(Q^b, W^b)} = \mathbf{F}^{(Q^b, W^b, Y)} = \mathbf{F}^{(Q^b, W^b, W)} = \mathbf{F}^{(W_1, W_2)}$. Thus $W^b(\cdot)$ complements the Brownian motion $Q^b(\cdot)$ in $\mathbf{F}^{(W_1, W_2)}$.

Recall now the one-dimensional Brownian motion $Q(\cdot)$ of (3.5), which is independent of $V^b(\cdot)$; it is also independent of the one-dimensional Brownian motion $W(\cdot)$, and indeed we have $\mathfrak{F}^{(W_1, W_2)}(t) = \mathfrak{F}^{(W, Q)}(t)$, $0 \leq t < \infty$ from Proposition 4.2. In other words, the one-dimensional Brownian motion $Q(\cdot)$ is complementable (by $W(\cdot)$), therefore also maximal, in the two-dimensional filtration $\mathbf{F}^{(W_1, W_2)}$.

Let us consider now the one-dimensional Brownian motions $V_1(\cdot)$ and $V_2(\cdot)$ of (3.3); we claim that they are complementable (by $W_2(\cdot)$ and $W_1(\cdot)$, respectively; though *not by one another!*), therefore also maximal, in the two-dimensional filtration $\mathbf{F}^{(W_1, W_2)}$. To see the first claim (the second one is argued in a completely analogous manner), let us observe that we have the equalities $dY(t) + \lambda \operatorname{sgn}(Y(t)) dt = dW(t) = \rho \operatorname{sgn}(Y(t)) dV_1(t) + \sigma dW_2(t)$ by virtue of (3.1), (3.3), (1.4), therefore

$$dY(t) = \operatorname{sgn}(Y(t)) (\rho dV_1(t) - \lambda dt) + \sigma dW_2(t).$$

But this is an equation of the form (8.7) with $A(\cdot) \equiv 0$, for which Proposition 8.1 (iii) is satisfied, and the Brownian motions $V_1(\cdot)$ and $W_2(\cdot)$ are independent. Thus, pathwise uniqueness holds for this equation and, since the equation admits a weak solution, this solution is actually strong:

$$\mathfrak{F}^Y(t) \subseteq \mathfrak{F}^{(V_1, W_2)}(t), \quad 0 \leq t < \infty.$$

Furthermore, we deduce $\mathfrak{F}^{V_2}(t) \subseteq \mathfrak{F}^{(V_1, W_2)}(t)$ from the second equation in (3.3), therefore also

$$\mathfrak{F}^V(t) \subseteq \mathfrak{F}^{(V_1, W_2)}(t), \quad 0 \leq t < \infty;$$

it follows from the last two displayed inclusions and (4.13) that we have

$$\mathfrak{F}^{(W_1, W_2)}(t) = \mathfrak{F}^{(Y, V)}(t) \subseteq \mathfrak{F}^{(V_1, W_2)}(t) \subseteq \mathfrak{F}^{(W_1, W_2)}(t), \quad 0 \leq t < \infty,$$

and this proves the complementability of $V_1(\cdot)$ by $W_2(\cdot)$. As for the third claim of this paragraph, it is fairly straightforward from the strict inclusion in (4.13) that $V_1(\cdot)$ cannot be complemented by $V_2(\cdot)$ in the two-dimensional filtration $\mathbf{F}^{(W_1, W_2)}$.

Finally, let us turn to the Brownian motion $V(\cdot) = \rho V_1(\cdot) + \sigma V_2(\cdot)$ in (3.4). We follow the same approach as for the Brownian motion $Q^b(\cdot)$, only with the roles of

$\delta = 2\rho\sigma$ and $\gamma = \rho^2 - \sigma^2$ interchanged, and with $W^b(\cdot)$ replaced by $U(\cdot)$ from (3.2). A bit more precisely, we observe

$$\begin{aligned} \gamma \int_0^t \operatorname{sgn}(Y(s)) dV(s) + \delta U^b(t) &= \gamma(\rho W_1(t) - \sigma W_2(t)) + \delta(\sigma W_1(t) - \rho W_2(t)) = \\ &= \rho W_1(t) + \sigma W_2(t) = W(t) = Y(t) - y - \lambda \int_0^t \operatorname{sgn}(Y(s)) ds, \end{aligned}$$

or equivalently

$$Y(t) = y + \int_0^t \operatorname{sgn}(Y(s)) d(\gamma V(s) + \lambda s) + \delta U^b(t), \quad 0 \leq t < \infty.$$

Since we are assuming $\delta = 2\rho\sigma > 0$, it follows from Proposition 8.1 (iii) that pathwise uniqueness holds for the above equation (even when $\gamma = \rho^2 - \sigma^2 = 0$); and since the equation admits a weak solution, this solution is actually strong. To wit, the process $Y(\cdot)$ is adapted to $\mathbf{F}^{(V, U^b)}$, and we arrive at the filtration identities $\mathbf{F}^{(V, U^b)} = \mathbf{F}^{(Y, V, U^b)} = \mathbf{F}^{(W, V, U^b)} = \mathbf{F}^{(W_1, W_2)}$ (recall Proposition 4.2). Thus $U^b(\cdot)$ complements the Brownian motion $V(\cdot)$ in $\mathbf{F}^{(W_1, W_2)}$.

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References

- Banner, A.D., Fernholz, E.R., Karatzas, I. (2005) Atlas models of equity markets. *Ann. Appl. Probab.* 15(4):2296–2330, doi: [10.1214/105051605000000449](https://doi.org/10.1214/105051605000000449)
- Barlow, M.T. (1982) One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc.* (2) 26(2):335–347, doi: [10.1112/jlms/s2-26.2.335](https://doi.org/10.1112/jlms/s2-26.2.335)
- Barlow, M.T. (1988) Skew Brownian motion and a one-dimensional stochastic differential equation. *Stochastics* 25(1):1–2
- Bass, R.F., Pardoux, É. (1987) Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields* 76(4):557–572, doi: [10.1007/BF00960074](https://doi.org/10.1007/BF00960074)
- Brossard, J., Leuridan, C. (2008) Transformations browniennes et compléments indépendants: résultats et problèmes ouverts. In: *Séminaire de Probabilités XLI, Lecture Notes in Math.*, vol 1934, Springer, Berlin, pp 265–278, doi: [10.1007/978-3-540-77913-1_13](https://doi.org/10.1007/978-3-540-77913-1_13)
- Burdzy, K., Nualart, D. (2002) Brownian motion reflected on Brownian motion. *Probab. Theory Related Fields* 122(4):471–493, doi: [10.1007/s004400100165](https://doi.org/10.1007/s004400100165)
- Chernyĭ, A.S. (2001) On strong and weak uniqueness for stochastic differential equations. *Teor. Veroyatnost. i Primenen.* 46(3):483–497, doi: [10.1137/S0040585X97979093](https://doi.org/10.1137/S0040585X97979093)
- Engelbert, H.J. (1991) On the theorem of T. Yamada and S. Watanabe. *Stochastics Stochastics Rep.* 36(3-4):205–216, doi: [10.1080/17442509108833718](https://doi.org/10.1080/17442509108833718)

- Fernholz, E.R. (2002) *Stochastic Portfolio Theory, Applications of Mathematics* vol 48. Springer-Verlag, New York
- Harrison, J.M., Shepp, L.A. (1981) On skew Brownian motion. *Ann. Probab.* 9(2):309–313, doi: [10.1214/aop/1176994472](https://doi.org/10.1214/aop/1176994472),
- Haussmann, U.G., Pardoux, É. (1986) Time reversal of diffusions. *Ann. Probab.* 14(4):1188–1205, doi: [10.1214/aop/1176992362](https://doi.org/10.1214/aop/1176992362),
- Ichiba, T., Papathanakos, V., Banner, A., Karatzas, I., Fernholz, E.R. (2011) Hybrid Atlas models. *Ann. Appl. Probab.* 21(2):609–644, doi: [10.1214/10-AAP706](https://doi.org/10.1214/10-AAP706), arXiv: [0909.0065](https://arxiv.org/abs/0909.0065)
- Karatzas, I., Shreve, S.E. (1984) Trivariate density of Brownian motion, its local and occupation times, with application to stochastic control. *Ann. Probab.* 12(3):819–828, doi: [10.1214/aop/1176993230](https://doi.org/10.1214/aop/1176993230),
- Karatzas, I., Shreve, S.E. (1991) *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics, vol 113, 2nd edn. Springer-Verlag, New York
- Krylov, N.V. (2004) On weak uniqueness for some diffusions with discontinuous coefficients. *Stochastic Process. Appl.* 113(1):37–64, doi: [10.1016/j.spa.2004.03.012](https://doi.org/10.1016/j.spa.2004.03.012)
- Le Gall, J.F. (1983) Applications du temps local aux équations différentielles stochastiques unidimensionnelles. In: *Séminaire de Probabilités, XVII*, Lecture Notes in Math., vol 986, Springer, pp 15–31
- Lejay, A. (2006) On the constructions of the skew Brownian motion. *Probab. Surv.* 3:413–466 (electronic), doi: [10.1214/154957807000000013](https://doi.org/10.1214/154957807000000013)
- McKean, H.P. Jr. (1969) *Stochastic Integrals*. Probability and Mathematical Statistics, No. 5, Academic Press, New York
- Meyer, P.A. (1994) Sur une transformation du mouvement brownien due à Jeulin et Yor. In: *Séminaire de Probabilités, XXVIII*, Lecture Notes in Math., vol 1583, Springer, Berlin, pp 98–101, doi: [10.1007/BFb0073836](https://doi.org/10.1007/BFb0073836)
- Nakao, S. (1972) On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations. *Osaka J. Math.* 9:513–518
- Pal, S., Pitman, J. (2008) One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab.* 18(6):2179–2207, doi: [10.1214/08-AAP516](https://doi.org/10.1214/08-AAP516), arXiv: [0704.0957](https://arxiv.org/abs/0704.0957)
- Pardoux, É. (1986) Grossissement d’une filtration et retournement du temps d’une diffusion. In: *Séminaire de Probabilités, XX*, 1984/85, Lecture Notes in Math., vol 1204, Springer, Berlin, pp 48–55, doi: [10.1007/BFb0075711](https://doi.org/10.1007/BFb0075711)
- Perkins, E. (1982) Local time and pathwise uniqueness for stochastic differential equations. In: *Seminar on Probability, XVI*, Lecture Notes in Math., vol. 920, Springer, Berlin, pp 201–208
- Prokaj, V. (2011) The solution of the perturbed Tanaka-equation is pathwise unique, arXiv: [1104.0740](https://arxiv.org/abs/1104.0740)
- Soucaliuc, F., Werner, W. (2002) A note on reflecting Brownian motions. *Electron. Comm. Probab.* 7:117–122 (electronic)
- Soucaliuc, F., Tóth, B., Werner, W. (2000) Reflection and coalescence between independent one-dimensional Brownian paths. *Ann. Inst. H. Poincaré Probab. Statist.* 36(4):509–545, doi: [10.1016/S0246-0203\(00\)00136-9](https://doi.org/10.1016/S0246-0203(00)00136-9)
- Stroock, D.W., Varadhan, S.R.S. (1979) *Multidimensional Diffusion Processes*. Grundlehren der Mathematischen Wissenschaften, vol 233. Springer-Verlag, Berlin
- Veretennikov, A.Y. (1979) Strong solutions of stochastic differential equations. *Teor. Veroyatnost. i Primenen.* 24(2):348–360; translation in *Theory Probab. Appl.* (1979) 24:354–366
- Veretennikov, A.Y. (1980) Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb. (N.S.)* 111(153)(3):434–452; translation in *Math. USSR-Sb.* (1981) 39(3):387–403, doi: [10.1070/SM1981v039n03ABEH001522](https://doi.org/10.1070/SM1981v039n03ABEH001522)
- Veretennikov, A.Y. (1982) Criteria for the existence of a strong solution of a stochastic equation. *Teor. Veroyatnost. i Primenen.* 27(3):417–424; translation in *Theory Probab. Appl.* (1982) 27:441–449
- Walsh, J.B. (1978) A diffusion with a discontinuous local time. In: *Temps Locaux*. Astérisque 52-53:37–45.
- Zvonkin, A.K. (1974) A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)* 93(135):129–149; translation in *Math. USSR Sb.* (1974) 22:129–149, doi: [10.1070/SM1974v022n01ABEH001689](https://doi.org/10.1070/SM1974v022n01ABEH001689)