Portfolio Generating Functions

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Abstract

A general method is presented for constructing dynamic equity portfolios through the use of mathematical generating functions. The return on these functionally generated portfolios is related to the return on the market portfolio by a stochastic differential equation. Under appropriate conditions, this equation can be used to establish a dominance relationship between a functionally generated portfolio and the market portfolio.

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1 Introduction

Functionally generated equity portfolios were used in Fernholz (1998b) to study market diversity and in Fernholz (1997) to establish conditions under which arbitrage will exist in equity markets. In this paper we present a general discussion of these portfolios and the mathematical functions which generate them. We show that the return on such a portfolio is related to the return on the market portfolio by a stochastic differential equation. Under appropriate conditions, this equation can be used to establish a dominance relationship between a functionally generated portfolio and the market portfolio.

Functionally generated portfolios are not merely mathematical curiosities. Besides having been applied in studying theoretical questions such as market diversity and arbitrage, these portfolios have been used for actual equity investments. In fact, an institutional investment product based on a functionally generated portfolio of the stocks in the S&P 500 Index was introduced in 1996 (see Fernholz, Garvy, and Hannon (1998)). Moreover, Fernholz (1998a) has shown that a phenomenon similar to that which governs the behavior of functionally generated portfolios may explain the size effect, the historical tendency of smaller stocks to have higher returns than larger stocks.

The rest of this paper is organized as follows: In Section 2 there is a review of those elements of stochastic portfolio theory which we shall need later. Section 3 presents a formal development of the theory of *portfolio generating functions* (Definition 3.1). In Section 4, functionally generated portfolios are used to determine conditions under which portfolio dominance will exist. In particular, we show that a class of functions called *measures of diversity* (Definition 4.3) generate portfolios which will dominate the market portfolio under antitrust type restrictions. Section 5 is a summary.

We shall use a model of stock price processes represented by continuous semimartingales which is fairly standard in continuous-time financial theory (see, e.g., Karatzas and Shreve (1991) or Duffie (1992)). We shall make certain simplifying assumptions, among them:

1. Companies do not merge or break up, and the total number of shares of a company remains constant. The list of companies in the market is fixed.
2. There are no dividend payments.
3. There are no transaction costs, taxes, or problems with the indivisibility of shares.

2 Review of stochastic portfolio theory

In this section we shall review the basic definitions and results needed in the later sections. Much of the material in this section can also be found in Fernholz (1998b), but for completeness it is presented here also. We shall generally follow the definitions and notation used in Karatzas and Shreve (1991).

Let

$$W = \{W(t) = (W_1(t), \ldots, W_n(t)), \mathcal{F}_t, t \in [0, \infty)\}$$

be a standard n-dimensional Brownian motion defined on a probability space $\{\Omega, \mathcal{F}, P\}$ where $\{\mathcal{F}_t\}$ is the augmentation under P of the natural filtration $\{\mathcal{F}_t^W = \sigma(W(s); 0 \le s \le t)\}$. We say that a process $\{X(t), \mathcal{F}_t, t \in [0, \infty)\}$ is *adapted* if $X(t)$ is $\mathcal{F}_t$-measurable for $t \in [0, \infty)$. If X and Y are processes defined on $\{\Omega, \mathcal{F}, P\}$, we shall use the notation $X = Y$ if

$$P\{X(t) = Y(t), \quad t \in [0, \infty)\} = 1.$$

For continuous, square-integrable martingales $\{M(t), \mathcal{F}_t, t \in [0, \infty)\}$ and $\{N(t), \mathcal{F}_t, t \in [0, \infty)\}$, we can define the *cross-variation process* $\langle M, N \rangle$. The cross-variation process is adapted, continuous, and of bounded variation, and the operation $\langle \cdot, \cdot \rangle$ is bilinear on the real vector space of continuous, square-integrable martingales. If $M = N$, we shall use the notation $\langle M \rangle = \langle M, M \rangle$; $\langle M \rangle$ is called the *quadratic variation process* of M, and has continuous, nondecreasing sample paths. The Brownian motion process defined above is a continuous, square-integrable martingale, and it is characterized by its cross-variation processes

$$\langle W_i, W_j \rangle_t = \delta_{ij} t, \quad t \in [0, \infty),$$

where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise.

A *continuous semimartingale* $X = \{X(t), \mathcal{F}_t, t \in [0, \infty)\}$ is a measurable, adapted process which has the decomposition,

$$X(t) = X(0) + M_X(t) + V_X(t), \quad t \in [0, \infty), \quad \text{a.s.,} \tag{2.1}$$

where $\{M_X(t), \mathcal{F}_t, t \in [0, \infty)\}$ is a continuous, square-integrable martingale and $\{V_X(t), \mathcal{F}_t, t \in [0, \infty)\}$ is a continuous, adapted process which is locally of bounded variation. It can be shown that this decomposition is a.s. unique (see Karatzas and Shreve (1991)), so we can define the cross-variation process for continuous semimartingales X and Y by

$$\langle X, Y \rangle = \langle M_X, M_Y \rangle,$$

where M_X and M_Y are the martingale parts of X and Y, respectively.

Definition 2.1. Let X_0 be a positive number. A *stock* $X = \{X(t), \mathcal{F}_t, t \in [0, \infty)\}$ is a process of the form

$$X(t) = X_0 \exp\left(\int_0^t \gamma(s)\, ds + \int_0^t \sum_{\nu=1}^n \xi_\nu(s)\, dW_\nu(s) \right), \quad t \in [0, \infty), \tag{2.2}$$

where $\gamma = \{\gamma(t), \mathcal{F}_t, t \in [0, \infty)\}$ is measurable, adapted, and satisfies $\int_0^t |\gamma(s)| ds < \infty$, for all $t \in [0, \infty)$, a.s., and for $\nu = 1, \ldots, n$, $\xi_\nu = \{\xi_\nu(t), \mathcal{F}_t, t \in [0, \infty)\}$ is measurable, adapted, and satisfies $\int_0^t \xi_\nu^2(s) ds < \infty$ for all $t \in [0, \infty)$, a.s., and such that there exists a number $\varepsilon > 0$ for which $\xi_1^2(t) + \cdots + \xi_n^2(t) > \varepsilon$, $t \in [0, \infty)$, a.s.

It follows directly from Definition 2.1 that X is adapted, that $X(t) > 0$ for all $t \in [0, \infty)$, a.s., and that X has initial value $X(0) = X_0$. We shall set the initial value X_0 to be the total capitalization of the company represented by X at time $t = 0$, and we shall assume that this total capitalization is positive. This is equivalent to assuming that there is a single share of stock outstanding, and $X(t)$ represents its price at time t. We assume that stock shares are infinitely divisible, so there is no loss of generality in assuming a single share outstanding. The process γ is called the *growth rate (process)* of X and, for each ν, the process ξ_ν represents the sensitivity of X to the ν-th source of uncertainty W_ν.

Suppose that we have a family of stocks $X_i, i = 1, \ldots, n$,

$$X_i(t) = X_0^i \exp\left(\int_0^t \gamma_i(s)\, ds + \int_0^t \sum_{\nu=1}^n \xi_{i\nu}(s)\, dW_\nu(s) \right), \quad t \in [0, \infty). \tag{2.3}$$

Consider the matrix valued process ξ defined by $\xi(t) = (\xi_{i\nu}(t))_{1\le i,\nu\le n}$ and define the *covariance process* σ where $\sigma(t) = \xi(t)\xi^T(t)$. The cross-variation processes for $\log X_i$ and $\log X_j$ are related to σ by

$$\langle \log X_i, \log X_j\rangle_t = \int_0^t \sigma_{ij}(s)\,ds, \quad t\in[0,\infty), \quad \text{a.s.} \tag{2.4}$$

Since the processes $\xi_{i\nu}$ are assumed to be square integrable in Definition 2.1, it follows that for all i and j,

$$\int_0^t \sigma_{ij}(s)\,ds < \infty, \quad t\in[0,\infty), \quad \text{a.s.}$$

Definition 2.2. A *market* is a family $\mathcal{M} = \{X_i, \dots, X_n\}$ of stocks, defined as in (2.3), for which there is a number $\varepsilon > 0$ such that

$$x\sigma(t)x^T \ge \varepsilon\,\|x\|^2, \quad x\in\mathbb{R}^n, t\in[0,\infty), \quad \text{a.s.} \tag{2.5}$$

The *strong nondegeneracy* condition (2.5) is fairly common and can be found, for example, in Karatzas and Shreve (1991) and Karatzas and Kou (1996), and as *uniform ellipticity* in Duffie (1992).

Definition 2.3. Let $\mathcal{M}$ be a market of n stocks. A *portfolio* in $\mathcal{M}$ is a measurable, adapted process $\pi = \{\pi(t) = (\pi_1(t), \dots, \pi_n(t)), \mathcal{F}_t, t\in[0,\infty)\}$ such that $\pi(t)$ is bounded on $[0,\infty)\times\Omega$ and

$$\pi_1(t) + \cdots + \pi_n(t) = 1, \quad t\in[0,\infty), \quad \text{a.s.}$$

The processes π_i represent the respective *proportions*, or *weights*, of each stock in the portfolio. A negative value for $\pi_i(t)$ indicates a short sale. Suppose $Z_\pi(t)$ represents the value of an investment in π at time t. Then the amount invested in the i-th stock X_i will be

$$\pi_i(t)Z_\pi(t),$$

so if the price of X_i changes by $dX_i(t)$, the induced change in the portfolio value will be

$$\pi_i(t)Z_\pi(t)\frac{dX_i(t)}{X_i(t)}.$$

Hence the total change in the portfolio value at time t will be

$$dZ_\pi(t) = \sum_{i=1}^n \pi_i(t)Z_\pi(t)\frac{dX_i(t)}{X_i(t)},$$

or, equivalently,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^n \pi_i(t)\frac{dX_i(t)}{X_i(t)}. \tag{2.6}$$

Since we are interested in the behavior of portfolios, we are interested in solutions to (2.6). The following proposition and corollary are proved in Fernholz (1998b).

Proposition 2.1. *Let π be a portfolio and let*

$$\gamma_\pi(t) = \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t), \tag{2.7}$$

where

$$\gamma^*_\pi(t) = \frac{1}{2}\Big(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)\Big). \tag{2.8}$$

Then, for any positive initial value Z^π_0, the process Z_π defined by

$$Z_\pi(t) = Z^\pi_0 \exp\Big(\int_0^t \gamma_\pi(s)\,ds + \int_0^t \sum_{i,\nu=1}^n \pi_i(s)\xi_{i\nu}(s)\,dW_\nu(s)\Big), \quad t \in [0,\infty), \tag{2.9}$$

is a strong solution of (2.6).

Corollary 2.1. *Let π be a portfolio and Z_π be its value process. Then for $t \in [0,\infty)$,*

$$d\log Z_\pi(t) = \sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \gamma^*_\pi(t)\,dt. \tag{2.10}$$

The process γ_π in (2.7) is called the *portfolio growth rate (process)* of the portfolio π, and γ^*_π in (2.8) is called the *excess growth rate (process)*. It was proved in Fernholz (1998b) that for portfolios with non-negative weights, the excess growth rate is non-negative, and is positive unless the portfolio consists of a single stock.

The total capitalization of the market can be represented by a portfolio. Let us assume from now on that the market is $\mathcal{M} = \{X_1, \dots, X_n\}$, with n stocks.

Definition 2.4. The portfolio

$$\mu = \{\mu(t) = (\mu_1(t), \dots, \mu_n(t)), \mathcal{F}_t, t \in [0,\infty)\},$$

where

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \tag{2.11}$$

for $i = 1, \dots, n$, is called the *market portfolio (process)*.

It is clear that the μ_i defined by (2.11) satisfy the requirements of Definition 2.3. If we let

$$Z(t) = X_1(t) + \cdots + X_n(t), \tag{2.12}$$

then $Z(t)$ satisfies (2.6) with weights $\mu_i(t)$ given by (2.11). Hence, the value of the market portfolio represents the combined capitalization of all the stocks in the market. We shall let μ exclusively represent the market portfolio and Z its value process.

For any stock X_i and portfolio π we can consider the quotient process X_i/Z_π defined by

$$\log\big(X_i(t)/Z_\pi(t)\big) = \log X_i(t) - \log Z_\pi(t). \tag{2.13}$$

This process is a continuous semimartingale with

$$\langle \log(X_i/Z_\pi), \log(X_j/Z_\pi)\rangle_t = \langle \log X_i, \log X_j\rangle_t - \langle \log X_i, \log Z_\pi\rangle_t - \langle \log X_j, \log Z_\pi\rangle_t + \langle \log Z_\pi\rangle_t. \tag{2.14}$$

If we define the process $\sigma_{i\pi}$ by

$$\sigma_{i\pi}(t) = \sum_{j=1}^{n} \pi_j(t)\sigma_{ij}(t),$$

for $i = 1, \dots, n$, then

$$\langle \log X_i, \log Z_\pi \rangle_t = \int_0^t \sigma_{i\pi}(s)\, ds.$$

Define the *relative covariance (process)* τ^π to be the matrix valued process

$$\tau^\pi(t) = (\tau^\pi_{ij}(t))_{1 \le i,j \le n},$$

where

$$\tau^\pi_{ij}(t) = \sigma_{ij}(t) - \sigma_{i\pi}(t) - \sigma_{j\pi}(t) + \sigma_{\pi\pi}(t), \tag{2.15}$$

for $i, j = 1, \dots, n$, where $\sigma_{\pi\pi}(t) = \pi(t)\sigma(t)\pi^T(t)$, $t \in [0, \infty)$. Then for all i and j,

$$\langle \log(X_i/Z_\pi), \log(X_j/Z_\pi) \rangle_t = \int_0^t \tau^\pi_{ij}(s)\, ds. \tag{2.16}$$

In the case that $i = j$, we know that $\langle \log(X_i/Z_\pi) \rangle_t$ is non-decreasing, so

$$\tau^\pi_{ii}(t) \ge 0, \quad t \in [0, \infty), \quad \text{a.s.}$$

We shall use $\tau = \tau^\mu$ to represent the relative covariance process of the market portfolio μ, and τ_{ij} to represent its ij-th component.

Let $\eta = \{\eta(t) = (\eta_1(t), \dots, \eta_n(t)), \mathcal{F}_t, t \in [0, \infty)\}$ be a portfolio. Then, by Corollary 2.1,

$$d \log Z_\eta(t) = \sum_{i=1}^{n} \eta_i(t)\, d \log X_i(t) + \gamma^*_\eta(t)\, dt,$$

so

$$d \log\big(Z_\eta(t)/Z_\pi(t)\big) = \sum_{i=1}^{n} \eta_i(t)\, d \log\big(X_i(t)/Z_\pi(t)\big) + \gamma^*_\eta(t)\, dt. \tag{2.17}$$

The *relative variance (process)* of η and π is defined by

$$\begin{aligned} \tau^\pi_{\eta\eta}(t) &= \big(\eta(t) - \pi(t)\big)\sigma(t)\big(\eta(t) - \pi(t)\big)^T \\ &= \eta(t)\tau^\pi(t)\eta^T(t). \end{aligned} \tag{2.18}$$

Lemma 2.1. *The rank of $\tau^\pi(t)$ is $n - 1$, for $t \in [0, \infty)$, a.s. The null space of $\tau^\pi(t)$ is spanned by $\pi(t)$.*

Proof. From (2.18) and condition (2.5), it follows that

$$\eta(t)\tau^\pi(t)\eta^T(t) = 0$$

if and only if $\eta(t) = \pi(t)$, for all $t \in [0, \infty)$, a.s. □

The following lemma expresses the excess growth in terms of the relative covariance process.

Lemma 2.2. *Let π and η be portfolios. Then for $t \in [0,\infty)$,*

$$\gamma_\eta^*(t) = \frac{1}{2}\Big(\sum_{i=1}^n \eta_i(t)\tau_{ii}^\pi(t) - \tau_{\eta\eta}^\pi(t)\Big).$$

Proof. The proof is a direct calculation using (2.15) and (2.18). □

Lemma 2.3. *Let π be a portfolio. Then there exists an $\varepsilon > 0$ such that for $i = 1, \dots, n$,*

$$\tau_{ii}^\pi(t) \geq \varepsilon\big(1 - \pi_i(t)\big)^2, \quad t \in [0,\infty), \quad \text{a.s.} \tag{2.19}$$

Proof. For any i and $t \in [0,\infty)$, let $x(t) = (\pi_1(t), \dots, \pi_i(t) - 1, \dots, \pi_n(t))$. Then,

$$\begin{aligned}\tau_{ii}^\pi(t) &= \sigma_{ii}(t) - 2\sigma_{i\pi}(t) + \sigma_{\pi\pi}(t)\\ &= x(t)\sigma(t)x^T(t)\\ &\geq \varepsilon\,\|x(t)\|^2, \quad t \in [0,\infty), \quad \text{a.s.},\end{aligned}$$

where ε is chosen as in (2.5). Since,

$$\|x(t)\|^2 \geq \big(1 - \pi_i(t)\big)^2,$$

the lemma follows. □

Lemma 2.4. *Let π be a portfolio with non-negative weights, and let $\pi_{\max}(t) = \max_{1\leq i\leq n}\pi_i(t)$. Then there exists an $\varepsilon > 0$ such that*

$$\gamma_\pi^*(t) \geq \varepsilon(1 - \pi_{\max}(t))^2.$$

Proof. If we let $\eta = \pi$ in Lemma 2.2, then Lemma 2.1 implies that

$$\begin{aligned}\gamma_\pi^*(t) &= \frac{1}{2}\sum_{i=1}^n \pi_i(t)\tau_{ii}^\pi(t)\\ &\geq \frac{\varepsilon}{2}(1 - \pi_{\max}(t))^2,\end{aligned}$$

where ε is chosen as in Lemma 2.3, since the $\pi_i(t)$ are non-negative. □

3 Portfolio generating functions

In this section we shall show that certain real-valued functions of the market weights can be used to generate portfolios, and we shall study the properties these functions and the portfolios they generate. Functionally generated portfolios are of interest because under certain market conditions it can be shown that a dominance relationship exists between such a portfolio and the market portfolio. This relationship will be discussed in the next section; this section will be devoted to the basic properties of generating functions and the portfolios they generate.

We shall consider real-valued functions defined on the open simplex

$$\Delta^n = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 1, \quad 0 < x_i < 1, \quad i = 1, \dots, n\}.$$

It will be convenient to use the standard coordinate system in $\mathbb{R}^n$, even though it is not a coordinate system on Δ^n. For this reason we shall consider functions that are defined in an open neighborhood $U \subset \mathbb{R}^n$ of Δ^n. A real-valued function defined on a subset of $\mathbb{R}^n$ is C^2 if it is twice continuously differentiable in all variables. We shall use the notation D_i for the partial derivative with respect to the i-th variable, and D_{ij} for the second partial derivative with respect to the i-th and j-th variables.

Definition 3.1. Let U be an open neighborhood of Δ^n and $\mathbf{S}$ be a positive C^2 function defined in U. Then $\mathbf{S}$ is the *generating function* of the portfolio π if there exists a measurable, adapted process $\Theta = \{\Theta(t), \mathcal{F}_t, t \in [0,\infty)\}$ such that

$$d \log\big(Z_\pi(t)/Z(t)\big) = d \log \mathbf{S}(\mu(t)) + \Theta(t)dt, \quad t \in [0,\infty), \quad \text{a.s.} \tag{3.1}$$

Θ is called the *drift process* corresponding to $\mathbf{S}$.

We shall also say that the function $\mathbf{S}$ *generates* π, and that π is *functionally generated*. Definition 3.1 can be extended to include time dependent generating functions, but we shall not consider such functions here.

Proposition 3.1. *Suppose that $\mathbf{S}_1$ and $\mathbf{S}_2$ generate π_1 and π_2 with drift processes Θ_1 and Θ_2, respectively. Then*

$$d \log\big(Z_{\pi_1}(t)/Z_{\pi_2}(t)\big) = d \log\big(\mathbf{S}_1(\mu(t))/\mathbf{S}_2(\mu(t))\big) + \big(\Theta_1(t) - \Theta_2(t)\big)dt, \quad t \in [0,\infty), \quad \text{a.s.}$$

Proof. The proof follows directly from Definition 3.1. □

What follows is the main theorem on portfolio generating functions.

Theorem 3.1. *Let $\mathbf{S}$ be a positive C^2 function defined on a neighborhood U of Δ^n such that for $i = 1, \dots, n$, $x_i D_i \log \mathbf{S}(x)$ is bounded on Δ^n. Then $\mathbf{S}$ generates the portfolio π with weights*

$$\pi_i(t) = \Big(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}(\mu(t))\Big)\mu_i(t), \quad t \in [0,\infty), \quad \text{a.s.}, \tag{3.2}$$

for $i = 1, \dots, n$, and drift process

$$\Theta(t) = \frac{-1}{2\,\mathbf{S}(\mu(t))} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t), \quad t \in [0,\infty), \quad \text{a.s.} \tag{3.3}$$

Proof. The weight process μ_i is a quotient process with $\mu_i(t) = X_i(t)/Z(t)$ for all t. By (2.16) it follows that

$$d\langle \log \mu_i, \log \mu_j \rangle_t = \tau_{ij}(t)\, dt, \quad t \in [0,\infty), \quad \text{a.s.},$$

so by Itô's Lemma,

$$d\mu_i(t) = \mu_i(t)\, d \log \mu_i(t) + \frac{1}{2}\mu_i(t)\tau_{ii}(t)\, dt, \quad t \in [0,\infty), \quad \text{a.s.}, \tag{3.4}$$

and

$$d\langle \mu_i, \mu_j \rangle_t = \mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt, \quad t \in [0,\infty), \quad \text{a.s.} \tag{3.5}$$

Itô's lemma, along with (3.5), implies that a.s. for all $t \in [0, \infty)$,

$$d \log \mathbf{S}(\mu(t)) = \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} D_{ij} \log \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt.$$

Now,

$$D_{ij} \log \mathbf{S}(\mu(t)) = \frac{D_{ij}\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(t))} - D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t)),$$

so, a.s., for all $t \in [0, \infty)$,

$$\begin{aligned} d \log \mathbf{S}(\mu(t)) = \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t) + \frac{1}{2\,\mathbf{S}(\mu(t))} \sum_{i,j=1}^{n} D_{ij} \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt \\ - \frac{1}{2} \sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt. \end{aligned} \tag{3.6}$$

In order for (3.1) to hold, the martingale parts of $\log \mathbf{S}(\mu(t))$ and $\log\big(Z_\pi(t)/Z(t)\big)$ must be equal. Corollary 2.1 implies that for the portfolio π, a.s. for all $t \in [0, \infty)$,

$$\begin{aligned} d \log\big(Z_\pi(t)/Z(t)\big) &= \sum_{i=1}^{n} \pi_i(t)\, d \log\big(X_i(t)/Z(t)\big) + \gamma_\pi^*(t)\, dt \\ &= \sum_{i=1}^{n} \pi_i(t)\, d \log \mu_i(t) + \gamma_\pi^*(t)\, dt \\ &= \sum_{i=1}^{n} \frac{\pi_i(t)}{\mu_i(t)}\, d\mu_i(t) - \frac{1}{2} \sum_{i,j=1}^{n} \pi_i(t) \pi_j(t) \tau_{ij}(t)\, dt \end{aligned} \tag{3.7}$$

by Lemma 2.2. Suppose that

$$\pi_i(t) = \big(D_i \log \mathbf{S}(\mu(t)) + \varphi(t)\big) \mu_i(t), \tag{3.8}$$

where $\varphi(t)$ is chosen such that $\sum_{i=1}^{n} \pi_i(t) = 1$. Then, a.s. for all $t \in [0, \infty)$,

$$\begin{aligned} \sum_{i=1}^{n} \frac{\pi_i(t)}{\mu_i(t)}\, d\mu_i(t) &= \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t) + \varphi(t) \sum_{i=1}^{n} d\mu_i(t) \\ &= \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t), \end{aligned}$$

since $\sum_{i=1}^{n} d\mu_i(t) = 0$. Hence, the martingale parts of $\log \mathbf{S}(\mu(t))$ and $\log\big(Z_\pi(t)/Z(t)\big)$ are equal. If

$$\varphi(t) = 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log \mathbf{S}(\mu(t)),$$

then $\sum_{i=1}^{n} \pi_i(t) = 1$, and (3.2) is proved.

If $\pi_i(t)$ satisfies (3.8), then a.s. for all $t \in [0,\infty)$,

$$\begin{aligned}\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t) &= \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &\qquad + 2\varphi(t)\sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t) + \varphi^2(t)\sum_{i,j=1}^n \mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &= \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t),\end{aligned}$$

since $\mu(t)$ is in the null space of $\tau(t)$ by Lemma 2.1. Hence, a.s. for all $t \in [0,\infty)$,

$$d\log\big(Z_\pi(t)/Z(t)\big) = \sum_{i=1}^n D_i \log \mathbf{S}(\mu(t))\, d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt.$$

This equation and (3.6) imply that a.s. for all $t \in [0,\infty)$,

$$d\log\big(Z_\pi(t)/Z(t)\big) = d\log \mathbf{S}(\mu(t)) - \frac{1}{2\,\mathbf{S}(\mu(t))}\sum_{i,j=1}^n D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt,$$

so (3.3) is proved. The process Θ defined by (3.3) is clearly measurable and adapted. □

Example 3.1. Here are a few examples of generating functions and the portfolios they generate.

1. $\mathbf{S}(x) = 1$ generates the market portfolio μ with $\Theta(t) = 0$.
2. $\mathbf{S}(x) = c_1x_1 + \cdots + c_nx_n$, where $c_1, \ldots, c_n$ are constants, generates the buy-and-hold portfolio which holds c_i of the capitalization of the the i-th stock. Here $\Theta(t) = 0$. This type of portfolio is commonly held by investors, at least in a piecewise manner.
3. $\mathbf{S}(x) = (x_1 \cdots x_n)^{1/n}$ generates the equal weighted portfolio with $\Theta(t) = \gamma^*_\pi(t)$. The Value Line Index is such a portfolio.
4. $\mathbf{S}(x) = x_1^{p_1} \cdots x_n^{p_n}$, where $p_1, \ldots, p_n$ are constant and $p_1 + \cdots + p_n = 1$, generates the constant weight portfolio with weights $\pi_i(t) = p_i$ and $\Theta(t) = \gamma^*_\pi(t)$.

Corollary 3.1. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ generate portfolios π_1 and π_2, respectively. Then for constants p_1 and p_2 such that $p_1 + p_2 = 1$, the function*

$$\mathbf{S} = \mathbf{S}_1^{p_1}\mathbf{S}_2^{p_2}$$

generates a portfolio Y with weights

$$\pi_i = p_1\pi_{1i} + p_2\pi_{2i},$$

where π_{1i} and π_{2i} are the weights of π_1 and π_2, respectively.

Proof. The proof follows directly from (3.2) of Theorem 3.1. □

Not all portfolios are functionally generated; let us now characterize those that are. Recall that a differential is *exact* if it is of the form $\sum_i D_iG(x)\,dx_i$ for some differentiable function G (see Spivak (1965)).

Proposition 3.2. *Let π be a portfolio. Then π is functionally generated if and only if there exist continuously differentiable real-valued functions $F, f_1, \dots, f_n$ defined in a neighborhood of Δ^n such that $\pi_i(t) = f_i(\mu(t))$ for all $t \in [0, \infty)$, a.s., for $i = 1, \dots, n$, and*

$$\sum_{i=1}^{n} \left(\frac{f_i(x)}{x_i} + F(x) \right) dx_i \tag{3.9}$$

is an exact differential.

Proof. Suppose that $\pi_i(t) = f_i(\mu(t))$ for $i = 1, \dots, n$, and the differential in (3.9) is exact. Then there is a function G such that

$$D_i G(x) = \frac{f_i(x)}{x_i} + F(x),$$

for $i = 1, \dots, n$. Hence, Itô's lemma implies that, a.s., for all $t \in [0, \infty)$,

$$\begin{aligned} dG(\mu(t)) &= \sum_{i=1}^{n} \left(\frac{f_i(\mu(t))}{\mu_i(t)} + F(\mu(t)) \right) d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} D_{ij} G(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt \\ &= \sum_{i=1}^{n} \pi_i(t) \frac{d\mu_i(t)}{\mu_i(t)} + \frac{1}{2} \sum_{i,j=1}^{n} D_{ij} G(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt, \end{aligned} \tag{3.10}$$

since $\sum_i d\mu_i(t) = 0$. By (3.10) and (3.7), it follows that a.s., for all $t \in [0, \infty)$,

$$dG(\mu(t)) = d \log\big(Z_\pi(t)/Z(t)\big) + \frac{1}{2} \sum_{i,j=1}^{n} \Big(\pi_i(t) \pi_j(t) \tau_{ij}(t) + D_{ij} G(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) \Big)\, dt.$$

Hence, the function $\mathbf{S} = e^G$ generates π.

Now suppose that π has a generating function $\mathbf{S}$ defined in a neighborhood U of Δ^n. For $x \in U$, define

$$f_i(x) = \Big(D_i \log \mathbf{S}(x) + 1 - \sum_{j=1}^{n} x_j\, D_j \log \mathbf{S}(x) \Big) x_i,$$

for $i = 1, \dots, n$, and let

$$F(x) = -1 + \sum_{j=1}^{n} x_j\, D_j \log \mathbf{S}(x).$$

By Theorem 3.1 the weights π_i satisfy $\pi_i(t) = f_i(\mu(t))$, for $i = 1, \dots, n$, for all $t \in [0, \infty)$, a.s. Then for $x \in U$, the differential

$$\begin{aligned} \sum_{i=1}^{n} \left(\frac{f_i(x)}{x_i} + F(x) \right) dx_i &= \sum_{i=1}^{n} D_i \log \mathbf{S}(x)\, dx_i \\ &= d \log \mathbf{S}(x) \end{aligned}$$

is exact, so the proposition is proved. □

Example 3.2. Here is an example of a portfolio π whose weights depend differentiably on the market portfolio weights, but is not functionally generated. For $x \in \mathbb{R}^n$, let

$$\begin{aligned} f_1(x) &= x_1, \\ f_2(x) &= x_2 + \cdots + x_n, \\ f_i(x) &= 0, \qquad \text{for } i = 3, \dots, n, \end{aligned}$$

and let π be the portfolio defined by $\pi_i(t) = f_i(\mu(t))$, for $i = 1, \ldots, n$. It can be shown (see Spivak (1965)) that if the differential in (3.9) is exact, then for all i and j,

$$D_j\left(\frac{f_i(x)}{x_i} + F(x)\right) = D_i\left(\frac{f_j(x)}{x_j} + F(x)\right).$$

No function F will satisfy these equations for all i and j, so π has no generating function.

It would seem reasonable that only the values of a generating function on Δ^n should affect the portfolio it generates. To prove this, we first need a couple of lemmas.

Lemma 3.1. *Let $f_1, \ldots, f_n$ be continuous real-valued functions on Δ^n and g be a continuous real-valued function on $\Delta^n \times [0, \infty)$. If*

$$\sum_{i=1}^n f_i(\mu(t))\, d\mu_i(t) = g(\mu(t), t)\, dt, \quad t \in [0, \infty), \quad \text{a.s.,}$$

then for $x \in \Delta^n$, $f_i(x) = f_j(x)$ for all i and j, and $g = 0$.

Proof. For $t \in [0, \infty)$, let

$$Y(t) = \int_0^t \sum_{i=1}^n f_i(\mu(s))\, d\mu_i(s).$$

Then Y is a semimartingale with

$$d\langle Y\rangle_t = \sum_{i,j=1}^n f_i(\mu(t)) f_j(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt, \quad t \in [0, \infty), \quad \text{a.s.,}$$

by (3.5) (see Karatzas and Shreve (1991)). By hypothesis

$$Y(t) = \int_0^t g(\mu(s), s)\, ds,$$

so the martingale part of Y in the decomposition (2.1) is null. Therefore, $\langle Y\rangle_t = 0$, so

$$\sum_{i,j=1}^n f_i(\mu(t)) f_j(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) = 0, \quad t \in [0, \infty), \quad \text{a.s.,}$$

and $(f_1(\mu(t))\mu_1(t), \ldots, f_n(\mu(t))\mu_n(t))$ is in the null space of $\tau(t)$. By Lemma 2.1 the null space of $\tau(t)$ is spanned by $(\mu_1(t), \ldots, \mu_n(t))$, so

$$f_1(\mu(t)) = f_2(\mu(t)) = \cdots = f_n(\mu(t)).$$

Hence, a.s., for all $t \in [0, \infty)$,

$$\begin{aligned}\sum_{i=1}^n f_i(\mu(t))\, d\mu_i(t) &= f_1(\mu(t)) \sum_{i=1}^n d\mu_i(t) \\ &= 0,\end{aligned}$$

since $\sum_{i=1}^n d\mu_i(t) = 0$, and therefore $g(\mu(t), t) = 0$. □

Lemma 3.2. *Let f be a continuously differentiable real-valued function defined in a neighborhood U of Δ^n. Then f is constant on Δ^n if and only if for all $x \in \Delta^n$, $D_i f(x) = D_j f(x)$ for all i and j.*

Proof. Parameterize Δ^n by positive real numbers $t_1, \ldots, t_{n-1}$ with $t_1 + \cdots + t_{n-1} < 1$ such that $x_i = t_i$ for $i = 1, \ldots, n-1$ and $x_n = 1 - t_1 - \cdots - t_{n-1}$. Then for $i = 1, \ldots, n-1$,

$$\frac{\partial f(x)}{\partial t_i} = D_i f(x) - D_n f(x)$$

for all $x \in \Delta^n$. If f is constant on Δ^n then all its partial derivatives with respect to the parameters t_i will be zero, which implies that $D_i f(x) = D_n f(x)$ for all i. Likewise if $D_i f(x) = D_j f(x)$ for all i and j, then the partial derivatives with respect to all the t_i will be zero, so f will be constant on Δ^n. □

Proposition 3.3. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be positive C^2 functions defined in an open neighborhood of Δ^n. Then $\mathbf{S}_1$ and $\mathbf{S}_2$ generate the same portfolio if and only if $\mathbf{S}_1/\mathbf{S}_2$ is constant on Δ^n.*

Proof. Suppose $\mathbf{S}_1$ and $\mathbf{S}_2$ are defined on the open neighborhood U of Δ^n, and that $\mathbf{S}_1/\mathbf{S}_2$ is constant on Δ^n. Define the function f for $x \in U$ by

$$f(x) = \log \mathbf{S}_1(x) - \log \mathbf{S}_2(x).$$

Then f is constant on Δ^n, so Lemma 3.2 implies that $D_i f(x) = D_j f(x)$ for all i and j, for all $x \in \Delta^n$. Therefore,

$$D_i \log \mathbf{S}_1(\mu(t)) - D_j \log \mathbf{S}_1(\mu(t)) = D_i \log \mathbf{S}_2(\mu(t)) - D_j \log \mathbf{S}_2(\mu(t)),$$

for all i and j. Hence, the difference $D_i \log \mathbf{S}_1(\mu(t)) - D_i \log \mathbf{S}_2(\mu(t))$ is the same for all i, and (3.2) of Theorem 3.1 implies that the weights generated by $\mathbf{S}_1$ and $\mathbf{S}_2$ will be the same. It follows that $\mathbf{S}_1$ and $\mathbf{S}_2$ generate the same portfolio.

Now suppose that $\mathbf{S}_1$ and $\mathbf{S}_2$ generate the same portfolio. By Itô's Lemma, a.s., for all $t \in [0, \infty)$,

$$\begin{aligned} d\log\big(\mathbf{S}_1(\mu(t))/\mathbf{S}_2(\mu(t))\big) &= \sum_{i=1}^n D_i \log\big(\mathbf{S}_1(\mu(t))/\mathbf{S}_2(\mu(t))\big)\, d\mu_i(t) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n D_{ij} \log\big(\mathbf{S}_1(\mu(t))/\mathbf{S}_2(\mu(t))\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt. \end{aligned}$$

Since the values of the portfolios generated by $\mathbf{S}_1$ and $\mathbf{S}_2$ will be equal, (3.1) implies that

$$d\log\big(\mathbf{S}_1(\mu(t))/\mathbf{S}_2(\mu(t))\big) = \big(\Theta_1(t) - \Theta_2(t)\big)\, dt, \quad t \in [0, \infty), \quad \text{a.s.},$$

so, a.s., for all $t \in [0, \infty)$,

$$\begin{aligned} &\sum_{i=1}^n D_i \log\big(\mathbf{S}_1(\mu(t))/\mathbf{S}_2(\mu(t))\big)\, d\mu_i(t) \\ &\qquad = \Big(\Theta_1(t) - \Theta_2(t) - \frac{1}{2}\sum_{i,j=1}^n D_{ij} \log\big(\mathbf{S}_1(\mu(t))/\mathbf{S}_2(\mu(t))\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\Big)\, dt. \end{aligned}$$

Then Lemma 3.1 implies that for all $x \in \Delta^n$,

$$D_i \log\big(\mathbf{S}_1(x)/\mathbf{S}_2(x)\big) = D_j \log\big(\mathbf{S}_1(x)/\mathbf{S}_2(x)\big),$$

for all i and j. It follows by Lemma 3.2 that $\log\big(\mathbf{S}_1(x)/\mathbf{S}_2(x)\big)$ is constant for $x \in \Delta^n$. □

This proposition implies that we could have defined generating functions on Δ^n rather than in a neighborhood. The difficulty is that on Δ^n there is no linear coordinate system which treats all the n stocks in the same manner.

For use in the next section, we need to establish conditions under which a generating function will have a positive drift process.

Proposition 3.4. *Let* $\mathbf{S}$ *be a generating function such that for all* $x \in \Delta^n$*, the matrix* $\big(D_{ij}\mathbf{S}(x)\big)$ *has at most one positive eigenvalue, and if there is a positive eigenvalue it corresponds to an eigenvector perpendicular to* Δ^n*. Let* π *be the portfolio generated by* $\mathbf{S}$*. Then* $\pi_i(t) \geq 0$*, for* $i = 1, \ldots, n$*, and* $\Theta(t) \geq 0$ *for all* $t \in [0, \infty)$*, a.s. If for all* $x \in \Delta^n$*,* $\operatorname{rank}\big(D_{ij}\mathbf{S}(x)\big) > 1$*, then* $\Theta(t) > 0$ *for all* $t \in [0, \infty)$*, a.s.*

Proof. Suppose that $\mathbf{S}$ is a generating function which satisfies the hypothesis of the proposition. For any $t \in [0, \infty)$, define $x(u) \in \Delta^n$ by

$$x(u) = uv_k + (1-u)\mu(t)$$

for $0 \leq u < 1$, where $v_k = (0, \ldots, 1, \ldots, 0)$ with 1 in the k-th position and 0 elsewhere. Let

$$f(u) = \mathbf{S}(x(u)),$$

so

$$f'(u) = D_k\mathbf{S}(x(u)) - \sum_{i=1}^n \mu_i(t) D_i\mathbf{S}(x(u)), \tag{3.11}$$

and

$$\begin{aligned} f''(u) &= (v_k - \mu(t))\big(D_{ij}\mathbf{S}(x(u))\big)(v_k - \mu(t))^T \\ &\leq 0, \end{aligned}$$

since $v_k - \mu(t)$ is parallel to Δ^n and hence is composed of eigenvectors of $\big(D_{ij}\mathbf{S}(x(u))\big)$ which have non-positive eigenvalues. The usual convexity arguments imply that

$$f(u) \leq f(0) + uf'(0), \qquad 0 \leq u < 1,$$

so,

$$0 < f(0) + uf'(0), \qquad 0 \leq u < 1.$$

It follows from this and (3.11) that

$$0 \leq \mathbf{S}(\mu(t)) + D_k\mathbf{S}(\mu(t)) - \sum_{i=1}^n \mu_i(t) D_i\mathbf{S}(\mu(t)).$$

Hence, by (3.2) of Theorem 3.1, $\pi_k(t) \geq 0$, $k = 1, \ldots, n$, for all $t \in [0, \infty)$, a.s.

For any $t \in [0, \infty)$, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\big(D_{ij}\mathbf{S}(\mu(t))\big)$. Let $e_k = (e_{k1}, \ldots, e_{kn})$ be a normalized eigenvector corresponding to the eigenvalue λ_k, for $k = 1, \ldots, n$. For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x\big(D_{ij}\mathbf{S}(\mu(t))\big)y^T &= \sum_{k=1}^n \lambda_k x e_k^T e_k y^T \\ &= \sum_{k=1}^n \lambda_k \sum_{i,j=1}^n x_i y_j e_{ki} e_{kj}. \end{aligned}$$

It follows that

$$\sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t) = \sum_{k=1}^{n} \lambda_k \sum_{i,j=1}^{n} \mu_i(t)\mu_j(t)e_{ki}e_{kj}\tau_{ij}(t). \tag{3.12}$$

If one of the λ_i is positive, we can assume without loss of generality that it is λ_1 with $e_1 = \pm(n^{-1/2}, \ldots, n^{-1/2})$. If $(\tau_{ij}(t))$ is positive semi-definite with null space generated by $\mu(t)$, as in Lemma 2.1, then

$$\sum_{i,j=1}^{n} \mu_i(t)\mu_j(t)e_{ki}e_{kj}\tau_{ij}(t) \geq 0 \tag{3.13}$$

for $k = 1, \ldots, n$, and

$$\sum_{i,j=1}^{n} \mu_i(t)\mu_j(t)e_{1i}e_{1j}\tau_{ij}(t) = 0$$

if $e_1 = \pm(n^{-1/2}, \ldots, n^{-1/2})$. Hence, by Lemma 2.1 and (3.12),

$$\sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t) \leq 0, \quad t \in [0, \infty) \quad \text{a.s.}$$

By (3.3) of Theorem 3.1, $\Theta(t) \geq 0$, for all $t \in [0, \infty)$, a.s.

If for all $x \in \Delta^n$, $\operatorname{rank}\big(D_{ij}\mathbf{S}(x)\big) > 1$, then at least one of the eigenvalues $\lambda_2, \ldots, \lambda_n$ is negative. Since the eigenvectors e_k are pairwise orthogonal, only e_1 can be perpendicular to Δ^n, so (3.13) will be positive for at least one $k \geq 2$. Therefore, Lemma 2.1 and (3.12) imply that

$$\sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t) < 0, \quad t \in [0, \infty) \quad \text{a.s.}$$

Hence, $\Theta(t) > 0$, for all $t \in [0, \infty)$, a.s. □

4 Portfolio dominance and measures of diversity

In this section we shall use the stochastic differential equation (3.1) to establish dominance relationships between certain functionally generated portfolios and the market portfolio. We shall show that if there are appropriate bounds on a generating function and the corresponding drift process, then such a relationship will hold. In particular, we shall show that functions which are *measures of diversity* will generate portfolios that dominate the market portfolio if appropriate bounds exist on the concentration of market capital. Our purpose here is not to achieve maximum generality, but rather to explore the use of generating functions for constructing dominant portfolios. Accordingly, we shall entertain a bias toward simplicity over generality.

Definition 4.1. Let η and ξ be portfolios. Then η *strictly dominates* ξ if there is a number $t > 0$ such that

$$P\{Z_\eta(t)/Z_\eta(0) > Z_\xi(t)/Z_\xi(0)\} = 1. \tag{4.1}$$

This definition is stronger than than the usual definition of "dominates" in which (4.1) is replaced by $P\{Z_\eta(t)/Z_\eta(0) \geq Z_\xi(t)/Z_\xi(0)\} = 1$ and $P\{Z_\eta(t)/Z_\eta(0) > Z_\xi(t)/Z_\xi(0)\} > 0$.

Lemma 4.1. *Let* $\mathbf{S}$ *be the generating function of the portfolio* π *with drift process* Θ*, and suppose that* c_1 *and* c_2 *are constants. If for all* $t > 0$*,* $\mathbf{S}(\mu(t)) > c_1 > 0$*, a.s., and* $\Theta(t) > c_2 > 0$*, a.s., then* π *strictly dominates* μ*. If for all* $t > 0$*,* $\mathbf{S}(\mu(t)) < c_1$*, a.s., and* $\Theta(t) < c_2 < 0$*, a.s., then* μ *strictly dominates* π*.*

Proof. Suppose the set of conditions with $c_1 > 0$ and $c_2 > 0$ hold. It follows from (3.1) that a.s., for $T > 0$,

$$\log\big(Z_\pi(T)/Z_\pi(0)\big) - \log\big(Z(T)/Z(0)\big) > \log c_1 - \log \mathbf{S}(\mu(0)) + c_2 T.$$

Therefore, if

$$T > \big(\log \mathbf{S}(\mu(0)) - \log c_1\big)/c_2,$$

then,

$$\log\big(Z_\pi(T)/Z_\pi(0)\big) > \log\big(Z(T)/Z(0)\big)$$

almost surely. The proof for the second set of conditions is similar. □

Dominance relationships between pairs of functionally generated portfolios can similarly be established by applying Proposition 3.1 if appropriate bounds exist.

Example 4.1. Consider the function

$$\mathbf{S}(x) = x_1^2.$$

Theorem 3.1 implies that $\mathbf{S}$ generates a portfolio π with weights

$$\pi_1(t) = 2 - \mu_1(t),$$

and

$$\pi_i(t) = -\mu_i(t),$$

for $i = 2, \dots, n$. The drift process is

$$\Theta(t) = -\tau_{11}(t).$$

In this case $\mathbf{S}(\mu(t)) < 1$, so we can apply Lemma 4.1 if we can establish a negative upper bound on Θ. Lemma 2.3 shows that for all $t > 0$, $\tau_{11}(t) > \varepsilon\big(1 - \mu_1(t)\big)^2$, a.s., where ε is that of (2.5). Hence, if $\mu_1(t)$ can be bounded away from 1, the market portfolio will strictly dominate π. This brings us to the concept of *market diversity*.

Definition 4.2. The market $\mathcal{M}$ is *diverse* if there exists a number $\delta > 0$ such that for $i = 1, \dots, n$,

$$\mu_i(t) \leq 1 - \delta, \quad t \in [0, \infty), \quad \text{a.s.} \tag{4.2}$$

This definition was given in Fernholz (1998b). The condition it imposes, (4.2), is observable and broadly consistent with the actual behavior of equity markets in industrial economies, especially if there are antitrust laws. It was shown in Fernholz (1998b) that a portfolio generated by the entropy function (see Example 4.2 below) will strictly dominate the market portfolio in a diverse market. The entropy function is the archetypal measure of diversity; here we wish to give a general definition of these measures. Recall that a real-valued function F defined on a subset of $\mathbb{R}^n$ is *symmetric* if it is invariant under permutations of the variables $x_i, i = 1, \dots, n$, and *concave* if for $0 < p < 1$ and $x, y \in \mathbb{R}^n$, $F(px + (1-p)y) > pF(x) + (1-p)F(y)$.

Definition 4.3. A positive C^2 function defined on an open neighborhood of Δ^n is a *measure of diversity* if it is symmetric and concave.

In this definition, symmetry ensures that all stocks are treated in the same manner, and concavity implies that transferring capital from a larger company to a smaller one will increase the measure. The results of the previous section imply that measures of diversity can be used to generate portfolios.

Proposition 4.1. *Suppose that* $\mathbf{S}$ *is a measure of diversity which generates a portfolio* π *with drift process* Θ*. Then* $\Theta(t) \geq 0$ *for all* $t \in [0, \infty)$*, a.s., and* $\mu_i(t) \geq \mu_j(t)$ *implies that* $\pi_j(t)/\mu_j(t) \geq \pi_i(t)/\mu_i(t)$ *for all* $t \in [0, \infty)$*, a.s.*

Proof. If $\mathbf{S}$ is a measure of diversity, then by definition it is concave and C^2. It is well known that for such a function, the matrix $\big(D_{ij}\mathbf{S}(x)\big)$ is negative semi-definite. A negative semi-definite matrix has no positive eigenvalues, so Proposition 3.4 implies that $\Theta(t) \geq 0$ for all $t \in [0, \infty)$, a.s.

Now suppose that $x = (x_1, \ldots, x_n) \in \Delta^n$ with $x_i \leq x_j$ for some $i < j$. Define

$$x(u) = (x_1, \ldots, (1-u)x_i + ux_j, \ldots, ux_i + (1-u)x_j, \ldots, x_n),$$

so $x(0) = x$ and $x(1)$ is x with the i-th and j-th coordinates reversed. Define $f(u) = \mathbf{S}(x(u))$, so f is C^2 and concave, and since $\mathbf{S}$ is symmetric, $f(0) = f(1)$. Now,

$$f'(u) = (x_j - x_i)\big(D_i\mathbf{S}(x(u)) - D_j\mathbf{S}(x(u))\big),$$

and the concavity of f implies that $f'(0) \geq 0$. Since $x_i \leq x_j$, it follows that $D_i\mathbf{S}(x) \geq D_j\mathbf{S}(x)$. Then (3.2) of Theorem 3.1 implies that for $\mu_i(t) \geq \mu_j(t)$, $\pi_j(t)/\mu_j(t) \geq \pi_i(t)/\mu_i(t)$. □

This proposition shows that the weight ratios $\pi_i(t)/\mu_i(t)$ decrease with increasing market weight. Hence, if a stock's market weight increases, i.e., the stock goes up relative to the market, then the portfolio π will sell some (fractional) shares of that stock.

Although Proposition 4.1 shows that a measure of diversity will have a non-negative drift process, this is not sufficient to ensure that the portfolio it generates will dominate the market portfolio. In order to apply Lemma 4.1, the drift process must have a positive lower bound. Let us now consider some examples of measures of diversity, and determine in which cases such a bound exists.

Example 4.2. The *entropy* function,

$$\mathbf{S}(x) = -\sum_{i=1}^{n} x_i \log x_i,$$

was studied in Fernholz (1998b). The weights of the portfolio it generates are

$$\pi_i(t) = -\frac{\mu_i(t)\log\mu_i(t)}{\mathbf{S}(\mu(t))},$$

and the drift process is

$$\Theta(t) = \frac{\gamma^*(t)}{\mathbf{S}(\mu(t))}.$$

Lemma 2.4 implies that in a diverse market $\gamma^*(t)$, and hence, $\Theta(t)$, has a positive lower bound. It is not difficult to show that the same is true for $\mathbf{S}(\mu(t))$, so Lemma 4.1 can be applied.

Example 4.3. The geometric mean in Example 3.1 is a measure of diversity which generates a portfolio with all weights equal to n^{-1} and drift process $\Theta(t) = \gamma_\pi^*(t)$. The value of the geometric mean of the $\mu_i(t)$ will be bounded away from zero only if each individual $\mu_i(t)$ is so bounded. This condition is quite restrictive, and does not necessarily hold even in a diverse market.

Example 4.4. For $0 < p < 1$, let

$$\mathbf{D}_p(x) = \Big(\sum_{i=1}^n x_i^p\Big)^{1/p}.$$

This is a measure of diversity, and has in fact been used to construct an institutional equity investment product (see Fernholz, Garvy, and Hannon (1998)). The portfolio generated by $\mathbf{D}_p$ has weights

$$\pi_i(t) = \frac{\mu_i^p(t)}{\big(\mathbf{D}_p(\mu(t))\big)^p}.$$

The drift process, which is positive, is

$$\Theta(t) = (1-p)\gamma_\pi^*(t).$$

Here $\mathbf{D}_p(\mu(t)) > 1$. Moreover, Proposition 4.1 implies that $\pi_{\max}(t) \le \mu_{\max}(t)$, so if the market is diverse, Lemma 2.4 implies that $\gamma_\pi^*(t)$ will have a positive lower bound. Therefore, Lemma 4.1 can be applied.

Example 4.5. Let

$$\mathbf{S}(x) = 1 - \frac{1}{2}\sum_{i=1}^n x_i^2.$$

This measure of diversity was used Fernholz (1997). Here,

$$\pi_i(t) = \Big(\frac{2-\mu_i(t)}{\mathbf{S}(\mu(t))} - 1\Big)\mu_i(t),$$

for $i = 1, \ldots, n$, with drift process

$$\Theta(t) = \frac{1}{2\,\mathbf{S}(\mu(t))}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t).$$

$\mathbf{S}(\mu(t)) > 1/2$ and in a diverse market, $\Theta(t)$ has a positive lower bound by Lemma 2.3.

Example 4.6. The *Gini coefficient* is frequently used to measure diversity. It is usually defined as

$$G(x) = \frac{1}{2}\sum_{i=1}^n |x_i - n^{-1}|.$$

If we modify it to

$$\mathbf{S}(x) = 1 - \frac{1}{2}\sum_{i=1}^n |x_i - n^{-1}|,$$

this comes closer to Definition 4.3, but fails to be C^2. Nevertheless a version of Theorem 3.1 is probably valid with portfolio weights

$$\pi_i(t) = \Big(\frac{\operatorname{sign}\big(n^{-1}-\mu_i(t)\big)}{2\,\mathbf{S}(\mu(t))} + 1 - \sum_{j=1}^n \frac{\mu_j(t)\operatorname{sign}\big(n^{-1}-\mu_j(t)\big)}{2\,\mathbf{S}(\mu(t))}\Big)\mu_i(t),$$

and a non-negative drift process Θ which depends on a local time measuring the time $\mu_i(t)$ spends near n^{-1}, for $i = 1, \ldots, n$. See Karatzas and Shreve (1991), Chapter 6, for details about Brownian local time.

Rather than attempt to analyze the Gini coefficient here, we shall settle for a quadratic version of it. Let

$$\mathbf{S}(x) = 1 - \frac{1}{2}\sum_{i=1}^{n}\big(x_i - n^{-1}\big)^2.$$

This is a measure of diversity with portfolio weights

$$\pi_i(t) = \left(\frac{n^{-1} - \mu_i(t)}{\mathbf{S}(\mu(t))} + 1 - \sum_{j=1}^{n}\frac{\mu_j(t)\big(n^{-1} - \mu_j(t)\big)}{\mathbf{S}(\mu(t))}\right)\mu_i(t),$$

and positive drift process

$$\Theta(t) = \frac{1}{2\,\mathbf{S}(\mu(t))}\sum_{i=1}^{n}\mu_i^2(t)\tau_{ii}(t).$$

Here $\mathbf{S}(\mu(t)) > 1/2$, and as in the previous example, in a diverse market $\Theta(t)$ will have a positive lower bound.

Example 4.7. The function

$$\mathbf{S}(x) = 1 - \frac{1}{4}\sum_{i=1}^{n}\big(x_i - n^{-1}\big)^4$$

is quite similar to that of Example 4.6, and is indeed a measure of diversity. Here $3/4 < \mathbf{S}(\mu(t)) < 1$, and the drift process is

$$\Theta(t) = \frac{3}{2\,\mathbf{S}(\mu(t))}\sum_{i=1}^{n}\big(\mu_i(t) - n^{-1}\big)^2\mu_i^2(t)\tau_{ii}(t).$$

Now suppose that the relative covariance processes τ_{ii} are uniformly bounded, say by $M > 0$. Then

$$\Theta(t) < 2M\max_{1\le i\le n}\big(\mu_i(t) - n^{-1}\big)^2.$$

Since there is a positive probability that all the $\mu_i(t)$ will remain arbitrarily close to n^{-1} for an arbitrarily long time, the contribution from $\Theta(t)\,dt$ in (3.1) can be minimal. If then one of the $\mu_i(t)$ quickly increases to almost $1 - \delta$, $\mathbf{S}(\mu(t))$ will decrease and π will have lower return than the market over the period. Hence, even in a diverse market, π does not dominate μ.

Example 4.8. The *Rényi entropy* is a generalization of the entropy function we considered above (see Rényi (1960)). For $p \ne 1$ it is defined by

$$\mathbf{S}_p(x) = \frac{1}{1-p}\log\sum_{i=1}^{n}x_i^p.$$

As $p \to 1$, $\mathbf{S}_p$ tends to the usual entropy function. It can be shown that for $p < 1$, $\mathbf{S}_p$ is a measure of diversity, but for $p > 1$, $\mathbf{S}_p$ is not concave. For $p > 1$, the weights of the portfolio $\mathbf{S}_p$ generates may be negative and the corresponding drift process may take on negative values.

Example 4.9. Suppose that $x_{(i)}$, $i = 1, \ldots, n$, are the order statistics, $x_{(i)} \leq x_{(i+1)}$. The function

$$\mathbf{S}(x) = x_{(1)} + \cdots + x_{(k)},$$

for $k < n$, is symmetric and concave, but not C^2. Nevertheless, it probably can be proved that $\mathbf{S}$ generates a portfolio π such that for $i = 1, \ldots, n$,

$$\pi_i(t) = \begin{cases} \mu_i(t)/\mathbf{S}(\mu(t)) & \text{if } \mu_i(t^-) \leq \mu_{(k)}(t^-), \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu_i(t^-) \leq \mu_{(k)}(t^-)$ if there is an $\varepsilon > 0$ such that $\mu_i(s) \leq \mu_{(k)}(s)$ for $0 < t - s < \varepsilon$. Then,

$$d \log(Z_\pi(t)/Z(t)) = d \log \mathbf{S}(\mu(t)) + d\Lambda_k(t),$$

where Λ_k is a local time measuring the time $\mu_{(k)}(t)$ spends near $\mu_{(k+1)}(t)$. This construction is related to the *size effect* discussed in Fernholz (1998a).

5 Conclusions

We have shown that positive C^2 functions of the market weights generate equity portfolios, and that the return on these portfolios is related to the market return by a stochastic differential equation. Under appropriate conditions, this equation can be used to establish a dominance relationship between a functionally generated portfolio and the market portfolio. In a diverse market, certain measures of market diversity generate portfolios that dominate the market portfolio.

References

Duffie, D. (1992). *Dynamic Asset Pricing Theory.* Princeton, NJ: Princeton University Press.

Fernholz, R. (1997). Arbitrage in equity markets. Technical report, INTECH, Princeton, NJ.

Fernholz, R. (1998a, May/June). Crossovers, dividends, and the size effect. *Financial Analysts Journal 54*(3), 73–78.

Fernholz, R. (1998b). On the diversity of equity markets. *Journal of Mathematical Economics,* to appear.

Fernholz, R., R. Garvy, and J. Hannon (1998, Winter). Diversity weighted indexing. *Journal of Portfolio Management 24*(2), 74–82.

Karatzas, I. and S. G. Kou (1996). On the pricing of contingent claims under constraints. *The Annals of Applied Probability 6*, 321–369.

Karatzas, I. and S. Shreve (1991). *Brownian Motion and Stochastic Calculus.* New York: Springer-Verlag.

Rényi, A. (1960). On measures of entropy and information. In *Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics, and Probability*, pp. 547–561.

Spivak, M. (1965). *Calculus on Manifolds.* New York: Benjamin.