Probabilistic Aspects of Arbitrage

Daniel Fernholz and Ioannis Karatzas

Abstract Consider the logarithm $\log(1/U(T,\mathbf{z}))$ of the highest return on investment that can be achieved relative to a market with Markovian weights, over a given time-horizon $[0,T]$ and with given initial market weight configuration $\mathscr{X}(0)=\mathbf{z}$. We characterize this quantity *(i)* as the smallest amount of relative entropy with respect to the *Föllmer exit measure*, under which the market weight process $\mathscr{X}(\cdot)$ is a diffusion with values in the unit simplex Δ and the same covariance structure but zero drift; and *(ii)* as the smallest "total energy" expended during $[0,T]$ by the respective drift, over a class of probability measures which are absolutely continuous with respect to the exit measure and under which $\mathscr{X}(\cdot)$ stays in the interior Δ^o of the unit simplex at all times, almost surely. The smallest relative entropy, or total energy, corresponds to the conditioning of the exit measure on the event $\{\mathscr{X}(t)\in\Delta^o,\ \forall\ 0\le t\le T\}$; whereas, under this "minimal energy" measure, the portfolio $\widehat{\pi}(\cdot)$ generated by the function $U(\cdot,\cdot)$ has the numéraire and relative log-optimality properties. This same portfolio $\widehat{\pi}(\cdot)$ also attains the highest possible relative return on investment with respect to the market.

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1 Introduction

The pioneering work of Fernholz [6] demonstrated that, under appropriate conditions, it is possible systematically to outperform a market portfolio over sufficiently long time horizons. Since then there has been an effort to understand various aspects of such "relative arbitrage": its implementation on arbitrary time horizons, the nature and behavior of portfolios that implement it, pricing and hedging in the context of strict local martingale deflators, etcetera. These efforts are summarized in the survey paper [7].

The fairly recent article [5] addresses an issue that arises naturally in this context: if arbitrage exists on a given time-horizon relative to a given market, what is the "best possible" arbitrage of this type? how can it be characterized? what portfolio(s) implement it? Within a Markovian weight model and under suitable regularity conditions, it was shown in [5] that the reciprocal $U(T,\mathbf{z}) \in (0,1]$ of the highest relative return on investment, that can be achieved with respect to the market over a given time-horizon $[0,T]$ and with initial market weight configuration $\mathscr{X}(0) = \mathbf{z}$ in the interior Δ^o of the unit simplex Δ, is equal to the probability under an auxiliary measure $\widehat{\mathbb{Q}}$ that the process of relative market weights $\mathscr{X}(\cdot)$ stays in Δ^o throughout the time interval $[0,T]$. The probability measure $\widehat{\mathbb{Q}}$ is the celebrated *Föllmer exit measure* [9], [10]: the original measure $\mathbb{P}$ is absolutely continuous with respect to it; it renders $\mathscr{X}(\cdot)$ a $\Delta-$valued diffusion with the same covariance structure but zero drift, thus a *martingale;* and it anoints $\mathscr{X}(\cdot)$ with the "numéraire property", of a portfolio that cannot be outperformed. It was also shown in [5] that $U(\cdot\,,\cdot)$ satisfies a linear, second-order partial differential equation on $(0,\infty) \times \Delta^o$, determined entirely on the basis of the covariance structure of the underlying model. The function $U(\cdot\,,\cdot)$ is the smallest nonnegative supersolution of this equation subject to $U(0,\cdot) \equiv 1$ on Δ^o, and the portfolio $\widehat{\pi}(\cdot)$ which attains this best possible relative arbitrage is the one generated by this function in the sense of Fernholz [6], p. 46.

We show here that this portfolio $\widehat{\pi}(\cdot)$ has *itself* the numéraire and log-optimality properties, but under *yet another* probability measure $\mathbb{P}_\star(\cdot) = \widehat{\mathbb{Q}}(\cdot \,|\, \mathscr{X}(t) \in \Delta^o,\ \forall\ 0 \le t \le T)$, the conditioning of the Föllmer measure on the event that the boundary of the simplex is not attained by the process $\mathscr{X}(\cdot)$ during $[0,T]$. This $\mathbb{P}_\star$ has a further characterization as the probability measure whose associated drift $\vartheta^{\mathbb{P}_\star}(\cdot)$ keeps the process $\mathscr{X}(\cdot)$ inside of Δ^o throughout $[0,T]$ with the smallest amount of "total energy", or effort, $(1/2)\, \mathbb{E}^{\mathbb{P}} \int_0^T \|\vartheta^{\mathbb{P}}(t)\|^2 \, dt$, over all probability measures $\mathbb{P} \ll \widehat{\mathbb{Q}}$ with $\mathbb{P}(\mathscr{X}(t) \in \Delta^o,\ \forall\ 0 \le t \le T) = 1$; it is also the measure with the smallest relative entropy $H_T(\mathbb{P} \,|\, \widehat{\mathbb{Q}})$ over this same class, and this smallest value is the logarithm $\log(1/U(T,\mathbf{z}))$ of the highest achievable return on investment relative to the market. We call this measure $\mathbb{P}_\star$ "minimal energy measure".

The paper is organized as follows. Sections 2 and 3 provide necessary background material and set up the model. Section 4 introduces the notions of relative arbitrage and of the minimal capital necessary to implement it, as well as of the Föllmer exit measure and its properties. We also describe in section 4 the analytical properties of the function $U(\cdot\,,\cdot)$, and of the "optimal arbitrage" portfolio $\widehat{\pi}(\cdot)$ generated by it. The conditional measure $\mathbb{P}_\star$ is introduced in section 5, and under it the numéraire and relative log-optimality properties of $\widehat{\pi}(\cdot)$ are developed. Section 6 establishes the minimum relative entropy and stochastic control characterizations of $\mathbb{P}_\star$, building on a Schrödinger-type equation satisfied by the function $\log(1/U(\cdot,\cdot))$ and on the relation of this equation to one of the Hamilton-Jacobi-Bellman (HJB) type. We conclude in section 7 with several additional characterizations of the optimal arbitrage function, including one that involves a zero-sum stochastic game, and with a compilation of results.

2 Preliminaries

On a given filtered probability space $(\Omega,\mathscr{F},\mathbb{P})$, $\mathbb{F}=\{\mathscr{F}(t)\}_{0\le t<\infty}$ we consider a vector $\mathfrak{X}(\cdot)=\big(X_1(\cdot),\cdots,X_n(\cdot)\big)'$ of strictly positive semimartingales which represent the capitalizations of various assets in an equity market. We denote by $X(\cdot):=X_1(\cdot)+\cdots+X_n(\cdot)$ the total market capitalization, and by $Z_1(\cdot):=X_1(\cdot)/X(\cdot),\cdots,$ $Z_n(\cdot):=X_n(\cdot)/X(\cdot)$ the corresponding relative weights of the individual assets. The vector $\mathscr{Z}(\cdot)=\big(Z_1(\cdot),\cdots,Z_n(\cdot)\big)'$ of these weights is a semimartingale with values in the interior Δ^o of the simplex $\Delta:=\big\{\mathbf{z}=(z_1,\cdots,z_n)'\in[0,1]^n : \sum_{i=1}^n z_i=1\big\}$ in $n-1$ dimensions; we shall let $\Gamma:=\Delta\setminus\Delta^o$ denote the boundary of Δ. We shall take throughout $\mathscr{F}(0)=\{\emptyset,\Omega\}$ mod. $\mathbb{P}$.

One way to think of investment in such a market is in terms of selecting a *portfolio* $\pi(\cdot)=(\pi_1(\cdot),\cdots,\pi_n(\cdot))'$, i.e., an $\mathbb{F}-$predictable process with values in $\big\{\mathbf{z}\in\mathbb{R}^n : \sum_{i=1}^n z_i=1\big\}$, such that $\pi_i(\cdot)/X_i(\cdot)$ is an admissible integrand for the semimartingale $X_i(\cdot)$; then $\pi_i(t)$ stands for the proportion of wealth that gets invested at time $t>0$ in the i^{th} asset, for each $i=1,\cdots,n$. With this interpretation, the dynamics of the *wealth process* $V^{w,\pi}(\cdot)$, corresponding to portfolio $\pi(\cdot)$ and initial wealth $w\in(0,\infty)$, are given by

$$\frac{dV^{w,\pi}(t)}{V^{w,\pi}(t)}=\sum_{i=1}^n \pi_i(t)\,\frac{dX_i(t)}{X_i(t)}\,,\qquad V^{w,\pi}(0)=w\,. \tag{1}$$

The collection of all portfolios will be denoted by Π. Portfolios that take values in Δ will be called "long-only"; the most conspicuous of these is the *market portfolio* $\mathscr{Z}(\cdot)$, which actually takes values in Δ^o and generates wealth $V^{w,\mathscr{Z}}(\cdot)=wX(\cdot)/X(0)$ proportional to the total market capitalization at all times.

If one is interested in performance relative to the market, it makes sense to consider for any given portfolio $\pi(\cdot)\in\Pi$ the ratio, or relative performance,

$$Y^{q,\pi}(\cdot):=V^{qX(0),\pi}(\cdot)\,/\,X(\cdot)=V^{qX(0),\pi}(\cdot)\,/\,V^{X(0),\mathscr{Z}}(\cdot)\,; \tag{2}$$

here the scalar $q>0$ measures initial wealth $w=qX(0)$ as a proportion of total market capitalization at time $t=0$. The dynamics of this relative performance are shown to be

$$\frac{dY^{q,\pi}(t)}{Y^{q,\pi}(t)} = \sum_{i=1}^n \pi_i(t)\,\frac{dZ_i(t)}{Z_i(t)}, \qquad Y^{q,\pi}(0) = q. \tag{3}$$

2.1 Change of Variables

It is occasionally convenient to describe a portfolio $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))'$ in terms of the vector $\psi(\cdot) = (\psi_1(\cdot), \cdots, \psi_n(\cdot))'$ of *scaled relative weights*

$$\psi_i(\cdot) := \pi_i(\cdot)\,/\,Z_i(\cdot), \qquad i = 1, \cdots, n.$$

This allows us to rewrite the relative performance dynamics (3) simply as

$$\frac{dY^{q,\pi}(t)}{Y^{q,\pi}(t)} = \sum_{i=1}^n \psi_i(t)\,dZ_i(t) = \psi'(t)\,d\mathscr{Z}(t), \qquad Y^{q,\pi}(0) = q. \tag{4}$$

Since $\sum_{i=1}^n dZ_i(t) = 0$, one need specify the vector $\psi(t)$ of scaled portfolio weights only modulo a scalar factor, then recover from $\psi_1(t), \cdots, \psi_n(t)$ the ordinary portfolio weights

$$\pi_i(t) = Z_i(t)\Big(\psi_i(t) + 1 - \sum_{j=1}^n Z_j(t)\psi_j(t)\Big), \qquad i = 1, \cdots, n. \tag{5}$$

3 The Model

We shall postulate from now onwards an *Itô process model* for the Δ^o-valued relative market weights $\mathscr{Z}(\cdot) = \big(Z_1(\cdot), \cdots, Z_n(\cdot)\big)'$ process, of the form

$$d\mathscr{Z}(t) = \mathrm{s}\big(\mathscr{Z}(t)\big)\big(dW(t) + \vartheta(t)\,dt\big), \qquad \mathscr{Z}(0) = \mathbf{z} \in \Delta^o. \tag{6}$$

Here $W(\cdot)$ is an $n-$dimensional $\mathbb{P}-$Brownian motion; $\vartheta(\cdot)$ is an $n-$dimensional, progressively measurable and locally square-integrable process, i.e., $\int_0^T ||\vartheta(t)||^2 dt < \infty$ holds $\mathbb{P}-$a.s. for every $T \in (0, \infty)$; and the volatility structure is characterized in terms of $\mathrm{s}(\cdot) = (\mathrm{s}_{i\nu}(\cdot))_{1 \le i,\nu \le n}$, a matrix-valued function with continuous components $\mathrm{s}_{i\ell} : \Delta \to \mathbb{R}$ that satisfy the condition $\sum_{i=1}^n \mathrm{s}_{i\ell}(\cdot) \equiv 0$ for all $\ell = 1, \cdots, n$.

We shall assume that the corresponding *covariance matrix*

$$\mathrm{a}(\mathbf{z}) := \mathrm{s}(\mathbf{z})\,\mathrm{s}'(\mathbf{z}), \qquad \mathbf{z} \in \Delta \tag{7}$$

has rank $n-1$ for every $\mathbf{z} \in \Delta^o$ (cf. Lemma 3.1 in [7]), as well as rank k in the interior $\mathfrak{d}^o$ of every $k-$dimensional sub-simplex $\mathfrak{d} \subset \Gamma$, $k = 1, \cdots, n-2$.

If, on some given time-horizon $[0,T]$ of finite length, we can write $\vartheta(t) = \Theta(T-t, \mathscr{Z}(t))$, $0 \le t \le T$ for some continuous function $\Theta : [0,T] \times \Delta \to \mathbb{R}^n$, we shall say that the resulting diffusion

$$d\mathscr{Z}(t) = \mathrm{b}\big(T-t, \mathscr{Z}(t)\big)\, dt + \mathrm{s}(\mathscr{Z}(t))\, dW(t)\,, \qquad \mathscr{Z}(0) = \mathbf{z} \in \Delta^o\,, \tag{8}$$

with $\mathrm{b}(\tau,\mathbf{z}) := \mathrm{s}(\mathbf{z})\Theta(\tau,\mathbf{z})$, constitutes a *Markovian Market Weight (MMW)* model on this $[0,T]$.

3.1 Numéraire and Log-Optimality Properties

Consider now two portfolios $\pi(\cdot)$, $\nu(\cdot)$ with corresponding scaled relative weights $\psi_i(\cdot) = \pi_i(\cdot)/Z_i(\cdot)$ and $\varphi_i(\cdot) = \nu_i(\cdot)/Z_i(\cdot)$, $i = 1, \cdots, n$. An application of Itô's rule in conjunction with (4) gives

$$d\left(\frac{Y^{q,\pi}(t)}{Y^{q,\nu}(t)}\right) = \left(\frac{Y^{q,\pi}(t)}{Y^{q,\nu}(t)}\right) \big(\psi(t) - \varphi(t)\big)' \left[d\mathscr{Z}(t) - \mathrm{a}\big(\mathscr{Z}(t)\big)\, \varphi(t)\, dt \right]; \tag{9}$$

whereas, on the strength of (6), the last term (in brackets) is

$$\mathrm{s}\big(\mathscr{Z}(t)\big)\, dW(t) + \mathrm{s}\big(\mathscr{Z}(t)\big) \left(\vartheta(t) - \big(\mathrm{s}\big(\mathscr{Z}(t)\big)\big)' \varphi(t)\right) dt\,.$$

The finite variation part of this expression vanishes, if we select the portfolio $\nu(\cdot)$ as in (5) with scaled relative weights $\varphi(\cdot) = \big(\varphi_1(\cdot), \cdots, \varphi_n(\cdot)\big)'$ that satisfy

$$\big(\mathrm{s}\big(\mathscr{Z}(\cdot)\big)\big)' \varphi(\cdot) = \vartheta(\cdot)\,. \tag{10}$$

Thus, for $\nu(\cdot) \equiv \nu^{\mathbb{P}}(\cdot)$ selected this way, *the ratio* $Y^{q,\pi}(\cdot)/Y^{q,\nu^{\mathbb{P}}}(\cdot)$ *is, for any portfolio* $\pi(\cdot) \in \Pi$, a positive local martingale, thus also *a supermartingale.*

We express this by saying that the portfolio $\nu^{\mathbb{P}}(\cdot)$ has the "numéraire property" (e.g., [20], [22], [17] and the references there). As was observed in [7], [17], no arbitrage relative to such a portfolio is possible over any finite time-horizon. Note also that, if $\vartheta(\cdot) \equiv 0$, we can select $\varphi(\cdot) \equiv 0$ in (10), so the resulting market portfolio $\mathscr{Z}(\cdot)$ from (5) has then the numéraire property.

With $\nu^{\mathbb{P}}(\cdot)$ selected as in (5) from scaled relative weights $\varphi(\cdot) = (\varphi_1(\cdot), \cdots, \varphi_n(\cdot))'$ that satisfy (10), the expression of (9) becomes

$$d\left(Y^{q,\pi}(t) \,/\, Y^{q,\nu^{\mathbb{P}}}(t)\right) = \left(Y^{q,\pi}(t) \,/\, Y^{q,\nu^{\mathbb{P}}}(t)\right) \big(\psi(t) - \varphi(t)\big)' \mathrm{s}\big(\mathscr{Z}(t)\big)\, dW(t)$$

and yields

$$d\log\left(\frac{Y^{q,\pi}(t)}{Y^{q,\nu^{\mathbb{P}}}(t)}\right) = \big(\psi(t) - \varphi(t)\big)' \mathrm{s}\big(\mathscr{Z}(t)\big) dW(t) - \frac{1}{2}\big(\psi(t) - \varphi(t)\big)' \mathrm{a}\big(\mathscr{Z}(t)\big)\big(\psi(t) - \varphi(t)\big) dt\,.$$

We deduce as in [18], [17] (see also [2], [1]) the *relative log-optimality* property of $\nu^{\mathbb{P}}(\cdot)$: for every portfolio $\pi(\cdot) \in \Pi$ and $(T,q) \in (0,\infty)^2$, we have

$$\mathbb{E}^{\mathbb{P}}\Big[\log \frac{Y^{q,\pi}(T)}{q}\Big] \le \mathbb{E}^{\mathbb{P}}\Big[\log \frac{Y^{q,\nu^{\mathbb{P}}}(T)}{q}\Big] = \frac{1}{2}\,\mathbb{E}^{\mathbb{P}}\int_0^T \|\vartheta(t)\|^2\,dt\,. \tag{11}$$

4 Relative Arbitrage

Let us recall briefly some results from [5]. Within a market model as in the previous section, and for a given time-horizon $[0,T]$ and initial market weight configuration $\mathscr{X}(0) = \mathbf{z} \in \Delta^o$, we introduce the *relative arbitrage function*

$$U(T,\mathbf{z}) := \inf\big\{\, q > 0 : \exists\ \pi(\cdot) \in \Pi \ \text{ s.t. } \mathbb{P}\big(Y^{q,\pi}(T) \ge 1\big) = 1 \,\big\}\,. \tag{12}$$

In other words, $U(T,\mathbf{z}) \in (0,1]$ is the smallest relative initial wealth required at $t=0$, in order to attain at time $t=T$ relative wealth of (at least) 1 with respect to the market, $\mathbb{P}-$a.s. Equivalently, the quantity $\big(1/U(T,\mathbf{z})\big)$ gives the maximal relative amount by which the market portfolio can be outperformed over the time horizon $[0,T]$.

If $U(T,\mathbf{z}) = 1$, it is not possible to outperform the market over $[0,T]$; if, on the other hand, $U(T,\mathbf{z}) < 1$, then for every $q \in (U(T,\mathbf{z}),1)$ (and even for $q = U(T,\mathbf{z})$, when the infimum in (12) is attained) there exists a portfolio $\pi_q(\cdot) \in \Pi$ such that

$$\frac{V^{1,\pi_q}(T)}{V^{1,\mathscr{X}}(T)} \ge \frac{1}{q} > 1 \qquad \text{holds } \mathbb{P}-\text{a.s.}$$

In the terminology of [7], each such $\pi_q(\cdot)$ is then *strong arbitrage relative to the market portfolio* $\mathscr{X}(\cdot)$ over the time-horizon $[0,T]$.

Remark: It is shown in [7], section 11 (see also [5]) that a sufficient condition leading to $U(T,\mathbf{z}) < 1$, is that there exist a real constant $h > 0$ for which

$$\sum_{i=1}^n z_i\,\frac{\mathrm{a}_{ii}(\mathbf{z})}{z_i^2} \ge h\,, \quad \forall\ \mathbf{z} \in \Delta^o\,. \tag{13}$$

The expression on the left-hand side of this inequality is the average, weighted by market capitalization z_i, of the variances $\mathrm{a}_{ii}(\mathbf{z})/z_i^2$ of individual asset returns relative to the market and, as such, a measure of the market's "intrinsic volatility". The condition (13) posits a positive lower bound on this quantity as sufficient for $U(T,\mathbf{z}) < 1$, i.e., for the possibility of outperforming the market.

Here, we shall characterize $U(T,\mathbf{z})$ in various equivalent ways, thus providing necessary and sufficient conditions for this possibility.

4.1 The Föllmer "Exit Measure"

Under appropriate "canonical" conditions on the filtered measurable space $(\Omega, \mathscr{F})$, $\mathbb{F} = \{\mathscr{F}(t)\}_{0 \le t < \infty}$, there exists on it a probability measure $\widehat{\mathbb{Q}}$, the so-called *Föllmer exit measure*, with respect to which the original probability measure $\mathbb{P}$ is absolutely continuous and which has the following properties: The process

$$\widehat{W}(t) := W(t) + \int_0^t \vartheta(s)\, ds\,, \qquad 0 \le t < \infty\,,$$

whose differential appears in (6), is a $\widehat{\mathbb{Q}}-$Brownian motion; the exponential process

$$\Lambda(t) := \exp\left\{ \int_0^t \vartheta'(s)\, d\widehat{W}(s) - \frac{1}{2} \int_0^t ||\vartheta(s)||^2\, ds \right\}, \qquad 0 \le t < \infty \tag{14}$$

is a $\widehat{\mathbb{Q}}-$martingale, indeed we have $\mathbb{P}(A) = \int_A \Lambda(t)\, d\widehat{\mathbb{Q}}$ for $A \in \mathscr{F}(t)$; whereas the vector $\mathscr{Z}(\cdot) = \big(Z_1(\cdot), \cdots, Z_n(\cdot)\big)'$ of relative market weights is a $\widehat{\mathbb{Q}}-$martingale and a Markov process, with values in Δ and "purely diffusive" dynamics of the form

$$d\mathscr{Z}(t) = \mathrm{s}\big(\mathscr{Z}(t)\big)\, d\widehat{W}(t)\,, \qquad \mathscr{Z}(0) = \mathbf{z} \in \Delta^o. \tag{15}$$

In particular, viewed as a portfolio, $\mathscr{Z}(\cdot)$ has the numéraire property under $\widehat{\mathbb{Q}}$; to wit, we have $\mathscr{Z}(\cdot) \equiv \nu^{\widehat{\mathbb{Q}}}(\cdot)$ in the notation of subsection 3.1. We shall let

$$\mathscr{T} := \inf\big\{t \ge 0 : \mathscr{Z}(t) \in \Gamma\big\} = \min_{1 \le i \le n} \mathscr{T}_i\,, \qquad \mathscr{T}_i := \inf\big\{t \ge 0 : Z_i(t) = 0\big\} \tag{16}$$

be the first time the process $\mathscr{Z}(\cdot)$ reaches the boundary of the simplex Δ, with the usual convention $\inf \emptyset = \infty$. We have then the representation

$$U(T, \mathbf{z}) = \widehat{\mathbb{Q}}^{\mathbf{z}}\big(\mathscr{T} > T\big)\,, \qquad (T, \mathbf{z}) \in (0, \infty) \times \Delta^o \tag{17}$$

for the relative arbitrage function of (12), as the probability under the Föllmer measure (which we index here by the starting position $\mathscr{Z}(0) = \mathbf{z}$ of the diffusion in (15)) that $\mathscr{Z}(\cdot)$ has not reached the boundary Γ of the simplex by time $t = T$. Since each $Z_i(\cdot)$ is a nonnegative $\widehat{\mathbb{Q}}-$martingale, it is clear that on $\{\mathscr{T}_i < \infty\}$ we have $Z_i(\mathscr{T}_i + u) = 0$, $\forall\, u \ge 0$, $\widehat{\mathbb{Q}}-$a.s. Thus, the process $\mathscr{Z}(\cdot)$ stays $\widehat{\mathbb{Q}}-$a.s. on Γ once it gets there; and so on by induction, regarding the boundaries $\gamma = \mathfrak{d} \setminus \mathfrak{d}^o$ of lower-dimensional subsimplices $\mathfrak{d}$ of Γ.

For proofs of these claims we refer the reader to [5], as well as to [3], [4], [21], [23] and of course to the original work of Föllmer [9], [10]. It is also seen in [5] that $\mathscr{T} = \inf\{t \ge 0 : \Lambda(t) = 0\}$ holds $\widehat{\mathbb{Q}}-$a.s., and that the relative arbitrage function of (12) admits the representation

$$U(T, \mathbf{z}) = \mathbb{E}^{\mathbb{P}^{\mathbf{z}}}\left[\frac{1}{\Lambda(T)}\right], \tag{18}$$

where now we index by the starting position $\mathscr{X}(0) = \mathbf{z}$ of the process $\mathscr{X}(\cdot)$ in (6) the original probability measure $\mathbb{P}$. Under this measure, the *deflator* process

$$\frac{1}{\Lambda(\cdot)} = \exp\left\{ -\int_0^{\cdot} \vartheta'(t)\, dW(t) - \frac{1}{2}\int_0^{\cdot} ||\vartheta(t)||^2\, dt \right\} \equiv \frac{q}{Y^{q,\nu^{\mathbb{P}}}(\cdot)} \tag{19}$$

appearing in (18) is, of course, a strictly positive local martingale and supermartingale; here $q > 0$ is a constant, and $\nu^{\mathbb{P}}(\cdot)$ the numéraire portfolio of subsection 3.1.

It may be helpful to think of the passage from the original measure $\mathbb{P}$ to the Föllmer measure $\widehat{\mathbb{Q}}$, as a generalized Girsanov-like change of probability that "removes the drift" in (6), when the deflator process $1/\Lambda(\cdot)$ of (19) is a strict local martingale under $\mathbb{P}$, i.e., when $U(T,\mathbf{z}) < 1$. As $\Lambda(\cdot)$ can reach the origin with positive $\widehat{\mathbb{Q}}-$probability, this is in general *not* an equivalent change of measure. Nonetheless, the market weight process $\mathscr{X}(\cdot)$ is a $\widehat{\mathbb{Q}}-$martingale, so the Föllmer measure $\widehat{\mathbb{Q}}$ can be thought of as a "martingale measure" for the model under consideration.

4.2 The Functionally-Generated Portfolio

It is shown in [5] that, under appropriate regularity conditions on the covariance structure $\mathrm{a}(\cdot)$ and on the relative drift $\vartheta(\cdot)$, the function $U(\cdot,\cdot)$ defined in (12) is of class $\mathscr{C}^{1,2}$ on $(0,\infty) \times \Delta^o$ and satisfies on this domain the equation

$$D_\tau U(\tau,\mathbf{z}) = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \mathrm{a}_{ij}(\mathbf{z})\, D^2_{ij} U(\tau,\mathbf{z}), \tag{20}$$

or equivalently $D_\tau U = \frac{1}{2}\mathrm{Tr}\big(\mathrm{a} D^2 U\big)$; and that $U(\cdot,\cdot)$ is also the smallest nonnegative supersolution of this equation, subject to the initial condition

$$U(0,\mathbf{z}) \equiv 1, \qquad \forall\ \mathbf{z} \in \Delta^o. \tag{21}$$

We shall use throughout the notation $D_\tau f = \partial f/\partial\tau$, $D_i f = \partial f/\partial x_i$, $Df = \big(\partial f/\partial x_1, \cdots, \partial f/\partial x_n\big)'$ for the gradient, and $D^2 f = \big(D^2_{ij} f\big)_{1\le i,j\le n}$ with $D^2_{ij} f = \partial^2 f/\partial x_i \partial x_j$ for the Hessian. Let us consider the process

$$\widehat{Y}(t) := U(T-t, \mathscr{X}(t))\, 1_{\{\mathscr{T} > t\}}, \qquad 0 \le t \le T; \tag{22}$$

it satisfies $\widehat{Y}(0) = U(T,\mathbf{z})$ and $\widehat{Y}(T) = 1$, $\mathbb{P}-$a.s. An application of Itô's rule leads now, in conjunction with (20) and (6), to the $\mathbb{P}-$dynamics

$$d\widehat{Y}(t) = \widehat{Y}(t)\,\big(\widehat{\psi}(t)\big)'\, d\mathscr{X}(t) \tag{23}$$

as in (4), with

$$\widehat{\psi}_i(t) := \left(\frac{D_i U}{U}\right)\big(T-t, \mathscr{Z}(t)\big) = D_i \log U\big(T-t, \mathscr{Z}(t)\big), \qquad i = 1, \cdots, n \quad (24)$$

and, via (5), to the *functionally-generated* (in the terminology of [6], p. 46) portfolio

$$\widehat{\pi}_i(t) := Z_i(t) \left[D_i \log U\big(T-t, \mathscr{Z}(t)\big) + 1 - \sum_{j=1}^{n} Z_j(t) D_j \log U\big(T-t, \mathscr{Z}(t)\big) \right]. \quad (25)$$

If $U(T, \mathbf{z}) < 1$, this portfolio implements the optimal arbitrage under the original probability measure, in the sense that with $q = U(T, \mathbf{z})$ we have

$$\widehat{Y}(\cdot) \equiv Y^{q, \widehat{\pi}}(\cdot), \quad \mathbb{P} - \text{a.s.} \quad (26)$$

as in (2), (3). In particular, we see that the infimum in (12) is then attained.

5 Induced Drifts

One potentially intriguing aspect of the arbitrage function in (12), and of the portfolio (25) which is generated by it and implements the optimal arbitrage, is that they appear not to depend at all on the drift vector $\mathrm{s}(\mathscr{Z}(\cdot))\,\vartheta(\cdot)$ of the Itô process $\mathscr{Z}(\cdot)$ in (6). That is, as long as we know the covariance structure of (7), we can construct the function $U(\cdot,\cdot)$, at least in principle, as the smallest nonnegative (super)solution of the Cauchy problem (20), (21), and from $U(\cdot,\cdot)$ the portfolio $\widehat{\pi}(\cdot)$ of (25) that attains the prescribed amount of arbitrage. Even if we were given the drift as well, we would not, in general, be able to improve upon this optimal arbitrage.

Of course, the drift is not entirely irrelevant; we have assumed, for one, that it satisfies certain regularity conditions. More significantly, we have posited that under the measure $\mathbb{P}$ the market weight process $\mathscr{Z}(\cdot)$ never hits the boundary Γ of the simplex; this assumption alone places non-trivial restrictions on the drift term or, equivalently, on the "relative market price of risk" process $\vartheta(\cdot)$ of (6); cf. [13], chapters 9 and 11.

5.1 Conditioning

One way to construct such a drift is as follows. We reverse the above procedure and *start* with a probability measure $\widehat{\mathbb{Q}}$ on the canonical filtered measurable space $(\Omega, \mathscr{F})$, $\mathbb{F} = \{\mathscr{F}(t)\}_{0 \le t < \infty}$, under which the relative weight process $\mathscr{Z}(\cdot)$ is a Markovian diffusion and a martingale on the state space Δ with dynamics as in (15), and with $\widehat{W}(\cdot)$ an $n-$dimensional Brownian motion. For the given time-horizon $[0, T]$, we construct then a *new* probability measure $\mathbb{P}_\star$ on $\mathscr{F}(T)$ via the recipe

$$\mathbb{P}_\star(A) := \widehat{\mathbb{Q}}\big(A \mid \mathscr{T} > T\big), \qquad A \in \mathscr{F}(T) \quad (27)$$

in the notation of (16). In other words, the distribution of the market weight process $\mathscr{Z}(\cdot)$ under $\mathbb{P}_\star$ on $\mathscr{F}(T)$ is the same as the conditional distribution under $\widehat{\mathbb{Q}}$ of the process $\mathscr{Z}(\cdot)$, conditioned on the event that this process has not reached the boundary Γ of the simplex by time T.

An elementary computation gives

$$\frac{d\,\mathbb{P}_\star}{d\,\widehat{\mathbb{Q}}}\bigg|_{\mathscr{F}(t)} = \frac{U\big(T-t,\mathscr{Z}(t)\big)}{U(T,\mathbf{z})}\,1_{\{\mathscr{T}>t\}} = \frac{\widehat{Y}(t)}{\widehat{Y}(0)} =: \Lambda^{\mathbb{P}_\star}(t)\,, \quad 0\le t\le T\,, \tag{28}$$

$\widehat{\mathbb{Q}}-$a.s. In particular, the process $\widehat{Y}(\cdot)$ of (22) is a $\widehat{\mathbb{Q}}-$martingale, so the representation (17) follows then from optional sampling and (21).

Repeating the steps of our analysis in subsection 4.2, now in reverse, we see that the $\widehat{\mathbb{Q}}-$martingale $\Lambda^{\mathbb{P}_\star}(\cdot)$ of (28) can be cast in the manner of (14), with $\vartheta(\cdot)$ replaced by the $n-$dimensional process

$$\vartheta^{\mathbb{P}_\star}(\cdot) = \Theta_\star\big(T-\cdot\,,\mathscr{Z}(\cdot)\big)\,, \qquad \text{where} \quad \Theta_\star(\tau,\mathbf{z}) := \big(\mathrm{s}(\mathbf{z})\big)'\,D\log U(\tau,\mathbf{z})\,. \tag{29}$$

Moreover, (6) takes now the form of an *MMW model*

$$d\mathscr{Z}(t) = \mathrm{a}\big(\mathscr{Z}(t)\big)\,D\log U\big(T-t,\mathscr{Z}(t)\big)dt + \mathrm{s}\big(\mathscr{Z}(t)\big)\,dW^{\mathbb{P}_\star}(t)\,, \quad \mathscr{Z}(0) = \mathbf{z}\in\Delta^o \tag{30}$$

with $W^{\mathbb{P}_\star}(\cdot) := \widehat{W}(\cdot) - \int_0^\cdot \vartheta^{\mathbb{P}_\star}(t)\,dt$ an $n-$dimensional $\mathbb{P}_\star-$Brownian motion.

5.2 Numéraire and Log-Optimality Properties of $\widehat{\pi}(\cdot)$

It is clear from (29), (30), (10) and (24) that the portfolio $\widehat{\pi}(\cdot)$ of (25) has the numéraire and log-optimality properties *under the probability measure* $\mathbb{P}_\star$ of (27); i.e.,

the process $\;Y^{q,\pi}(t)/Y^{q,\widehat{\pi}}(t)\,,\;\; 0\le t\le T\;$ is a $\mathbb{P}_\star-$supermartingale, and

$$\mathbb{E}^{\mathbb{P}_\star}\Big[\log\frac{Y^{q,\pi}(t)}{q}\Big] \le \mathbb{E}^{\mathbb{P}_\star}\Big[\log\frac{Y^{q,\widehat{\pi}}(t)}{q}\Big] = \frac{1}{2}\,\mathbb{E}^{\mathbb{P}_\star}\int_0^t \|\vartheta^{\mathbb{P}_\star}(u)\|^2\,du \tag{31}$$

holds for $q>0\,,\; t\in[0,T]$ on the strength of (11), for every $\pi(\cdot)\in\Pi$. In the notation of subsection 3.1, we can thus write $\widehat{\pi}(\cdot)\equiv\nu^{\mathbb{P}_\star}(\cdot)$.

We may characterize the strategy that achieves optimal arbitrage over $[0,T]$ thus: *Start with a probability measure* $\widehat{\mathbb{Q}}$ *that generates* driftless, Markovian *market weights as in (15); then construct the measure* $\mathbb{P}_\star$ *by conditioning* $\widehat{\mathbb{Q}}$ *on the event* $\{\mathscr{T}>T\}$*; finally, find a portfolio* $\widehat{\pi}(\cdot)$ *that maximizes expected logarithmic relative returns under* $\mathbb{P}_\star$.

Let us also note that, in the context of the MMW model (30) with $\vartheta(\cdot)\equiv\vartheta^{\mathbb{P}_\star}(\cdot)$ as in (29), the representation (18), (19) can be cast on the strength of (28), (26) as

$$U(T,\mathbf{z}) = \frac{1}{\Lambda^{\mathbb{P}_\star}(T)} = \frac{q}{Y^{q,\widehat{\pi}}(T)}\,, \qquad \mathbb{P}_\star - \text{a.s.} \tag{32}$$

6 Relative Entropy

The definitions in (14) and (28) suggest a re-writing of (31) for $t=T,\ q>0$ as

$$\mathbb{E}^{\mathbb{P}_\star}\Big[\log\frac{Y^{q,\pi}(T)}{q}\Big] \le \mathbb{E}^{\mathbb{P}_\star}\Big[\log\frac{Y^{q,\widehat{\pi}}(T)}{q}\Big] = \mathbb{E}^{\mathbb{P}_\star}\big[\log\Lambda^{\mathbb{P}_\star}(T)\big]\,, \quad \forall\ \pi(\cdot)\in\Pi\,, \tag{33}$$

and thence a further interpretation of the arbitrage function and of the portfolio $\widehat{\pi}(\cdot)$ generated by it, this time in terms of *relative entropy*. In particular, we claim that with $L:=\log(1/U)$ we have the string of equalities

$$\begin{aligned} H_T(\mathbb{P}_\star\,|\,\widehat{\mathbb{Q}}) &:= \mathbb{E}^{\mathbb{P}_\star}\Big[\log\Big(\big(d\mathbb{P}_\star/d\widehat{\mathbb{Q}}\big)\big|_{\mathscr{F}(T)}\Big)\Big] \\ &= \mathbb{E}^{\mathbb{P}_\star}\big[\log\Lambda^{\mathbb{P}_\star}(T)\big] = \frac{1}{2}\,\mathbb{E}^{\mathbb{P}_\star}\int_0^T \|\vartheta^{\mathbb{P}_\star}(t)\|^2 dt \\ &= \frac{1}{2}\,\mathbb{E}^{\mathbb{P}_\star}\int_0^T \big(DL\big(T-t,\mathscr{Z}(t)\big)\big)'\mathrm{a}\big(\mathscr{Z}(t)\big)\,DL\big(T-t,\mathscr{Z}(t)\big)dt = \log\big(1/U(T,\mathbf{z})\big). \end{aligned} \tag{34}$$

Proof of (34), (33): Let us start by observing from the Cauchy problem of (20), (21) that the function $L=\log(1/U)$ satisfies the semilinear *Schrödinger-type equation*

$$D_\tau L(\tau,\mathbf{z}) = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \mathrm{a}_{ij}(\mathbf{z})\,\big(D^2_{ij}L - D_iL\cdot D_jL\big)(\tau,\mathbf{z}) \tag{35}$$

or equivalently $D_\tau L = \frac{1}{2}\mathrm{Tr}\big(\mathrm{a}D^2L\big) - \frac{1}{2}(DL)'\mathrm{a}(DL)$ on $(0,\infty)\times\Delta^o$, and the initial condition $L(0,\cdot)\equiv 0$ on Δ^o.

In conjunction with (30) and the notation of (29), this implies that the process

$$\frac{1}{2}\int_0^\cdot ||\Theta_\star\big(T-t,\mathscr{Z}(t)\big)||^2\,dt + L\big(T-\cdot,\mathscr{Z}(\cdot)\big) = L(T,\mathbf{z}) - M_\star(\cdot)\,, \tag{36}$$

with $M_\star(\cdot) := \int_0^\cdot \big(\Theta_\star\big(T-t,\mathscr{Z}(t)\big)\big)'\,dW^{\mathbb{P}_\star}(t)$, is a nonnegative $\mathbb{P}_\star-$local martingale, thus a $\mathbb{P}_\star-$supermartingale; in particular,

$$\mathbb{E}^{\mathbb{P}_\star}\int_0^T ||\Theta_\star\big(T-t,\mathscr{Z}(t)\big)||^2\,dt \le 2L(T,\mathbf{z})\,. \tag{37}$$

We have used here the identity $L(0,\cdot)\equiv 0$ on Δ^o, and the fact that the market weight process $\mathscr{Z}(\cdot)$ takes values in Δ^o, $\mathbb{P}_\star-$a.s.

The quadratic variation of the $\mathbb{P}_\star-$local martingale $M_\star(\cdot)$ of (36) is given by $\langle M_\star\rangle(\cdot) = \int_0^\cdot ||\Theta_\star\big(T-t,\mathscr{Z}(t)\big)||^2\,dt$, so (37) implies that $M_\star(\cdot)$ is actually a (square-integrable) martingale under $\mathbb{P}_\star$. In particular, (37) holds as equality, the last equation in (34) holds.

The only other claim in (34) that needs discussion, is its third equality; but now this follows from the work of H. Föllmer, namely, Proposition 2.11 in [12] and Lemma 2.6 in [11]. As for (33), this is now just a restatement of (31). □

6.1 Stochastic Control

Emboldened by all this, let us consider on the filtered measurable space $(\Omega, \mathscr{F})$, $\mathbb{F} = \{\mathscr{F}(t)\}_{0 \le t \le T}$ the collection $\mathfrak{P}$ of all probability measures that satisfy $\mathbb{P} \ll \widehat{\mathbb{Q}}$ on $\mathscr{F}(T)$ and $\mathbb{P}\big(\mathscr{X}(t) \in \Delta^o, \ \forall \ 0 \le t \le T\big) = 1$.

The measure $\mathbb{P}_\star$ of subsection 5.1 belongs to $\mathfrak{P}$, as does the generic probability measure of our model introduced in section 3. It is not hard to see that $\mathfrak{P}$ consists of precisely those probability measures $\mathbb{P}$ on $(\Omega, \mathscr{F})$ which are absolutely continuous with respect to $\mathbb{P}_\star$ on $\mathscr{F}(T)$. The elements of $\mathfrak{P}$ are absolutely continuous with respect to $\widehat{\mathbb{Q}}$ but, in general, *not* equivalent: the case $\widehat{\mathbb{Q}}^{\mathbf{z}}\big(\mathscr{X}(t) \in \Delta^o, \ \forall \ 0 \le t \le T\big) = U(T, \mathbf{z}) < 1$ is the most interesting.

For every $\mathbb{P} \in \mathfrak{P}$, the nonnegative $\widehat{\mathbb{Q}}-$martingale $(d\,\mathbb{P}/d\,\widehat{\mathbb{Q}})|_{\mathscr{F}(t)} = \Lambda^{\mathbb{P}}(t)$, $0 \le t \le T$ admits a representation of the form (14) for an appropriate $n-$dimensional process $\vartheta(\cdot) \equiv \vartheta^{\mathbb{P}}(\cdot)$ which: is $\mathbb{F}-$progressively measurable; is square-integrable on $[0,T]$, $\mathbb{P}-$a.s.; and satisfies $\int_0^T ||\vartheta(t)||^2\, dt < \int_0^{\mathscr{T}} ||\vartheta(t)||^2\, dt = \infty$, $\widehat{\mathbb{Q}}-$a.s. on $\{\mathscr{T} > T\}$. The vector $\mathscr{X}(\cdot)$ of relative market weights is under $\mathbb{P}$ an Itô process of the form $d\mathscr{X}(t) = \mathrm{s}(\mathscr{X}(t))\big(dW^{\mathbb{P}}(t) + \vartheta^{\mathbb{P}}(t)\, dt\big)$, $\mathscr{X}(0) = \mathbf{z} \in \Delta^o$ as in (6), with values in Δ^o and $W^{\mathbb{P}}(\cdot) \equiv W(\cdot)$ an $n-$dimensional $\mathbb{P}-$Brownian motion. Finally, just as before, the relative entropy of $\mathbb{P}$ with respect to the probability measure $\widehat{\mathbb{Q}}$ on $\mathscr{F}(T)$ is given as

$$H_T(\mathbb{P}\,|\,\widehat{\mathbb{Q}}) := \mathbb{E}^{\mathbb{P}}\Big[\log\Big(\big(d\,\mathbb{P}/d\,\widehat{\mathbb{Q}}\big)\big|_{\mathscr{F}(T)}\Big)\Big] = \mathbb{E}^{\mathbb{P}}\big[\log \Lambda^{\mathbb{P}}(T)\big] = \frac{1}{2}\,\mathbb{E}^{\mathbb{P}}\int_0^T \|\vartheta^{\mathbb{P}}(t)\|^2\, dt\,. \tag{38}$$

The crucial next step is a simple observation that goes back at least to to Holland [16] and Fleming [8], namely, that the Schrödinger equation (35) can be cast in the *Hamilton-Jacobi-Bellman (HJB)* form

$$D_\tau L = \frac{1}{2}\,\mathrm{Tr}\big(\mathrm{a} D^2 L\big) + \min_{\theta \in \mathbb{R}^n}\left[(DL)'\,\mathrm{s}\,\theta + \frac{1}{2}\,\|\theta\|^2\right];$$

and that the minimization is attained at $\theta_\star = -\,\mathrm{s}' DL$, just as in (29).

Of course, every time an HJB equation makes its appearance, an associated stochastic control problem cannot lag very far behind. In our context, this problem amounts to minimizing, over all probability measures $\mathbb{P} \in \mathfrak{P}$, the "total energy" $(1/2)\,\mathbb{E}^{\mathbb{P}}\int_0^T \|\vartheta^{\mathbb{P}}(t)\|^2\, dt$ during $[0,T]$, of the drift term in (6) that keeps the relative weight process $\mathscr{X}(\cdot)$ inside of Δ^o during this time-horizon. Equivalently, from

(38), this amounts to finding a probability measure that minimizes the relative entropy $H_T(\mathbb{P}\,|\,\widehat{\mathbb{Q}})$ over all $\mathbb{P} \in \mathfrak{P}$. The answer to (both incarnations of) this question should now be obvious; it is the subject of the proposition that follows.

It is instructive to consider first the subclass $\mathfrak{P}^\dagger \subseteq \mathfrak{P}$ of probability measures

$$\mathfrak{P}^\dagger := \left\{ \mathbb{P} \in \mathfrak{P} \,:\, \mathbb{E}^{\mathbb{P}} \left(\int_0^T \big\| \Theta_\star\big(T-t, \mathscr{X}(t)\big) \big\|^2 dt \right)^{1/2} < \infty \right\}. \tag{39}$$

Clearly $\mathbb{P}_\star$ belongs to $\mathfrak{P}^\dagger$ because for it we have the property $\mathbb{E}^{\mathbb{P}_\star} \int_0^T || \Theta_\star\big(T - t, \mathscr{X}(t)\big)||^2 dt < \infty$ from (37). This implies, furthermore, that every probability measure $\mathbb{P}$ with $\mathbb{P} \ll \mathbb{P}_\star$ on $\mathscr{F}(T)$ and $\mathbb{E}^{\mathbb{P}_\star}\big[\big((d\mathbb{P}/d\mathbb{P}_\star)|_{\mathscr{F}(T)}\big)^2\big] < \infty$ also belongs to $\mathfrak{P}^\dagger$.

Let us observe that $\mathbb{P}_\star$ minimizes the "total energy" $\frac{1}{2}\,\mathbb{E}^{\mathbb{P}} \int_0^T \|\vartheta^{\mathbb{P}}(t)\|^2 dt$ over all probability measures in $\mathfrak{P}^\dagger$, i.e.,

$$\min_{\mathbb{P}\in\mathfrak{P}^\dagger} \frac{1}{2}\,\mathbb{E}^{\mathbb{P}} \int_0^T \|\vartheta^{\mathbb{P}}(t)\|^2 dt \;=\; \frac{1}{2}\,\mathbb{E}^{\mathbb{P}_\star} \int_0^T \|\vartheta^{\mathbb{P}_\star}(t)\|^2 dt = \log\big(1/U(T,\mathbf{z})\big)\,. \tag{40}$$

Proof of (40): This amounts to a standard verification argument, one part of which has already been established in the proof of (34). There, we obtained in (36) the $\mathbb{P}_\star-$semimartingale decomposition of the process $L\big(T-\cdot\,, \mathscr{X}(\cdot)\big)$; now we obtain for this process its $\mathbb{P}-$semimartingale decomposition

$$\frac{1}{2}\int_0^{\cdot} \|\vartheta^{\mathbb{P}}(t)\|^2 dt + L\big(T-\cdot\,,\mathscr{X}(\cdot)\big) = L(T,\mathbf{z}) + \frac{1}{2}\int_0^{\cdot} \|\vartheta^{\mathbb{P}}(t) - \vartheta^{\mathbb{P}_\star}(t)\|^2 dt \,-M(\cdot)\,, \tag{41}$$

where the $\mathbb{P}-$local martingale $M(\cdot) := \int_0^{\cdot}(\vartheta^{\mathbb{P}^\star}(t))' dW^{\mathbb{P}}(t)$ has quadratic variation $\langle M\rangle(\cdot) = \int_0^{\cdot} || \Theta_\star\big(T-t, \mathscr{X}(t)\big)||^2 dt$. But the condition $\mathbb{E}^{\mathbb{P}}\big(\sqrt{\langle M\rangle(T)}\,\big) < \infty$ of (39) implies that $M(\cdot)$ is $\mathbb{P}-$uniformly integrable, by the Burkholder-Davis-Gundy inequalities (e.g., Theorem 3.3.28 in [19]), thus a bona-fide $\mathbb{P}-$martingale (ibid., Problem 1.5.19(i)). This makes the right-hand side of (41) a $\mathbb{P}-$submartingale, and leads to the comparison $(1/2)\,\mathbb{E}^{\mathbb{P}}\int_0^T \|\vartheta^{\mathbb{P}}(t)\|^2 dt \geq L(T,\mathbf{z}) = -\log U(T,\mathbf{z})$, completing the verification argument.

We have used here yet again the identity $L(0,\cdot) \equiv 0$ on Δ^o in conjunction with the fact that $\mathscr{X}(\cdot)$ takes values in Δ^o, $\mathbb{P}-$a.s. $\square$

It might well be possible to strengthen this argument, and show that $\mathfrak{P}^\dagger$ can be replaced in (40) by the larger set $\mathfrak{P}$. This would be the case if, for instance, one could show that the function $\Theta_\star(\cdot\,,\cdot)$ of (29) is bounded uniformly over $[0,T]\times\Delta^o$; then $\mathfrak{P}^\dagger = \mathfrak{P}$ would follow from (39). Here is, however, a totally different and elementary argument based on first principles, that establishes this stronger result.

Proposition 1: *The measure $\mathbb{P}_\star$ of (27) has the smallest relative entropy with respect to the Föllmer measure $\widehat{\mathbb{Q}}$ of subsection 4.1, among all probability measures $\mathbb{P} \in \mathfrak{P}$; whereas its associated drift $\vartheta^{\mathbb{P}_\star}(\cdot)$ in (29) keeps the process $\mathscr{X}(\cdot)$ inside Δ^o during this time-horizon $[0,T]$ with the smallest amount of total energy. These two minima* *are equal, and*

$$\log\big(1/U(T,\mathbf{z})\big) \,=\, \min_{\mathbb{P}\in\mathfrak{P}} H_T(\mathbb{P}\,|\,\widehat{\mathbb{Q}}) \,=\, H_T(\mathbb{P}_\star\,|\,\widehat{\mathbb{Q}}) \tag{42}$$

$$= \min_{\mathbb{P}\in\mathfrak{P}^\dagger} \frac{1}{2}\,\mathbb{E}^{\mathbb{P}}\int_0^T \|\vartheta^{\mathbb{P}}(t)\|^2\,dt \,=\, \frac{1}{2}\,\mathbb{E}^{\mathbb{P}_\star}\int_0^T \|\vartheta^{\mathbb{P}_\star}(t)\|^2\,dt$$

$$= \mathbb{E}^{\mathbb{P}_\star}\Big[\log\frac{Y^{q,\widehat{\pi}}(T)}{q}\Big] \,=\, \max_{\pi(\cdot)\in\Pi} \mathbb{E}^{\mathbb{P}_\star}\Big[\log\frac{Y^{q,\pi}(T)}{q}\Big], \qquad q>0\,.$$

Proof: In light of (33) and (34), only the second equation needs justification. For an arbitrary $\mathbb{P}\in\mathfrak{P}$ we have $\mathbb{P}(\mathscr{T}>T)=1$, so

$$H_T(\mathbb{P}\,|\,\widehat{\mathbb{Q}}) \,=\, \mathbb{E}^{\mathbb{P}}\Big[\log\Big(\big(d\,\mathbb{P}/d\,\widehat{\mathbb{Q}}\big)\big|_{\mathscr{F}(T)}\Big)\Big] \,=\, \int_{\{\mathscr{T}>T\}} \log\Big(\big(d\,\mathbb{P}/d\,\widehat{\mathbb{Q}}\big)\big|_{\mathscr{F}(T)}\Big)\,d\,\mathbb{P}.$$

But (28) gives $\big(d\,\mathbb{P}_\star/d\,\widehat{\mathbb{Q}}\big)\big|_{\mathscr{F}(T)}=1/U(T,\mathbf{z})$, $\widehat{\mathbb{Q}}-$a.s. on the event $\{\mathscr{T}>T\}$, thus

$$H_T(\mathbb{P}\,|\,\widehat{\mathbb{Q}}) \,=\, \int_{\{\mathscr{T}>T\}} \Big(\log\big(d\,\mathbb{P}/d\,\mathbb{P}_\star\big)\big|_{\mathscr{F}(T)} + \log\big(1/U(T,\mathbf{z})\big)\Big)\,d\,\mathbb{P}$$

$$= H_T(\mathbb{P}\,|\,\mathbb{P}_\star) + \log\big(1/U(T,\mathbf{z})\big) \,\geq\, \log\big(1/U(T,\mathbf{z})\big) \,=\, H_T(\mathbb{P}_\star\,|\,\widehat{\mathbb{Q}})\,. \qquad \square$$

According to this result, the quantity $\log\big(1/U(T,\mathbf{z})\big)$ – logarithm of the maximal relative amount by which the market portfolio can be outperformed over the horizon $[0,T]$, $\mathbb{P}-$a.s. for every $\mathbb{P}\in\mathfrak{P}$ – is also the maximal $\mathbb{P}_\star-$expected-log-outperformance of the market, during the horizon $[0,T]$, that can be achieved over all portfolios $\pi(\cdot)\in\Pi$.

On the other hand, we have shown that the probability measure $\mathbb{P}_\star$ of (27)has the smallest relative entropy with respect to the Föllmer measure $\widehat{\mathbb{Q}}$, over the collection $\mathfrak{P}$; and that its associated drift $\vartheta^{\mathbb{P}_\star}(\cdot)$ keeps the relative weight process $\mathscr{Z}(\cdot)$ inside Δ^o with the smallest amount of total energy. For this reason we shall call $\mathbb{P}_\star$ *minimal energy measure* for the process $\mathscr{Z}(\cdot)$.

This terminology echoes that of Frittelli [14] (see also [15]), where the term "minimal entropy martingale measure" is used in a different context (the valuation of contingent claims in incomplete markets) and with an eye towards banishing arbitrage rather than implementing it in the best way when it exists. In that context, the Minimal Entropy *Martingale* (MEM) measure renders the underlying price process a martingale, which our $\mathbb{P}_\star$ in general does not.

6.2 Stochastic Game

Having come thus far, let us also note that (33) can be written as

$$\mathbb{E}^{\mathbb{P}_\star}\Big[\log\frac{Y^{q,\pi}(T)}{q}\Big] \le \mathbb{E}^{\mathbb{P}_\star}\Big[\log\frac{Y^{q,\widehat{\pi}}(T)}{q}\Big] \tag{43}$$

$$= \mathbb{E}^{\mathbb{P}_\star}\big[\log \Lambda^{\mathbb{P}_\star}(T)\big] = H_T(\mathbb{P}_\star\,|\,\widehat{\mathbb{Q}}) = \log\big(1/U(T,\mathbf{z})\big)$$

$$= \mathbb{E}^{\mathbb{P}}\Big[\log\frac{Y^{q,\widehat{\pi}}(T)}{q}\Big], \quad \forall\ \big(\mathbb{P},\pi(\cdot)\big)\in\mathfrak{P}\times\Pi\,.$$

For this last equality we have used (28) in the form

$$\Lambda^{\mathbb{P}_\star}(T) = Y^{q,\widehat{\pi}}(T)/q = \widehat{Y}(T)/\widehat{Y}(0) = \big(U(0,\mathscr{X}(T))/U(T,\mathbf{z})\big)\,1_{\{\mathscr{T}>T\}}\,, \quad \mathbb{P}-\text{a.s.}$$

and yet again (21), (26) as well as the fact $\mathbb{P}(\mathscr{X}(t)\in\Delta^o,\ \ \forall\ 0\le t\le T)=1$, to obtain that $\Lambda^{\mathbb{P}_\star}(T) = Y^{q,\widehat{\pi}}(T)/q = 1/U(T,\mathbf{z})$ holds $\mathbb{P}-$a.s.

This gives also the following strengthening of (32), (18): for every $q>0$, $\mathbb{P}\in\mathfrak{P}$ we have

$$U(T,\mathbf{z}) = \mathbb{E}^{\mathbb{P}}\Big[\frac{1}{\Lambda^{\mathbb{P}}(T)}\Big] = \mathbb{E}^{\mathbb{P}}\Big[\frac{q}{Y^{q,\nu^{\mathbb{P}}}(\cdot)}\Big] = \frac{1}{\Lambda^{\mathbb{P}_\star}(T)} = \frac{q}{Y^{q,\widehat{\pi}}(T)}\,, \quad \mathbb{P}-\text{a.s.} \tag{44}$$

• We conclude from (43) that the pair $\big(\mathbb{P}_\star\,,\widehat{\pi}(\cdot)\big)$ is a saddle point for the *Stochastic Game* with value

$$\log\big(1/U(T,\mathbf{z})\big) = \min_{\mathbb{P}\in\mathfrak{P}}\ \max_{\pi(\cdot)\in\Pi}\ \mathbb{E}^{\mathbb{P}}\Big[\log\frac{Y^{q,\pi}(T)}{q}\Big] = \max_{\pi(\cdot)\in\Pi}\ \min_{\mathbb{P}\in\mathfrak{P}}\ \mathbb{E}^{\mathbb{P}}\Big[\log\frac{Y^{q,\pi}(T)}{q}\Big].$$

The "maximizer" in this zero-sum game is an investor who tries to outperform the market by his choice of portfolio $\pi(\cdot)\in\Pi$; whereas we can think of the "minimizer" as Nature, or the goddess *Tyche* herself, who tries to thwart the investor's efforts by choosing the probability measure $\mathbb{P}\in\mathfrak{P}$, or equivalently its induced drift $\vartheta^{\mathbb{P}}(\cdot)\equiv\vartheta(\cdot)$, to the investor's detriment.

7 Conclusion

We recapitulate and summarize by listing together in a single proposition the various interpretations of the arbitrage function we have discussed.

Proposition 2: *Consider a Markovian Market Weight model as in (8), and subject to the regularity assumptions of [5]. Then the relative arbitrage function*

$$U(T,\mathbf{z}) := \inf\big\{\,q>0:\ \exists\ \pi(\cdot)\in\Pi\ \ \text{s.t.}\ \mathbb{P}\big(Y^{q,\pi}(T)\ge 1\big)=1\,\big\}, \quad (T,\mathbf{z})\in(0,\infty)\times\Delta^o$$

of (12) admits the following representations, in terms of:

1. *The Cauchy problem of (20)–(21), as its smallest nonnegative classical solution;*
2. *The probability under the Föllmer exit measure* $\widehat{\mathbb{Q}}$ (*which makes the relative market weight process* $\mathscr{X}(\cdot)$ *a diffusion with zero drift, thus a martingale*) *that*

$\mathscr{X}(\cdot)$ has not reached the boundary of the simplex by time T, i.e.,

$$U(T,\mathbf{z}) = \widehat{\mathbb{Q}}\big(\mathscr{X}(t) \in \Delta^o,\ \forall\ 0 \le t \le T\big)\,;$$

3. *The expected log-relative-return of $\widehat{\pi}(\cdot) \equiv \nu^{\mathbb{P}_\star}(\cdot)$, the numéraire portfolio under the probability measure $\mathbb{P}_\star(\cdot) := \widehat{\mathbb{Q}}\big(\,\cdot \mid \mathscr{X}(t) \in \Delta^o,\ \forall\ 0 \le t \le T\big)$ of (27); this quantity is also the maximal $\mathbb{P}_\star-$expected log-relative-return over all portfolios, to wit, for $q>0$,*

$$\log\big(1/U(T,\mathbf{z})\big) = \mathbb{E}^{\mathbb{P}_\star}\Big[\log\big(Y^{q,\widehat{\pi}}(T)/q\big)\Big] = \max_{\pi(\cdot)\in\Pi} \mathbb{E}^{\mathbb{P}_\star}\Big[\log\big(Y^{q,\pi}(T)/q\big)\Big]\,;$$

4. *The relative return of the portfolio $\widehat{\pi}(\cdot)$ in item 3, which is also the maximal achievable relative return over $[0,T]$, in the sense*

$$\frac{1}{U(T,\mathbf{z})} = \sup\big\{\, b>0\,:\ \exists\ \pi(\cdot)\in\Pi \quad \text{s.t. } \mathbb{P}\big(Y^{q,\pi}(T) \ge q\,b\big) = 1\,\big\}$$

$$= \Big(\mathbb{E}^{\mathbb{P}}\big[\big(\Lambda^{\mathbb{P}}(T)\big)^{-1}\big]\Big)^{-1} = \Big(\mathbb{E}^{\mathbb{P}}\big[\big(Y^{q,\nu^{\mathbb{P}}}(T)/q\big)^{-1}\big]\Big)^{-1} = \frac{Y^{q,\widehat{\pi}}(T)}{q}\,, \quad \mathbb{P}-\text{a.s.};$$

5. *The value of the zero-sum stochastic game*

$$\log\big(1/U(T,\mathbf{z})\big) = \min_{\mathbb{P}\in\mathfrak{P}}\ \max_{\pi(\cdot)\in\Pi} \mathbb{E}^{\mathbb{P}}\Big[\log\frac{Y^{q,\pi}(T)}{q}\Big] = \max_{\pi(\cdot)\in\Pi}\ \min_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}^{\mathbb{P}}\Big[\log\frac{Y^{q,\pi}(T)}{q}\Big]\,,$$

   which has the pair $\big(\mathbb{P}_\star\,,\widehat{\pi}(\cdot)\big)$ as saddle point in $\mathfrak{P}\times\Pi$;

6. *The relative entropy $H_T(\mathbb{P}_\star\,|\,\widehat{\mathbb{Q}})$ of $\mathbb{P}_\star$ with respect to the Föllmer exit measure, which is also the smallest such relative entropy attainable among probability measures in $\mathfrak{P}$, namely*

$$\log\big(1/U(T,\mathbf{z})\big) = H_T(\mathbb{P}_\star\,|\,\widehat{\mathbb{Q}}) = \min_{\mathbb{P}\in\mathfrak{P}} H_T(\mathbb{P}\,|\,\widehat{\mathbb{Q}})\,; \quad \textit{as well as}$$

7. *The "total energy" $(1/2)\mathbb{E}^{\mathbb{P}_\star}\int_0^T \|\vartheta^{\mathbb{P}_\star}(t)\|^2\,dt$ expended over $[0,T]$ by the $\mathbb{P}_\star-$induced drift $\vartheta^{\mathbb{P}_\star}(\cdot)$ as in (29) to keep the relative market weight diffusion process $\mathscr{X}(\cdot)$ in the interior of the simplex, which is also the smallest such total energy attainable among probability measures in the set $\mathfrak{P}$, that is,*

$$\log\big(1/U(T,\mathbf{z})\big) = \frac{1}{2}\,\mathbb{E}^{\mathbb{P}_\star}\int_0^T \|\vartheta^{\mathbb{P}_\star}(t)\|^2\,dt = \min_{\mathbb{P}\in\mathfrak{P}} \frac{1}{2}\,\mathbb{E}^{\mathbb{P}}\int_0^T \|\vartheta^{\mathbb{P}}(t)\|^2\,dt\,.$$

In items (3)–(5) the portfolio $\widehat{\pi}(\cdot)$ is given by (25), and Π denotes the class of all portfolios; whereas in item (5) the set $\mathfrak{P}$ is the collection of all probability measures $\mathbb{P} \ll \widehat{\mathbb{Q}}$ which are absolutely continuous with respect to the Föllmer exit measure on $\mathscr{F}(T)$ and satisfy $\mathbb{P}\big(\mathscr{X}(t)\in\Delta^o,\ \forall\ 0\le t\le T\big) = 1$.

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