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Portfolio Growth Rates In the Presence of Value Estimation Error

Robert Ferguson
Senior Investment Officer, INTECH

Dean Leistikow
Professor of Finance, Fordham University

Joel Rentzler
Professor of Finance, Baruch University

Susana Yu
Associate Professor of Finance, Montclair State University

All correspondence to
Robert Ferguson
644 Fern St.
West Palm Beach, FL 33401
561 835 8806
bobferg13@adelphia.net

Abstract

This paper analyzes the impact of value estimation error on portfolios' growth rates. Previous studies have attempted this in the context of cross sectional statistical analyses. However, the phenomenon is process driven, hence is best analyzed using Stochastic Portfolio Theory. The paper uses Stochastic Portfolio Theory to determine how value estimation error effects portfolios' growth rates. Several classes of portfolio weights are compared to the corresponding capitalization weights. Examples include estimation error neutral weights, market value neutral weights, Fundamental weights, and Diversity Weights. The paper also provides a theory of a traditional size effect.

Introduction

This paper analyzes the impact of value estimation error on portfolios' growth rates. Previous studies have attempted this in the context of cross sectional statistical analyses. However, the phenomenon is process driven, hence is best analyzed using Stochastic Portfolio Theory (Fernholz [2002]).

The paper uses Stochastic Portfolio Theory to examine how value estimation error effects portfolios' growth rates. Several classes of portfolio weights are analyzed with respect to the corresponding capitalization weights. Examples include estimation error neutral weights, market value neutral weights (Arnott, et al [2005], and Treynor [2005]), Fundamental weights (Arnott, et al [2005], and Treynor [2005]), and Diversity Weights (Fernholz [2002]).

The paper also provides a theory of a traditional size effect.

Stock price can be considered the sum of value plus estimation error. Estimation error produces mispricing. A stock is termed undervalued if its price is below value or overvalued if its price is above value.

As Arnott et al [2005] notes, capitalization weighted benchmarks overweight overvalued stocks and underweight undervalued stocks.[1] Consequently, (see Treynor [2005]) a $1.00 investment in a capitalization weighted benchmark buys less than $1.00 of value. In contrast, a $1.00 investment in a value weighted portfolio buys $1.00 of value. If there were no estimation error, then stock price would equal value, $1.00 invested in any portfolio would buy $1.00 of value, and the market portfolio would be both a capitalization weighted portfolio and a value weighted portfolio.

Since price includes estimation error and price is higher (lower) than value if estimation error is positive (negative), there is a positive correlation between price and estimation error.[2] A capitalization weighted benchmark's larger stocks tend to be overvalued relative to its smaller stocks. Thus, a capitalization weighted portfolio suffers a drag due to mispricing corrections. The drag tends to be less for a portfolio that overweights its capitalization weighted benchmark's smaller stocks. Such a portfolio benefits from mispricing corrections.

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1 Arnott's characterization is not quite correct because the capitalization weighted benchmark itself may be overvalued or undervalued. The correct statement is that capitalization weighted benchmarks overweight relatively overvalued stocks relative to relatively undervalued stocks. A similar qualification applies to Treynor's characterization. For example, if the capitalization weighted benchmark is undervalued, an investment in the capitalization weighted benchmark buys more than $1.00 of value and an investment in a value weighted portfolio buys even more.

2 Think of a three variable regression of price against value and estimation error. The estimation error coefficient is positive, hence there is a positive (partial) correlation between price and estimation error. This holds even if the capitalization weighted benchmark is overpriced or underpriced.

Fernholz and Shay [1982] denote a stock's or portfolio's logarithmic drift rate by "growth rate". Growth rates, not expected arithmetic returns, correspond to long-term returns. Fernholz and Shay show that a portfolio's growth rate is the weighted average of its stocks' growth rates plus an Excess Growth Rate. The Excess Growth Rate depends only on the portfolio's weights and stocks' covariance matrix. It has nothing to do with stocks' expected returns or growth rates.

Estimation errors effect a portfolio's growth rate in two ways, through mispricing corrections and estimation error volatility. Mispricing corrections are a component of stocks' growth rates, hence effect a portfolio's weighted average stock growth rate. The impact of mispricing corrections on a portfolio's weighted average stock growth rate can be positive or negative. Estimation error volatility effects a portfolio's Excess Growth Rate. The impact is positive for long portfolios.[3]

Treynor's and Arnott et al's analyses of the impact of estimation error parallels this one, but does not take a "process" approach, hence cannot address Excess Growth Rates.

There exist some portfolio weights where the direct effect of mispricing corrections and estimation error volatility on the portfolio's growth rate net to zero.[4] These weights are termed "estimation error neutral weights". Roughly, estimation error neutrality requires that a portfolio's weights be positively correlated with estimation errors. The corresponding required positive covariance is large only to the extent that estimation error variance is large in relation to the mispricing correction rate and the number of stocks is small. For portfolios of many stocks and reasonable estimation error characteristics, the required positive covariance is close to zero.

A rough approximation is that estimation error neutral weights will tend to have higher growth rates than capitalization weights if the half-life of mispricing corrections is shorter than 1.4 years.[5]

Capitalization weighted portfolios suffer a drag from mispricing corrections and gain from the estimation error volatility's contribution to their Excess Growth Rates. The net effect depends on the balance between estimation error volatility and the speed of mispricing corrections. A relatively high (low) rate of mispricing corrections leads to a net negative (positive) impact. A rough approximation for broad diversified portfolios is that

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[3] The intuition is provided later. The paper considers only long portfolios of many stocks in markets where all relevant covariance matrices are nonsingular. This suffices to assure that the impact on the portfolio's Excess Growth Rate is, for all practical purposes, positive.

[4] Direct estimation error effects are defined as those associated with growth rate components that involve the estimation errors' characteristics directly. Indirect estimation error effects are defined as those associated with growth rate components that depend on estimation error only through the portfolio's weights.

[5] Technically, the condition is that the mispricing correction rate parameter defined in the estimation error process corresponds to a half-life of less than 1.4 years.

the net direct impact of estimation errors on a capitalization weighted portfolio's growth rate is negative if estimation error corrections have a half-life of less than 1.4 years.

Relative to value weighted portfolios, capitalization weighted portfolios tend to suffer drag from mispricing corrections and, if they are diversified, might be expected to have about the same or lower Excess Growth Rates.[6] A rough approximation is that value weighted and capitalization weighted portfolios of the same stocks have similar Excess Growth Rates. If so, value weighted portfolios tend to have higher growth rates than capitalization weighted portfolios due to less mispricing correction drag.

There also exist portfolio weights that are independent of estimation error, in the sense of a zero cross sectional (point in time) correlation between portfolio weights and estimation errors. One example is an equal-weight portfolio, which has a zero correlation with anything. Randomly chosen weights are another example, but only on average. Such weights are termed "market value neutral" weights and are related to what Treynor [2005] refers to as "market value indifferent" weights. Market value neutral portfolio weights eliminate, or eliminate on average, the drag due to mispricing corrections that capitalization weights suffer from and, if diversified, tend to have similar or higher Excess Growth Rates.[7] Consequently, many market value neutral weights have higher growth rates than capitalization weights.

Arnott's Fundamental Indexes have weights that are functions of fundamental quantities, such as book value, cash flow, dividends, or sales. In general, fundamental weights need not be estimation error neutral or market value neutral. To the extent they are less correlated with estimation error than capitalization weights, they will tend to have a higher growth rate than capitalization weights. This should be true for choices of fundamental quantities that reflect value because then the Fundamental weights may be closer to market value neutral than capitalization weights.

Fernholz [2002] defines a passive portfolio termed a Diversity Weighted Portfolio. A Diversity Weighted Portfolio's weights are functions of an associated capitalization weighted benchmark's weights. Given the benchmark's weights, the Diversity Weighted Portfolio's weights are determined. A Diversity Weighted Portfolio's weights, as a percentage of its benchmark weights, monotonically decrease with increases in its benchmark weights. A benchmark's larger stocks are underweighted and its smaller stocks are overweighted. Fernholz shows that a Diversity Weighted Portfolio's relative return can be perfectly attributed to the change in the benchmark's Diversity (a function that measures the extent to which capital is spread across the benchmark's stocks) and a positive drift termed the Kinetic Differential. Fernholz shows that Diversity is bounded; hence a

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6 There is a tendency for diversified portfolios with higher allocations to smaller stocks to have higher Excess Growth Rates. In the presence of value estimation error, value weighted portfolios tend to have higher allocations to smaller stocks than capitalization weighted portfolios.

7 Market value neutral weights are likely to overweight smaller stocks relative to capitalization weights. As noted earlier, this tends to increase a portfolio's Excess Growth Rate.

Diversity Weighted Portfolio outperforms its benchmark over time. In the context of this paper, a Diversity Weighted Portfolio has less expected mispricing correction drag than its capitalization weighted benchmark and could be expected to have a higher Excess Growth Rate.

The analysis provides a partial explanation of the traditional size effect. It is widely believed that portfolios that overweight smaller stocks relative to capitalization weights have higher long-term returns, hence higher growth rates. In the context of this paper's analysis, capitalization weights typically are positively correlated with estimation errors. The positive correlation induces a drag due to mispricing corrections. Portfolio weights that are less positively correlated with estimation errors will tend to suffer less mispricing correction drag. One way of reducing the correlation is to overweight smaller stocks relative to capitalization weights. Portfolios that overweight smaller stocks relative to capitalization weights tend to suffer less mispricing drag and, if diversified, tend to have higher Excess Growth Rates, hence tend to have higher growth rates.

The remainder of the paper is organized as follows. Section two provides the mathematical background. The general results are applied to capitalization weights in section three. Estimation-error-neutral portfolio weights are examined in section four. Section five analyzes market-value-neutral portfolio weights. Fundamentally weighted portfolios are discussed in section six. Section seven discusses Fernholz's Diversity Weighted Portfolios. Section eight presents the argument for considering mispricing corrections as a traditional size effect. Section nine is the conclusion. An appendix is available from the authors that contains the mathematical development.

Mathematical background

This paper presumes that stock price is the sum of value plus estimation error. [8] Random processes for value and estimation error are posited and formulas and approximations for portfolios' growth rates are found. The value processes are assumed independent of the estimation error processes. Both the value and estimation error processes provide for correlations among stocks' returns. The estimation error processes are mean reverting and are the source of mispricing corrections. These mispricing corrections are the basis for using fundamental and quantitative analysis to construct portfolios that beat a benchmark.

The estimation error processes are not designed to model benchmark overvaluation or undervaluation, although they permit it.

Fernholz [2002] addresses stock market stability. A stock market is stable if stocks' market weights have an asymptotic (over time) distribution that spreads the market's capital across the stocks, as opposed to one that has virtually all the market's capital in one stock. The price processes often assumed in the finance literature lead to unstable mar-

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[8] Some might argue that only price is observable, hence that defining price as the sum of value and estimation error is futile. However, this breakdown is the sine qua non of investment analysis and the success of some fundamental analysts argue for taking it seriously.

kets where, almost always over the long term, one stock has weight approximately 1.0 and all other stocks have weight approximately 0.0 (this does not preclude the lead stock changing over time).

Real stock markets are stable. Their capitalization is spread across many stocks and the functional relationship between stocks' capitalization weights and size rank is mean reverting, as opposed to divergent. In order for capitalization weights to be stable, it is necessary that both the value and estimation error processes be stable. This is assumed in the paper.

The logarithmic value process for stock i is:[9]

$$d\log V_i(t) = \gamma_{Vi}(t)\,dt + \sigma_{Vi}(t)\,dW_{Vi}(t) \tag{1}$$

$V_i(t) \equiv$	Stock i's value at time t.
$\gamma_{Vi}(t) \equiv$	Stock i's logarithmic value growth rate (logarithmic value drift rate) at time t.
$\sigma_{Vi}(t) \equiv$	Stock i's logarithmic value volatility rate at time t.
$dW_{Vi}(t) \equiv$	Stock i's logarithmic value Brownian motion at time t.

An example of a seemingly innocuous price process that implies an unstable value market is when all of the value growth rates, $\gamma_{Vi}(t)$, are the same and constant and all of the value volatility rates, $\sigma_{Vi}(t)$, are the same and constant. A market composed of statistically identical stocks can be unstable.

Stability requires restrictions on the $\gamma_{Vi}(t)$ and $\sigma_{Vi}(t)$. An example of a stable value market is when stocks' logarithmic value growth rates are given by $\gamma_{Vi}(t) = (1/n) - \pi_{Vi}(t)$, where n is the number of securities, $\pi_{Vi}(t)$ are the stocks' value weights, and all stocks have the same value volatility rate, $\sigma_{Vi}(t) = \sigma_V$. Smaller stocks have higher value growth

9 The logarithmic change in value corresponds to a continuous return. The value process presumes that, over a short time interval, the continuous change (return) in value is the sum of two components. The first is a trend, equal to the product of a continuous return rate, $\gamma_{Vi}(t)$, and the time interval, dt. The second is a random disturbance equal to the product of a standard deviation rate, $\sigma_{Vi}(t)$, and a random disturbance for the interval, $dW_{Vi}(t)$.

rates than larger stocks, providing a tendency toward an equal weight value market. This balances the tendency of the random disturbances to drive stocks away from an equal weight market.

The estimation error process for stock i is:[10]

$$d\varepsilon_i(t) = -\gamma_\varepsilon \varepsilon_i(t)\,dt + \sigma_{\varepsilon i}(t)\,dW_{\varepsilon i}(t) \tag{2}$$

$\varepsilon_i(t) \equiv$	Stock i's logarithmic estimation error at time t.
$\gamma_\varepsilon \equiv$	Stocks' mispricing correction rate. This is a logarithmic drift rate.
$\sigma_{\varepsilon i}(t) \equiv$	Stock i's logarithmic estimation error volatility rate at time t.
$dW_{\varepsilon i}(t) \equiv$	Stock i's logarithmic estimation error Brownian motion at time t.

Stability for the estimation error process requires restrictions on the $\sigma_{\varepsilon i}(t)$. An example is when $\sigma_{\varepsilon i}(t) = \sigma_\varepsilon$.[11]

The Brownian motions $W_{Vi}(t)$ and $W_{\varepsilon i}(t)$ are independent. Value estimation error is independent of value. This assures that each stock's expected price is its value and that there is no tendency for stocks' prices to differ from their values on average over time. There are correlations among the value processes and among the estimation error processes taken separately.

There is no loss of generality from assuming that the value and estimation error Brownian motion processes are independent. Any reasonable operational definition of value should exhibit such independence.[12]

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10 The first term on the right of Equation (2) constitutes a trend component in the direction of zero estimation error. This term eliminates the proportion γ_ε of last period's estimation error this period. The second term adds a new estimation error this period.

11 The estimation error process is unstable if $\gamma_\varepsilon = 0$, even if stocks' logarithmic estimation error volatility rates are the same and constant.

In the absence of the disturbance term in Equation (2), estimation error corrects exponentially to zero with a half-life of T .

$$\varepsilon_i(t) = \varepsilon_i(0)e^{-\gamma_\varepsilon t} \tag{3}$$

$$T = -\frac{\ln(0.5)}{\gamma_\varepsilon} \tag{4}$$

This shows that the mispricing correction half-life is large (small) to the extent that the mispricing correction rate is small (large).

The logarithmic stock price process, $d\log X_i(t)$, is:

$$\log X_i(t) = \log V_i(t) + \varepsilon_i(t) \tag{5}$$

$$d\log X_i(t) = d\log V_i(t) + d\varepsilon_i(t) \tag{6}$$

and is stable in the sense of Fernholz [2002] because it is the sum of two independent stable processes.

The stock price process can be rewritten as follows.

$$d\log X_i(t) = \left[\gamma_{Vi}(t) - \gamma_\varepsilon \varepsilon_i(t)\right]dt + \left[\sigma_{Vi}(t)\,dW_{Vi}(t) + \sigma_{\varepsilon i}(t)\,dW_{\varepsilon i}(t)\right] \tag{7}$$

The stock's growth rate, denoted by $\gamma_{Xi}(t)$, is the sum of the growth rates of the value and estimation error processes.

$$\gamma_{Xi}(t) \equiv \gamma_{Vi}(t) - \gamma_\varepsilon \varepsilon_i(t)$$

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12 If value is correlated with estimation error, then large and small value stocks will be mispriced on average, which is unreasonable. Such a correlation suggests misspecification that justifies redefining value to include the portion of estimation error responsible for the correlation.

(8)

Taking into account the assumed independence between the value and estimation error processes, the stock's variance rate, denoted by $\sigma_{X_i}^2(t)$, is the sum of the variance rates of the value and estimation error processes.

$$\sigma_{X_i}^2(t) = \sigma_{X_iX_i}(t) = \sigma_{V_i}^2(t) + \sigma_{\varepsilon_i}^2(t) = \sigma_{V_iV_i}(t) + \sigma_{\varepsilon_i\varepsilon_i}(t)$$

(9)

Equation (7) becomes:

$$d\log X_i(t) = \gamma_{Xi}(t)\,dt + \sigma_{X_i}(t)\,dW_{Xi}(t)$$

(10)

It is shown in the appendix (available from the authors) that a portfolio's price process, $Z_{\pi_X}(t)$, is given by:

$$d\log Z_{\pi_X}(t) = \left[\sum_i \pi_{Xi}(t)\gamma_{Vi}(t) - \gamma_\varepsilon \sum_i \pi_{Xi}(t)\varepsilon_i(t) + \gamma_{\pi_X}^*(t)\right]dt + \sum_i \pi_{Xi}(t)\sigma_{X_i}(t)\,dW_{Xi}(t)$$

(11)

$\pi_{Xi}(t) \equiv$ Stock i's price weight in the portfolio.

$\gamma_{\pi_X}^*(t) \equiv$ The portfolio's price Excess Growth Rate.

Equation (11) shows that the portfolio's growth rate, $\gamma_{\pi_X}(t)$, is the weighted average of the stock's growth rates plus the portfolio's Excess Growth Rate. A useful way of ordering the terms is:

$$\gamma_{\pi_X}(t) = \sum_i \pi_{Xi}(t)\gamma_{Vi}(t) + \gamma_{\pi_X}^*(t) - \gamma_\varepsilon \sum_i \pi_{Xi}(t)\varepsilon_i(t)$$

(12)

Fernholz (2002) shows that the portfolio's Excess Growth Rate is:

$$\gamma_{\pi_X}^*(t) = \frac{1}{2}\left(\sum_i \pi_{Xi}(t)\sigma_{X_iX_i}(t) - \sum_{ij} \pi_{Xi}(t)\pi_{Xj}(t)\sigma_{X_iX_j}(t)\right)$$

(13)

Fernholz also shows that Excess Growth Rates are positive for long portfolios. Long portfolios' growth rates exceed the weighted average growth rates of their stocks.[13]

The second term on the right of Equation (11) is the portfolio's random disturbance term.

The portfolio's variance rate, implied by the random disturbance term in Equation (11) is:

$$\sigma_{\pi_X}^2(t) = \sum_{ij} \pi_{Xi}(t)\pi_{Xj}(t)\sigma_{X_i}(t)\sigma_{X_j}(t)\rho_{X_iX_j}(t) = \sum_{ij} \pi_{Xi}(t)\pi_{Xj}(t)\sigma_{X_iX_j}(t) \tag{14}$$

where:

$$\sigma_{X_iX_j}(t) = \sigma_{V_iV_j}(t) + \sigma_{\varepsilon_i\varepsilon_j}(t) \tag{15}$$

The portfolio's Excess Growth Rate can be partitioned into the sum of two other excess growth rates, one for the value process, $\gamma_{\pi_X V}^*(t)$, and one for the estimation error process, $\gamma_{\pi_X \varepsilon}^*(t)$ (see the appendix).[14]

$$\gamma_{\pi_X}^*(t) = \gamma_{\pi_X V}^*(t) + \gamma_{\pi_X \varepsilon}^*(t) \tag{16}$$

$$\gamma_{\pi_X V}^*(t) \equiv \frac{1}{2}\left[\sum_i \pi_{Xi}(t)\sigma_{V_iV_i}(t) - \sum_{ij} \pi_{Xi}(t)\pi_{Xj}(t)\sigma_{V_iV_j}(t)\right] \tag{17}$$

$$\gamma_{\pi_X \varepsilon}^*(t) \equiv \frac{1}{2}\left[\sum_i \pi_{Xi}(t)\sigma_{\varepsilon_i\varepsilon_i}(t) - \sum_{ij} \pi_{Xi}(t)\pi_{Xj}(t)\sigma_{\varepsilon_i\varepsilon_j}(t)\right]$$

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13 Yes, it seems weird. Here is some intuition. For log normally distributed stocks and portfolios, the long-term return corresponds to the logarithmic drift rate. This is equal to expected arithmetic return less one-half of variance. Consider an equal weight portfolio of statistically identical and independent stocks. Its expected return is that of its stocks, but its variance is much less. Consequently, its long-term return is much higher than that of its stocks. Note that trading is required to maintain equal weights.

14 This reflects the independence of the value and estimation error processes.

(18)

The portfolio's growth rate can be rewritten as:

$$\gamma_{\pi_X}(t) = \sum_i \pi_{Xi}(t)\gamma_{Vi}(t) + \gamma^*_{\pi_X V}(t) + \gamma^*_{\pi_X \varepsilon}(t) - \gamma_\varepsilon \sum_i \pi_{Xi}(t)\varepsilon_i(t)$$

(19)

Equation (19) shows that the direct impact of estimation error on a portfolio's growth rate is:[15]

$$\gamma^*_{\pi_X \varepsilon}(t) - \gamma_\varepsilon \sum_i \pi_{Xi}(t)\varepsilon_i(t)$$

(20)

The paper presumes long portfolios of many stocks and with all the relevant covariance matrices (value, estimation error, price) nonsingular. In this case, excess growth rates are positive (see Fernholz [2002]). Consequently, the first term in Expression (20) is positive.

The second term (including the minus sign) in Expression (20) can be positive, zero, or negative as the portfolio's weights have a negative, zero, or positive cross sectional correlation with estimation errors at time t. This effect is due to exposure to mispricing corrections.

The net direct impact of estimation error on the portfolio can be negative only if there is a strong enough positive relationship between the portfolio's weights and estimation errors and the mispricing correction rate is fast enough. If the mispricing correction rate is zero, there is a net benefit from estimation errors through the estimation error excess growth rate. However, the estimation error process is unstable if $\gamma_\varepsilon = 0$.

The term $\sum_i \pi_{Xi}(t)\varepsilon_i(t)$ is related to the cross sectional covariance between the portfolio's weights and the estimation errors at a point in time, $\sigma_{\pi_X \varepsilon}(t)$. The paper focuses on what is typical, and it seems reasonable in this case to proxy "typical" by expected value.[16]

[15] This defines the direct impact of estimation error on a portfolio's growth rate.

[16] In log form, time averages are indicative of long-term return, hence expectations provide a measure of typical that is useful.

$$\sigma_{\pi_X \varepsilon}(t) = E\left[\left(\pi_{Xi}(t) - \frac{1}{n}\right)\varepsilon_i(t)\right] = E\left[\pi_{Xi}(t)\varepsilon_i(t)\right] \approx \frac{\sum_i \pi_{Xi}(t)\varepsilon_i(t)}{n}$$

(21)

It is presumed that the number of stocks is large enough so that the approximation is useful. Consequently:

$$-\gamma_\varepsilon \sum_i \pi_{Xi}(t)\varepsilon_i(t) \approx -\gamma_\varepsilon n \sigma_{\pi_X \varepsilon}(t) = -\gamma_\varepsilon n \rho_{\pi_X \varepsilon}(t)\sigma_{\pi_X}(t)\sigma_\varepsilon(t)$$

(22)

According to Expressions (20) and (22), the direct impact of estimation error is approximately:

$$\gamma^*_{\pi_X \varepsilon}(t) - \gamma_\varepsilon n \sigma_{\pi_X \varepsilon}(t) = \gamma^*_{\pi_X \varepsilon}(t) - \gamma_\varepsilon n \rho_{\pi_X \varepsilon}(t)\sigma_{\pi_X}(t)\sigma_\varepsilon(t)$$

(23)

Expression (23) suggests:

- Portfolio weights that are positively correlated with estimation errors suffer a drag from mispricing corrections.

- Presuming a fixed positive correlation between portfolio weights and estimation errors, mispricing correction drag is most likely to offset the gain from estimation error Excess Growth Rate when the mispricing correction rate is large.[17]

- Presuming a fixed positive covariance between portfolio weights and estimation errors, mispricing correction drag is most likely to offset the gain from the estimation error Excess Growth Rate when the number of securities and the mispricing correction rate are large.[18]

- Portfolio weights that are negatively correlated with estimation errors enjoy a gain from mispricing corrections.

The net direct impact of estimation error will be negative if, roughly (see the appendix):

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[17] Other things equal, a fixed correlation between portfolio weights and estimation error implies a decreasing standard deviation of portfolio weights as the number of stocks increases (roughly as 1/n). Therefore, the dependence on the number of stocks is likely to be immaterial.

[18] A fixed covariance strikes the authors as unrealistic.

$$\sigma_{\pi_X \varepsilon}(t) > \frac{\max\left(\frac{\sigma_{\varepsilon_i \varepsilon_i}(t)}{n}\right)}{2\gamma_{\varepsilon}}$$

(24)

This suggests a crucial role for the ratio of portfolio size adjusted estimation error variance to the mispricing correction rate. According to Inequality (24), the required correlation is close to zero if there are a lot of stocks in the portfolio. For portfolios of many stocks, the direct impact of estimation error will be negative if the correlation between the portfolio's weights and estimation error is positive.

The remainder of the paper presumes:

- A common estimation error variance rate.

$$\sigma_{\varepsilon_i \varepsilon_i}(t) = \sigma_{\varepsilon\varepsilon}(t)$$

(25)

- A common value variance rate:

$$\sigma_{V_i V_i}(t) = \sigma_{VV}(t)$$

(26)

With these assumptions, Equations (17), (18), and (19) become:

$$\gamma^*_{\pi_X V}(t) = \frac{1}{2}\left[1 - \sum_{ij} \pi_{Xi}(t)\pi_{Xj}(t)\rho_{V_i V_j}(t)\right]\sigma_v^2(t)$$

(27)

$$\gamma^*_{\pi_X \varepsilon}(t) = \frac{1}{2}\left[1 - \sum_{ij} \pi_{Xi}(t)\pi_{Xj}(t)\rho_{\varepsilon_i \varepsilon_j}(t)\right]\sigma_{\varepsilon}^2(t)$$

(28)

$$\begin{aligned}\gamma_{\pi_X}(t) = &\sum_i \pi_{Xi}(t)\gamma_{Vi}(t) \\ &+\frac{1}{2}\left[1-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\rho_{V_iV_j}(t)\right]\sigma_v^2(t) \\ &+\frac{1}{2}\left[1-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\rho_{\varepsilon_i\varepsilon_j}(t)\right]\sigma_\varepsilon^2(t) \\ &-\gamma_\varepsilon\sum_i \pi_{Xi}(t)\varepsilon_i(t)\end{aligned} \tag{29}$$

From Expressions (22) and (29):

$$\begin{aligned}\gamma_{\pi_X}(t) \approx &\sum_i \pi_{Xi}(t)\gamma_{Vi}(t) \\ &+\frac{1}{2}\left[1-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\rho_{V_iV_j}(t)\right]\sigma_v^2(t) \\ &+\frac{1}{2}\left[1-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\rho_{\varepsilon_i\varepsilon_j}(t)\right]\sigma_\varepsilon^2(t) \\ &-\gamma_\varepsilon n\sigma_{\pi_X\varepsilon}(t)\end{aligned} \tag{30}$$

Under these circumstances, a sufficient condition for negative net direct estimation error impact is:[19]

$$\sigma_{\pi_X\varepsilon}(t) > \frac{\sigma_\varepsilon^2(t)/n}{2\gamma_\varepsilon} \tag{31}$$

Capitalization weighted portfolios

For a capitalization weighted portfolio (see the appendix):

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19 Alternatively, $\rho_{\pi_X\varepsilon}(t) > \frac{1}{2\gamma_\varepsilon}\frac{\sigma_\varepsilon(t)/n}{\sigma_{\pi_X}(t)}$, where, presumably, $\sigma_\varepsilon(t)/n$ is commensurate with $\sigma_{\pi_X}(t)$ as a function of n because the latter declines as roughly 1/n.

$$\sum_i \pi_{X_i}(t)\varepsilon_i(t) = \frac{\sum_i \pi_{V_i}(t) e^{\varepsilon_i(t)} \varepsilon_i(t)}{\sum_j \pi_{V_j}(t) e^{\varepsilon_j(t)}}$$

(32)

Here, π_{Vi} is stock i's weight in a value weighted portfolio of the same stocks.

Equation (32) makes clear that there is, typically, a positive relationship between capitalization weights and estimation errors. This is obvious, since capitalization weights are increased (decreased) by positive (negative) estimation errors.[20] The positive relationship virtually assures that[21]

$$-\gamma_\varepsilon \sum_i \pi_{X_i}(t)\varepsilon_i(t) < 0$$

(33)

Consequently, mispricing corrections typically reduce the capitalization weighted portfolio's growth rate from what it would otherwise be and the reduction is greater the larger the mispricing correction rate. If the mispricing correction rate, γ_ε, is large enough, the mispricing correction drag in Expression (33) will more than offset the estimation error excess growth rate, $\gamma^*_{\pi_X \varepsilon}(t)$, and the capitalization weighted portfolio's growth rate will be less than it would be without estimation errors.

Suppose that $\gamma_\varepsilon = 0$.[22] This corresponds to no estimation error mean reversion to zero, i.e., no mispricing corrections. The mispricing correction rate term is zero and the direct impact of estimation error on the portfolio's growth rate becomes its estimation error excess growth rate, which is positive.[23] Consequently, the capitalization weighted portfolio's growth rate exceeds what it would be without estimation error.

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20 If log value and estimation error are taken to be bivariate normal, then the regression of estimation error on price has a positive slope and the probability of positive estimation error given price above its mean is greater than 0.5.

21 It is possible that all stocks have positive estimation errors or all stocks have negative estimation errors or all large stocks have negative estimation errors and all small stocks have positive estimation errors. However, these and similar outcomes are unlikely.

22 This sounds like an innocuous assumption, but it implies a non stable estimation error process. Therefore, the characterization is not for an indefinite period.

23 Eventually, this kind of estimation error specification drives the market to be dominated by a single stock. As this happens, the estimation error excess growth rate is driven to zero. Therefore, this example behaves as described only for a while.

The intuition for this is that estimation error increases relative volatility among stocks, increasing the excess growth rate, while there is no drag due to mispricing correction.

Gaining more intuition about the impact of estimation errors on a capitalization weighted portfolio's growth rate requires evaluating the right side of Equation (32). The paper uses an approximation approach suggested by the Great Pumpkin.[24] The approximations provide worthwhile insight for broad diversified portfolios, but are rough enough so that caution is advised.

The first approximation is (see the appendix):

$$-\gamma_{\varepsilon}\sum_{i}\pi_{X_i}\left(t\right)\varepsilon_i\left(t\right)\approx -2\gamma_{\varepsilon}\gamma^{*}_{\pi_V\varepsilon}\left(t\right)<0 \tag{34}$$

Here, $\gamma^{*}_{\pi_V\varepsilon}\left(t\right)$ is in the form of $\gamma^{*}_{\pi_X\varepsilon}\left(t\right)$, but with value weights, not capitalization weights. $\gamma^{*}_{\pi_V\varepsilon}\left(t\right)$ is termed the value weighted estimation error excess growth rate.

Expression (34) suggests that for capitalization weighted portfolios, the $-\gamma_{\varepsilon}\sum_{i}\pi_{Xi}\left(t\right)\varepsilon_i\left(t\right)$ term in Equation (19) reduces the portfolio's growth rate from what it would be without estimation error. This reduction is drag due to mispricing corrections.

The second approximation is that $\gamma^{*}_{\pi_X\varepsilon}\left(t\right)$ can be roughly approximated by $\gamma^{*}_{\pi_V\varepsilon}\left(t\right)$ as follows (see the appendix).

$$\gamma^{*}_{\pi_X\varepsilon}\left(t\right)=\frac{1}{2}\left[1-\sum_{ij}\pi_{X_i}\left(t\right)\pi_{X_j}\left(t\right)\rho_{\varepsilon_i\varepsilon_j}\left(t\right)\right]\sigma^2_{\varepsilon}\left(t\right)\approx\frac{1}{2}\left[1-\sum_{ij}\pi_{V_i}\left(t\right)\pi_{V_j}\left(t\right)\rho_{\varepsilon_i\varepsilon_j}\left(t\right)\right]\sigma^2_{\varepsilon}\left(t\right)=\gamma^{*}_{\pi_V\varepsilon}\left(t\right) \tag{35}$$

The approximation in Equation (35) equates the capitalization weighted and value weighted estimation error excess growth rates. This is likely to be a better approximation for these portfolios than if the value weighted portfolio is replaced by one purposely constructed to outperform the capitalization weighted one (e.g., using specific knowledge of the estimation error covariance structure to construct an optimum portfolio).

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[24] Responsibility for any inadequacy of the approximations remains with the Great Pumpkin. Simulations suggest an average percentage error of about 75% with a standard deviation of about 40% in Approximation (34). For example, the approximate half lives of 1.4 years referenced in the paper could be about 2.5 years. The same simulations suggest similar or much smaller average errors and standard deviations of error in related approximations in the paper that use the Great Pumpkin's approach.

The net approximation from Expressions (20), (34), and (35) is:

$$\gamma^{*}_{\pi_X \varepsilon}(t) - \gamma_{\varepsilon} \sum_{i} \pi_{X_i}(t)\varepsilon_i(t) \approx (1 - 2\gamma_{\varepsilon})\gamma^{*}_{\pi_V \varepsilon}(t)$$

(36)

This is such an aesthetic result, that the Great Pumpkin's approximation will be assumed to be worthwhile.[25]

Suppose $\gamma_{\varepsilon} = 1$. This corresponds to a high estimation error mean reversion rate to zero, i.e., a rapid correction of mispricing. Then, from Approximation (36):

$$\gamma^{*}_{\pi_X \varepsilon}(t) - \gamma_{\varepsilon} \sum_{i} \pi_{X_i}(t)\varepsilon_i(t) \approx -\gamma^{*}_{\pi_V \varepsilon}(t) < 0$$

(37)

and the net direct effect of estimation error is negative and the capitalization weighted portfolio's growth rate is less than what it would be without estimation error. The intuition for this is that the impact of rapid mispricing correction represents an insurmountable drag that the positive impact of estimation error volatility on excess growth rate cannot offset.

From Approximation (36), the mispricing correction rate that leaves no direct estimation error impact on a broad capitalization weighted portfolio's growth rate is approximately:

$$(1 - 2\gamma_{\varepsilon})\gamma^{*}_{\pi_V \varepsilon}(t) = 0$$

(38)

$$\gamma_{\varepsilon} \approx 0.5$$

(39)

This mispricing correction rate corresponds to a half-life of about 1.4 years and about a 90% correction in five years.

Estimation error neutral portfolio weights

Portfolio weights such that the direct impact of estimation error is zero are defined as estimation-error neutral. Thus, from Expression (20), for estimation-error neutral portfolio weights:

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[25] Why bother with reality if it gets in the way of beauty.

$$\gamma^*_{\pi_X \varepsilon}(t) - \gamma_\varepsilon \sum_i \pi_{Xi}(t)\varepsilon_i(t) = 0$$

(40)

This produces a portfolio growth rate that approximates what it would be without estimation error.[26] Since $\gamma^*_{\pi_X \varepsilon}(t) > 0$, this requires, roughly, portfolio weights that have the following positive covariance with estimation error (see the appendix). Substituting from Expressions (21) and (28) into Equation (40):

$$\sigma_{\pi_X \varepsilon}(t) \approx \frac{\gamma^*_{\pi_X \varepsilon}(t)}{n\gamma_\varepsilon} \approx \frac{1}{2}\left[1 - \sum_{ij} \pi_{V_i}(t)\pi_{V_j}(t)\rho_{\varepsilon_i \varepsilon_j}(t)\right]\left(\frac{\sigma^2_\varepsilon(t)/n}{\gamma_\varepsilon}\right)$$

(41)

Equation (41) shows that the positive covariance required for estimation error neutrality depends importantly on the rapidity of mispricing corrections in relation to portfolio size adjusted estimation error variance. The required covariance is positively related to the estimation error variance and negatively related to the number of stocks in the portfolio and the mispricing correction rate. Estimation error neutral weights are approximately uncorrelated weights when the number of stocks is large.

Whether or not estimation error neutral weights tend to provide a higher or lower growth rate than capitalization weights also depends importantly on the rapidity of mispricing corrections in relation to the estimation error variance. Since the impact of the mispricing correction rate on capitalization weighted portfolios is negative, a large mispricing correction rate, γ_ε, tends to provide an advantage to estimation error neutral weights.

If the only material impact on the difference in growth rates between an estimation error neutral portfolio and a capitalization weighted portfolio is due to the $\gamma^*_{\pi_X \varepsilon}(t) - \gamma_\varepsilon \sum_i \pi_{Xi}(t)\varepsilon_i(t)$ term, then, from Expressions (36) and (40), the differential growth rate is approximately:

$$\gamma^*_{\pi_{EEN}}(t) - \gamma^*_{\pi_X}(t) \approx -(1 - 2\gamma_\varepsilon)\gamma^*_{\pi_V \varepsilon}(t)$$

(42)

This suggests that mispricing correction rates above about 0.5 give the advantage to estimation error neutral portfolio weights.

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26 The estimation errors in the portfolio weights typically ought not to matter much for the remaining terms in the portfolio's growth rate formula, $\sum_i \pi_{Xi}(t)\gamma_{Vi}(t)$ and $\gamma^*_{\pi_X V}(t)$.

Even though estimation error neutral weights eliminate most of the impact of estimation errors on the portfolio's growth rate, that does not imply that estimation error neutral weights will correspond to a higher or lower growth rate compared to particular other portfolios. This is because changing a portfolio's weights changes all the terms in the formula for a portfolio's growth rate, not just the ones that are directly impacted by estimation errors.

Market value neutral portfolio weights.

Some portfolio weights have a zero correlation with estimation errors. Such weights are termed "market value neutral portfolio weights" and are related to the "market value indifferent" weights analyzed in Treynor [2005]. For market value neutral weights, Expression (20) shows that the estimation error impact on the portfolio's growth rate is:

$$\gamma^*_{\pi_X\varepsilon}(t)-\gamma_\varepsilon\sum_i \pi_{Xi}(t)\varepsilon_i(t)=\gamma^*_{\pi_X\varepsilon}=\frac{1}{2}\left[1-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\rho_{\varepsilon_i\varepsilon_j}(t)\right]\sigma^2_\varepsilon(t)>0 \qquad (43)$$

The Great Pumpkin's approximation procedure suggests, using Approximation (35), that in many cases:

$$\gamma^*_{\pi_X\varepsilon}(t)-\gamma_\varepsilon\sum_i \pi_{Xi}(t)\varepsilon_i(t)\approx\gamma^*_{\pi_V\varepsilon}>0 \qquad (44)$$

Market value neutral portfolios do not suffer from mispricing correction drag yet do gain from the estimation error excess growth rate.

Two examples of market value neutral portfolios are value weighted portfolios and equal weight portfolio. Other examples (in an expected sense) are randomly chosen portfolios, and all portfolio weights that are functions of quantities that are unrelated to estimation errors.

Approximations (36) and (44) suggest that the approximate difference between market value neutral weighted and capitalization weighted portfolios is:

$$\gamma^*_{\pi_{MVN}}(t)-\gamma^*_{\pi_X}(t)\approx\gamma^*_{\pi_V\varepsilon}(t)-(1-2\gamma_\varepsilon)\gamma^*_{\pi_V\varepsilon}(t)=2\gamma_\varepsilon\gamma^*_{\pi_V\varepsilon}(t)>0 \qquad (45)$$

This suggests that a market value neutral portfolio can be expected to have a higher growth rate than a capitalization weighted portfolio.

An interesting implication of the foregoing is that all market value neutral portfolios experience the same expected mispricing correction drag advantage over their capitalization weighted counterparts.

Under atypical circumstances, estimation errors could make the mispricing correction rate contribute positively to a capitalization weighted portfolio, e.g., if small value firms had positive errors and large value firms had negative errors. For all practical purposes this possibility can be ignored and Expression (45) used as a guide.

Fundamentally weighted portfolios

Fundamental portfolio weights, introduced in Arnott [2005], are weights that are functions of fundamental quantities such as book value, cash flow, dividends, or sales. Fundamental weights may be estimation error neutral, market value neutral, or neither. Whether a particular fundamental weight scheme meets either criterion may vary over time.

Fundamental weights need not be good indications of value. Fundamental weights that are proportional to book value cannot discriminate between profitable firms and unprofitable firms. Fundamental weights will tend to underweight high value stocks and overweight low value stocks. A $1.00 investment in the Fundamentally-weighted portfolio could buy much less than $1.00 of value, possibly even less than a capitalization weighted portfolio.

Fundamental weights also could be more positively correlated with estimation errors than capitalization weights. Suppose that large stocks tend to have lower profitability than small stocks and that value per dollar of book value is positively related to profitability. Then Fundamental weights that are proportional to book value will tend to overweight large stocks and underweight small stocks relative to a value weighted portfolio. Thus, Fundamental weights will be positively correlated with estimation error and probably more so than capitalization weights, since the latter reflect profitability.

To the extent that fundamental weights have a lower correlation with estimation error than capitalization weights, Fundamental weights may have a higher growth rate than capitalization weights.

If fundamental weights reflect value, then they may be approximately market value neutral. Market value neutrality does not require systematically overweighting (underweighting) a capitalization weighted benchmark's smaller (larger) stocks. Market value neutrality can be achieved by overweighting (underweighting) undervalued (overvalued) stocks, relative to capitalization weights, throughout the capitalization weight domain. This is classified as security selection in many performance attribution schemes.

Diversity Weighted Portfolios

Fernholz [2002] defines a passive portfolio termed a Diversity Weighted Portfolio. A Diversity Weighted Portfolio's weights are functions of an associated capitalization weighted benchmark's weights. Given the benchmark's weights, the Diversity Weighted Portfolio's weights are determined. The Diversity Weighted Portfolio's weights are:

$$\pi_{pi} \equiv \frac{\mu_i^p}{\sum_j \mu_j^p}$$

(46)

$\pi_{pi} \equiv$ Stock i's weight in the Diversity Weighted Portfolio.

$\mu_i \equiv$ Stock i's weight in the capitalization weighted benchmark.

$p \equiv$ A parameter between 0 and 1.

When p=1, the Diversity Weighted Portfolio is the capitalization weighted benchmark. When p=0, the Diversity Weighted Portfolio is an equal weight portfolio.

A Diversity Weighted Portfolio's weights as a percentage of its benchmark weights monotonically decrease with increases in its benchmark weights. The benchmark's larger stocks are underweighted and its smaller stocks are overweighted.

Fernholz shows that a Diversity Weighted Portfolio's relative return can be perfectly attributed to the change in the benchmark's Diversity (a function that measures the extent to which capital is spread across the benchmark's stocks) and a positive drift termed the Kinetic Differential. The Diversity function, $D_p(\mu_i)$, is symmetric and concave.

$$D_p(\mu) \equiv \left(\sum_i \mu_i^p \right)^{\frac{1}{p}}$$

(47)

The Diversity Weighted Portfolio's relative return is given by:[27]

$$d \ln\left(\frac{Z_{\pi_p}}{Z_\mu} \right) = d \ln\left(D_p(\mu) \right) + (1-p)\gamma^*_{\pi_p}$$

(48)

Here, $\gamma^*_{\pi_p}$ is the Diversity Weighted Portfolio's Excess Growth Rate, which is positive.

27 There is an additional drag called leakage for open systems of stocks where stocks can enter or leave the benchmark.

Fernholz shows that Diversity is bounded, hence a Diversity Weighted Portfolio outperforms its benchmark over time.

A Diversity Weighted Portfolio's weights typically are positively correlated with estimation error with estimation error for p>0 when capitalization weights are positively correlated with estimation error. For p=0, the expected correlation is zero, in which case the Diversity Weighted Portfolio is the equal weight Market Value Neutral portfolio.

In the context of this paper, a Diversity Weighted Portfolio's weights tend to have a lower correlation with estimation error than a capitalization weighted portfolio when p<1. This implies less expected mispricing correction drag than its capitalization weighted benchmark. Thus, this paper's model also suggests that a Diversity Weighted Portfolio has a higher growth rate than its capitalization weighted benchmark.

A theoretical size effect

The mispricing correction impact on a portfolio's growth rate is given by:

$$-\gamma_{\varepsilon}\sum_{i}\pi_{Xi}(t)\varepsilon_{i}(t)$$

(49)

The expected value of this term is:

$$E\left[-\gamma_{\varepsilon}\sum_{i}\pi_{Xi}(t)\varepsilon_{i}(t)\right]\approx -n\gamma_{\varepsilon}\sigma_{\pi_{X}\varepsilon}(t)$$

(50)

For capitalization weights, $\sigma_{\pi_X\varepsilon}(t) > 0$ and the implied impact is negative. Alternatively, the capitalization weight approximation is:

$$-\gamma_{\varepsilon}\sum_{i}\pi_{Xi}(t)\varepsilon_{i}(t)\approx -2\gamma_{\varepsilon}\gamma^{*}_{\pi_{V}\varepsilon}(t) < 0$$

(51)

This also is negative.

This suggests that portfolio weights with a lower correlation with estimation errors than capitalization weights typically enjoy a growth rate advantage.

Portfolios that overweight smaller stocks relative to capitalization weights will tend to have a smaller correlation between portfolio weights and estimation errors than capitalization weights. If so, then mispricing correction drag typically will be less for portfolios that overweight smaller stocks relative to capitalization weighted portfolios. This theoretical size effect partially explains the traditional size effect noted by others, e.g., Fama and French [1992].

Arnott's Fundamental Indexes somewhat overweight their capitalization weighted benchmark's smaller stocks. Moreover, they may often approximate market value neutral weights. This suggests that there is often a positive impact of estimation errors on the difference between the returns of Arnott's Fundamental Indexes and capitalization weighted portfolios due to differential mispricing correction drag.

Conclusion

Estimation errors effect a portfolio's growth rate in two ways, through mispricing corrections and estimation error volatility, the latter through its Excess Growth Rate. Mispricing corrections are a component of stocks' growth rates, hence effect a portfolio's weighted average stock growth rate. The impact of mispricing corrections on a portfolio's weighted average stock growth rate can be positive or negative. Estimation error volatility effects a portfolio's Excess Growth Rate. The impact is positive for long portfolios.

There exist some portfolio weights where the direct effect of mispricing corrections and estimation error volatility on the portfolio's growth rate net to zero. These weights are termed "estimation error neutral weights". Roughly, estimation error neutrality requires that a portfolio's weights be positively correlated with estimation error. The required positive covariance is large to the extent that estimation error variance is large in relation to the mispricing correction rate and the number of stocks is small. For portfolios of many stocks and reasonable estimation error characteristics, the required positive correlation is close to zero.

A rough approximation is that estimation error neutral weights will tend to have higher growth rates than capitalization weights if the half life of mispricing corrections is shorter than 1.4 years.

Capitalization weighted portfolios suffer drag from mispricing corrections and gain from estimation error volatility's contribution to their Excess Growth Rates. The net effect depends on the balance between estimation error volatility and the speed of mispricing corrections. A relatively high (low) rate of mispricing corrections leads to a net negative (positive) impact. A rough approximation is that the net direct impact of estimation errors on a capitalization weighted portfolio's growth rate is negative if estimation error corrections have a half life of less than 1.4 years.

Relative to value weighted portfolios, capitalization weighted portfolios tend to suffer drag from mispricing corrections and may or may not have higher Excess Growth Rates. A rough approximation is that value weighted and capitalization weighted portfolios of the same stocks have similar Excess Growth Rates. If so, value weighted portfolios tend to have higher growth rates than capitalization weighted portfolios due to less mispricing correction drag.

There also exist portfolio weights that are independent of estimation error, in the sense of a zero cross sectional (point in time) correlation between portfolio weights and estimation errors. Randomly chosen weights are an example. Such weights are termed “market value neutral” weights and are related to what Treynor [2005] refers to as “market value indifferent” weights. Market value neutral portfolio weights eliminate the drag due to mispricing corrections that capitalization weights suffer and often have similar Excess Growth Rates. Consequently, many market value neutral weights have higher growth rates than capitalization weights.

Arnott’s Fundamental Indexes have weights that are functions of fundamental quantities, such as book value. In general, fundamental weights need not be estimation error neutral or market value neutral. To the extent they are less correlated with estimation error than capitalization weights, they probably have a higher growth rate than capitalization weights. This should be true for choices of fundamental quantities that reflect value because then the Fundamental weights probably are approximately market value neutral.

Fernholz’s Diversity Weighted Portfolios overweight (underweight) a capitalization weighted benchmark’s smaller (larger) stocks, hence typically have a lower correlation with estimation error than capitalization weights. In the context of this paper’s model, this implies a higher growth rate. This is consistent with Fernholz’s mathematical proof, using Stochastic Portfolio Theory, that Diversity Weighted Portfolios outperform their capitalization weighted benchmarks over time.

Portfolios that overweight smaller stocks relative to capitalization weights will tend to have a smaller correlation between portfolio weights and estimation errors than capitalization weights. If so, then mispricing correction drag typically will be less for portfolios that overweight smaller stocks relative to capitalization weighted portfolios. This theoretical size effect partially explains the traditional size effect noted by others, e.g., Fama and French [1992].

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Appendix

The portfolio's price process is developed as follows.

Equation (10) implies:[28]

$$\frac{dX_i(t)}{X_i(t)} = \left[\gamma_{Xi}(t) + \frac{\sigma^2_{X_i}(t)}{2}\right]dt + \sigma_{X_i}(t)\,dW_{Xi}(t)$$

(52)

For a portfolio:

$$\frac{dZ_{\pi_X}(t)}{Z_{\pi_X}(t)} = \sum_i \pi_{Xi}(t)\frac{dX_i(t)}{X_i(t)}$$

(53)

Transforming to logarithms:[29]

$$d\log Z_{\pi_X}(t) + \frac{\sigma^2_{Z_{\pi_X}}(t)}{2}dt = \sum_i \pi_{Xi}(t)\left[d\log X_i(t) + \frac{\sigma^2_{X_i}(t)}{2}dt\right]$$

(54)

$$d\log Z_{\pi_X}(t) = \sum_i \pi_{Xi}(t)\,d\log X_i(t) + \gamma^*_{\pi_X}(t)\,dt$$

(55)

$$\gamma^*_{\pi_X}(t) = \frac{1}{2}\left(\sum_i \pi_{Xi}(t)\sigma_{X_iX_i}(t) - \sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\sigma_{X_iX_j}(t)\right)$$

(56)

$\gamma^*_{\pi_X}(t)$ is termed the portfolio's excess growth rate (Fernholz [2002]).

$$d\log Z_{\pi_X}(t) = \sum_i \pi_{Xi}(t)\gamma_{Xi}(t)\,dt + \gamma^*_{\pi_X}(t)\,dt + \sum_i \pi_{Xi}(t)\sigma_{X_i}(t)\,dW_{Xi}(t)$$

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28 Use Ito's lemma to transform from log X to X.

29 Use Ito's lemma to transform from Y to log Y.

(57)

$$d\log Z_{\pi_X}(t)=\left[\sum_i \pi_{Xi}(t)\gamma_{Vi}(t)-\gamma_\varepsilon\sum_i \pi_{Xi}(t)\varepsilon_i(t)+\gamma^*_{\pi_X}(t)\right]dt+\sum_i \pi_{Xi}(t)\sigma_{X_i}(t)dW_{Xi}(t)$$

(58)

The portfolio's excess growth rate process can be developed as follow.

$$\gamma^*_{\pi_X}(t)=\frac{1}{2}\left[\sum_i \pi_{Xi}(t)\left(\sigma_{V_iV_i}(t)+\sigma_{\varepsilon_i\varepsilon_i}(t)\right)-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\left(\sigma_{V_iV_j}(t)+\sigma_{\varepsilon_i\varepsilon_j}(t)\right)\right]$$

(59)

$$\gamma^*_{\pi_X}(t)=\frac{1}{2}\left[\sum_i \pi_{Xi}(t)\sigma_{V_iV_i}(t)-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\sigma_{V_iV_j}(t)\right]+\frac{1}{2}\left[\sum_i \pi_{Xi}(t)\sigma_{\varepsilon_i\varepsilon_i}(t)-\sum_{ij}\pi_{Xi}(t)\pi_{Xi}(t)\sigma_{\varepsilon_i\varepsilon_j}(t)\right]$$

(60)

This shows that the portfolio's excess growth rate can be partitioned into two other excess growth rates, one for the value process, $\gamma^*_{\pi_X V}(t)$, and one for the estimation error process, $\gamma^*_{\pi_X\varepsilon}(t)$. These excess growth rate components are defined as follows.

$$\gamma^*_{\pi_X V}(t)\equiv\frac{1}{2}\left[\sum_i \pi_{Xi}(t)\sigma_{V_iV_i}(t)-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\sigma_{V_iV_j}(t)\right]$$

(61)

$\gamma^*_{\pi_X V}(t)$ is termed the portfolio's value excess growth rate.

$$\gamma^*_{\pi_X\varepsilon}(t)\equiv\frac{1}{2}\left[\sum_i \pi_{Xi}(t)\sigma_{\varepsilon_i\varepsilon_i}(t)-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\sigma_{\varepsilon_i\varepsilon_j}(t)\right]$$

(62)

$\gamma^*_{\pi_X\varepsilon}(t)$ is termed the portfolio's estimation error excess growth rate.

$$\gamma^*_{\pi_X}(t)=\gamma^*_{\pi_X V}(t)+\gamma^*_{\pi_X\varepsilon}(t)$$

(63)

The approximate necessary condition for the net direct impact of estimation error to be negative is developed as follows.

In view of Equation (18), the estimation error Excess Growth Rate term in Expression (23) cannot exceed $\max\left(\sigma_{\varepsilon_i\varepsilon_i}(t)\right)/2$. Therefore, the net direct impact of estimation error will be negative if, roughly:

$$\frac{\max\left(\sigma_{\varepsilon_i\varepsilon_i}(t)\right)}{2} - \gamma_\varepsilon n \sigma_{\pi_X \varepsilon}(t) < 0$$

(64)

$$\gamma_\varepsilon \sigma_{\pi_X \varepsilon}(t) > \frac{1}{2}\max\left(\frac{\sigma_{\varepsilon_i\varepsilon_i}(t)}{n}\right)$$

(65)

$$\sigma_{\pi_X \varepsilon}(t) > \frac{\max\left(\frac{\sigma_{\varepsilon_i\varepsilon_i}(t)}{n}\right)}{2\gamma_\varepsilon}$$

(66)

For a capitalization weighted portfolio:

$$\pi_{Xi}(t) = \frac{X_i(t)}{\sum_j X_j(t)} = \frac{X_i(t)}{Z_{\pi_X}(t)}$$

(67)

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) = \sum_i \left(\frac{X_i(t)}{Z_{\pi_X}(t)}\right)\varepsilon_i(t) = \sum_i \frac{e^{\log X_i(t)}}{\sum_j e^{\log X_j(t)}}\varepsilon_i(t) = \sum_i \frac{e^{\log V_i(t)+\varepsilon_i(t)}}{\sum_j e^{\log V_j(t)+\varepsilon_j(t)}}\varepsilon_i(t)$$

(68)

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) = \sum_i \frac{V_i(t)e^{\varepsilon_i(t)}\varepsilon_i(t)}{\sum_j V_j(t)e^{\varepsilon_j(t)}}$$

(69)

Dividing both top and bottom by $\sum_k V_k(t)$:

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) = \frac{\sum_i \pi_{Vi}(t)e^{\varepsilon_i(t)}\varepsilon_i(t)}{\sum_j \pi_{Vj}(t)e^{\varepsilon_j(t)}} \tag{70}$$

Here, π_{Vi} is stock i's weight in a value weighted portfolio of the same stocks.

The approximation used to evaluate the capitalization weighted portfolio's growth rate is developed as follows.

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) = \frac{\sum_i \pi_{Vi}(t)e^{\varepsilon_i(t)}\varepsilon_i(t)}{\sum_j \pi_{Vj}(t)e^{\varepsilon_j(t)}} \approx \frac{\sum_i \pi_{Vi}(t)\left(1+\varepsilon_i(t)\right)\varepsilon_i(t)}{\sum_j \pi_{Vj}(t)\left(1+\varepsilon_j(t)\right)} \tag{71}$$

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) \approx \frac{\sum_i \pi_{Vi}(t)\left(1+\varepsilon_i(t)\right)\varepsilon_i(t)}{\sum_j \pi_{Vj}(t) + \sum_j \pi_{Vj}(t)\varepsilon_j(t)} = \frac{\sum_i \pi_{Vi}(t)\left(1+\varepsilon_i(t)\right)\varepsilon_i(t)}{1+\sum_j \pi_{Vj}(t)\varepsilon_j(t)} \tag{72}$$

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) \approx \sum_i \pi_{Vi}(t)\left(1+\varepsilon_i(t)\right)\varepsilon_i(t)\left(1-\sum_j \pi_{Vj}(t)\varepsilon_j(t)\right) \tag{73}$$

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) \approx \sum_i \pi_{Vi}(t)\left(\varepsilon_i(t)+\varepsilon_i^2(t)\right)\left(1-\sum_j \pi_{Vj}(t)\varepsilon_j(t)\right) \tag{74}$$

Note that the term $\sum_j \pi_{Vj}(t)\varepsilon_j(t)$ is likely to be much smaller than 1, as it is the weighted average of a large number of typically weakly related mean zero error terms. Roughly speaking, its variance is commensurate with the variance of the average error, which is inversely proportional to the number of stocks.

Proxy what is typical with expected values and drop terms of higher order than two.

$$E\left[\sum_i \pi_{Xi}(t)\varepsilon_i(t)\right] \approx \sum_i \pi_{Vi}(t)\sigma_{\varepsilon\varepsilon}(t) - \sum_{ij}\pi_{Vi}(t)\pi_{Vj}(t)\rho_{\varepsilon_i\varepsilon_j}\sigma_{\varepsilon\varepsilon} = 2\gamma^*_{\pi_V\varepsilon}(t)$$

(75)

This suggests the following approximation as roughly typical.

$$-\gamma_\varepsilon \sum_i \pi_{Xi}(t)\varepsilon_i(t) \approx -2\gamma_\varepsilon \gamma^*_{\pi_V\varepsilon}(t) < 0$$

(76)

Here, $\gamma^*_{\pi_V\varepsilon}$ is in the form of $\gamma^*_{\pi_X\varepsilon}(t)$, but with value weights, not capitalization weights. $\gamma^*_{\pi_V\varepsilon}$ is termed the value weighted estimation error excess growth rate.

The approximation for $\gamma^*_{\pi_X\varepsilon}(t)$ is developed as follows.

$$E\left(\pi_{Xi}(t)\pi_{Xj}(t)\right) \approx \pi_{Vi}(t)\pi_{Vj}(t)\left(1+\sum_{kl}\pi_{Vk}(t)\pi_{Vl}(t)\sigma_{\varepsilon_k\varepsilon_l} - 2\sum_k \pi_{Vk}(t)\sigma_{\varepsilon_k\varepsilon_i} - 2\sum_k \pi_{Vk}(t)\sigma_{\varepsilon_k\varepsilon_j} + \sigma_{\varepsilon_i\varepsilon_j}\right)$$

(77)

The term $\sum_{kl}\pi_{Vk}(t)\pi_{Vl}(t)\sigma_{\varepsilon_k\varepsilon_l}$ is of the form of a portfolio variance where estimation error plays the part of return. Unlike the factors that drive stock prices and lead to systematic risk being a large part of a portfolio's risk, estimation error describing relative returns should be less correlated. If so, then this term is likely to be small enough to ignore. Similarly, the remaining terms, except for $\sigma_{\varepsilon_i\varepsilon_j}$ may be ignorable. The latter term is likely be well within the range -0.1 to 0.1, so probably is ignorable, too.[30] If so:

$$\gamma^*_{\pi_X\varepsilon}(t) = \frac{1}{2}\left[1-\sum_{ij}\pi_{Xi}(t)\pi_{Xj}(t)\rho_{\varepsilon_i\varepsilon_j}(t)\right]\sigma^2_\varepsilon(t) \approx \frac{1}{2}\left[1-\sum_{ij}\pi_{Vi}(t)\pi_{Vj}(t)\rho_{\varepsilon_i\varepsilon_j}(t)\right]\sigma^2_\varepsilon(t)$$

(78)

The approximate covariance between a portfolio's weights and estimation error required for estimation error neutrality is developed as follows.

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30 A typical stock's total standard deviation is about 0.35, hence its variance is about 0.1225. Broad equity benchmarks typical standard deviations approximate 0.15, or a variance of 0.0225. Since the average stock's beta is about 1, this suggests that the typical stock's non-systematic variance is about 0.1225-0.0225=0.1. This should be a ballpark estimate of $\sigma_{\varepsilon_i\varepsilon_i}$. $\sigma_{\varepsilon_i\varepsilon_j}$ should average considerably less than this in magnitude and could easily average about zero.

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) = \frac{\gamma^*_{\pi_X\varepsilon}(t)}{\gamma_\varepsilon}$$

(79)

$$\sum_i \pi_{Xi}(t)\varepsilon_i(t) \approx n\sigma_{\pi_X\varepsilon}(t)$$

(80)

$$\sigma_{\pi_X\varepsilon}(t) \approx \frac{\gamma^*_{\pi_X\varepsilon}(t)}{n\gamma_\varepsilon} = \frac{1}{2}\left[1 - \sum_{ij} \pi_{Xi}(t)\pi_{Xj}(t)\rho_{\varepsilon_i\varepsilon_j}(t)\right]\left(\frac{\sigma^2_\varepsilon(t)/n}{\gamma_\varepsilon}\right)$$

(81)

$$\sigma_{\pi_X\varepsilon}(t) \approx \frac{\gamma^*_{\pi_X\varepsilon}(t)}{n\gamma_\varepsilon} = \frac{1}{2}\left[1 - \sum_{ij} \pi_{Vi}(t)\pi_{Vj}(t)\rho_{\varepsilon_i\varepsilon_j}(t)\right]\left(\frac{\sigma^2_\varepsilon(t)/n}{\gamma_\varepsilon}\right)$$

(82)