Original: 9/1/08
Revised: 9/15/10.

Stochastic Portfolio Theory

vs.

Modern Portfolio Theory

And

The Implications For

The Capital Asset Pricing Model

By

Adrian Banner

Robert Ferguson

Jason Greene

Carl Moss

Correspondence to:
Robert Ferguson
Senior Investment Officer
INTECH
525 Okeechobee Blvd, Suite 1800, West Palm Beach, FL 33401
561-775-1100
rferguson@intechjanus.com

Abstract

This paper contrasts the perspectives provided by the traditional Modern Portfolio Theory (MPT) analysis, which uses arithmetic returns, and the Stochastic Portfolio Theory (SPT) analysis, which uses continuous returns. The MPT analysis implies that an efficient portfolio's reward is proportional to its risk and that its information ratio is independent of its risk. The SPT analysis implies that an efficient portfolio's reward is not proportional to its risk, first rising with risk and then declining with risk, and that its information ratio declines as its risk increases. The analysis also has implications for the Capital Asset Pricing Model (CAPM). According to the MPT analysis, a stock's expected excess return is equal to its beta times the market's expected excess return. The SPT analysis shows that a stock's expected excess arithmetic return is equal to its beta times the market's expected excess arithmetic return plus one-half the market's variance of return times the excess of the stock's beta over 1. Compared to the MPT version of CAPM, the SPT version of CAPM shows that high beta stocks offer more expected excess arithmetic return and low beta stocks offer less expected excess arithmetic return.

Introduction

Actively managed portfolios, including long-only, long/short extension, and market-neutral long/short portfolios are typically characterized by their expected arithmetic relative returns, standard deviations of arithmetic relative return (tracking error), and information ratios (expected arithmetic relative return/standard deviation of arithmetic relative return).[1 2] These characterizations are too optimistic because expected arithmetic relative return overstates what is likely to be achieved over time. Long-term arithmetic relative return is less than expected arithmetic relative return.[3] It is even possible that a portfolio with an attractive expected arithmetic relative return has a negative long-term arithmetic relative return.

The discrepancy between expected arithmetic relative return and long-term arithmetic relative return is eliminated when continuous returns are used.[4] A portfolio's expected continuous relative return is its long-term continuous relative return.

This paper contrasts the perspectives provided by the traditional Modern Portfolio Theory (MPT) analysis, which uses arithmetic returns, and the Stochastic Portfolio Theory (SPT) analysis, which uses continuous returns. The MPT analysis implies that an efficient portfolio's reward is proportional to its risk and that its information ratio is independent of its risk. The SPT analysis implies that an efficient portfolio's reward is not proportional to its risk, first rising with risk and then declining with risk, and that its information ratio declines as its risk increases. The SPT analysis shows that the MPT analysis leads to excessive risk and disappointing long-term return.[5] This is particularly relevant for investors considering very risky investment portfolios, e.g., very aggressive long/short extension and market-neutral strategies.

The analysis also has implications for the Capital Asset Pricing Model (CAPM).[6] According to the MPT analysis, a stock's expected excess return is equal to its beta times

---

1 Authors who characterize actively managed portfolios in this manner, including market-neutral long/short portfolios or long/short extension portfolios, include Leibowitz and Bova (2007), Clarke, et al. (2008), Clarke, et al. (2006) Grinold (2006).

2 Arithmetic return is defined as change in value divided by beginning value.

3 Authors who address this issue include Latane (1959), Fernholz and Shay (1982), Fernholz (2002), and Hakansson (1971). Latane was the earliest author to strongly advocate choosing portfolios that maximized geometric return, which is equivalent to maximizing long-term arithmetic return. Fernholz was the first to develop a unified theoretical portfolio perspective based on log drift rates, which correspond to long-term arithmetic return.

4 Denote arithmetic and continuous return by R and r, respectively. Then the relationship between continuous return and arithmetic return is $r = \log\left(1+R\right)$, where "log" denotes "natural logarithm". Continuous returns are often called logarithmic returns.

5 Long Term Capital is a good example of this phenomenon. Its expected return was large but its long-term return was negative.

6 Merton (1990) develops a continuous time version of CAPM. However, it reflects optimization in the arithmetic drift rate domain, not the log drift rate domain. The log drift rate perspective is required to address the discrepancy between long-term arithmetic return and expected arithmetic return.

the market's expected excess return.[7] The SPT analysis shows that a stock's expected excess arithmetic return is equal to its beta times the market's expected excess arithmetic return plus one-half the market's variance of return times the excess of the stock's beta over 1. Compared to the MPT version of CAPM, the SPT version of CAPM shows that high beta stocks offer more expected excess arithmetic return and low beta stocks offer less expected excess arithmetic return.

The paper's analysis is in terms of equity portfolios with equity benchmarks. However, it can be extended easily to other markets and other benchmarks, for example to market-neutral portfolios with cash benchmarks.

The paper is organized as follows. Section two explains why expected arithmetic return is a misleading guide to long-term arithmetic return. Section three summarizes the paper's nomenclature. Section four contrasts the MPT analysis and the SPT analysis. The MPT analysis is presented in Section five. Sections six and seven summarize the arithmetic and continuous return stock price and portfolio value process approach that is the basis for the SPT analysis. The SPT analysis is presented in section eight. Section nine contrasts the MPT and SPT versions of the CAPM. Section ten is the conclusion.

An appendix is available from the authors that provides the mathematical justification for the paper's claims.

Why expected arithmetic relative return is a misleading guide to long-term arithmetic return

If expected arithmetic relative return were a good guide to long-term arithmetic relative return, then it would pay a long-term investor to hold a one-stock portfolio invested in the stock with the highest expected arithmetic return. In terms of the mean-variance efficient frontier, more risky portfolios would eventually outperform less risky portfolios because they have higher expected arithmetic relative returns. It would pay a long-term investor to maximize risk in order to maximize long-term arithmetic relative return. However this argument for taking great risk in order to obtain a high long-term return fails because long-term arithmetic relative return typically is materially less than expected arithmetic relative return, and more so as risk increases.

Consider a portfolio with a 50% chance of a relative return of 20%+100%=120% and a 50% chance of a relative return of 20%-100%=-80%. This portfolio's expected arithmetic relative return is (120-80)/2=20%. Its expected arithmetic relative return is attractive.

Consider what happens over time. Each period corresponds to the flip of a coin. For the coin, there will be about 50% heads and 50% tails over the long term. Consequently, the portfolio's arithmetic relative return will be 120% about half the time and -80% about half the time. Over a typical two periods, there will be one 120% arithmetic relative return and one -80% arithmetic relative return. A dollar invested in the portfolio will have a typical two-period relative result of up 120% to 2.20 and then down 80% to 0.44 or

[7] Excess return is defined as the difference between a stock's return and the interest rate.

down 80% to 0.20 and then up by 120% to 0.44. In each case, one relative dollar typically becomes 0.44 relative dollars after two periods, a relative return of -56% every two periods, or -33.7% per period. The portfolio's long-term arithmetic relative return is -33.7% per period versus its expected arithmetic relative return of 20% per period. The portfolio is unattractive despite its attractive 20% per period expected arithmetic relative return.

The difference between expected arithmetic relative return and long-term arithmetic return is approximately one-half the variance of arithmetic relative return. This discrepancy is eliminated when continuous relative returns are used. Expected continuous relative return is long-term continuous relative return and is equivalent to long-term arithmetic relative return. In contrast to an MPT analysis, an SPT analysis focuses on continuous return, hence provides an accurate guide to long-term results.

Nomenclature

A market-neutral long/short portfolio is defined as a set of weights (allocation proportions) that sum to 0.[8]

A portfolio is defined as a set of weights that sum to 1.

A portfolio's active weights are the differences between its weights and a benchmark's weights.

A portfolio's weights are the sum of its benchmark's weights plus its active weights.

A portfolio's active weights sum to 0.

An active portfolio is defined as a set of active weights.

An active portfolio is a market-neutral long/short portfolio and a market-neutral long/short portfolio is an active portfolio. There is no difference between them.

Long/short extension portfolios of the form (1+X)/X, e.g., 130/30, are portfolios where some stocks have negative active weight magnitudes that exceed their benchmark weights and where the sum of the excess negative active weight magnitudes is X.

In what follows, the term "portfolio" is reserved for weights that sum to 1 and the terms "active portfolio" and "market-neutral portfolio" are reserved for weights that sum to 0.

Returns are presumed to be excess returns, i.e., net of the interest rate.

---

8 This definition does not imply that the portfolio's beta is zero. However, its beta is likely to be close to zero in practice if the portfolio is broadly diversified.

The contrast between the MPT analysis and the SPT analysis

In the MPT perspective, reward is defined as expected arithmetic relative return, risk is defined as the standard deviation of arithmetic relative return (which is arithmetic tracking error), and the information ratio is defined as reward divided by risk. The portfolio's reward, risk, and information ratio are those of its active portfolio.

Scaling a portfolio's active weights up or down scales its reward and risk by the same scale factor, hence does not change its information ratio. For example, doubling a portfolio's active weights doubles its reward and doubles its risk, leaving the information ratio unchanged.

In this paper, optimum MPT portfolios are defined as mean-variance efficient in arithmetic relative return space. The optimum portfolio at a given level of risk is the portfolio with the maximum reward.[9]

Similarly, optimum active portfolios are mean-variance efficient in arithmetic relative return space. The optimum active portfolio at a given level of risk is the active portfolio with the maximum reward.

The weights of all optimum active portfolios are in the same proportion to each other, i.e., their active weight ratios are the same. The weights of any optimum active portfolio can be obtained by scaling up or down the weights of any other optimum active portfolio. For example, if one optimum portfolio's active weights for stock 1 and stock 2 are 5% and -2.5%, with a ratio of -2.0, then the ratio of the stocks' active weights is -2.0 for all optimum portfolios. A more aggressive optimum portfolio might have active weights for the two stocks of 10% and -5%, with the same ratio of -2.0.

An optimum active portfolio's weights, reward, and risk do not depend on the benchmark's weights.

An optimum active portfolio's reward is proportional to its risk. Increasing (decreasing) risk by X% increases (decreases) reward by X%. For example, suppose an optimum active portfolio has tracking error of 4% and an expected arithmetic relative return of 2%. Then an optimum active portfolio with tracking error of 8% will have an expected arithmetic relative return of 4%.

The information ratio for all optimum active portfolios is the same. No optimum active portfolio is inherently better than another is. For example, suppose a conservative opti-

---

[9] The paper's focus on relative return space reflects its use in evaluating actively managed portfolios.

The traditional MPT definition is that optimum portfolios are mean-variance efficient in arithmetic return space. Each combination of a traditional optimum portfolio and a benchmark implies an optimum active portfolio, in the sense that adding this active portfolio to the benchmark produces the traditional optimum portfolio. This paper does not address whether this implied active portfolio is efficient in relative return space.

mum active portfolio's expected arithmetic relative return is 2% and its tracking error is 4%. Then its information ratio is 0.5. A more aggressive optimum active portfolio with 8% tracking error would have an expected arithmetic relative return of 4% and the same information ratio of 0.5.

An optimum active portfolio's information ratio cannot exceed the Sharpe ratio maximizing portfolio's Sharpe ratio.

Short positions are not a prerequisite for achieving a high information ratio. For every long/short extension portfolio with a given information ratio, there are less risky long-only portfolios with the same information ratio.

> Reasoning: The long/short extension portfolio has active weights. It has short positions because some of its negative active weights are large enough to offset the benchmark's positive weights. Scaling a portfolio's active weights does not change a portfolio's information ratio. Scale down the long/short extension portfolio's active weights sufficiently to eliminate its short positions. The resulting portfolio is a long-only portfolio with the same information ratio as the long/short extension portfolio's.

A long/short extension portfolio's information ratio is not related to its leverage (the X in (1+X)/X). For every long/short extension portfolio of the form (1+X)/X with a given information ratio there are long/short extension portfolios with more leverage and more risk and less leverage and less risk, all with the same information ratio.

> Reasoning: The long/short extension portfolio has active weights. Scaling the long/short extension portfolio's active weights upward (downward) increases (decreases) the magnitude of its short positions, hence increases (decreases) its leverage (the X in its (1+X)/X). Scaling a portfolio's active weights does not change a portfolio's information ratio.

The CAPM relation is that a stock's expected excess return is equal to its beta times the market's expected excess return.

Some of the SPT perspective's implications are the same as for MPT. Others are different. SPT's implications include the following (important differences from the MPT implications are shown in italics).

Reward is defined as expected continuous relative return (the current contribution to long-term continuous relative return and equivalent to the current contribution to long-term arithmetic relative return).

Risk is defined as the standard deviation of continuous relative return (equivalent to the standard deviation of arithmetic relative return), which is continuous tracking error.[10]

The information ratio is defined as reward divided by risk.

Scaling a portfolio's active weights up or down scales its risk by the same scale factor. For example, doubling a portfolio's active weights doubles its risk.

Scaling a portfolio's active weights up or down does not scale its reward by the same scale factor. Beginning with a scale factor of zero, increasing the scale factor initially increases the active portfolio's reward but subsequently decreases it. A large enough scale factor makes the reward negative. For example, begin with a portfolio with an expected continuous relative return of 3% and tracking error of 6%. Doubling the portfolio's active weights will double the tracking error to 12%, but will not double the reward to 6%. The new reward will be less than 6%. Sufficient upward scaling would make it negative.

Scaling a portfolio's active weights up (down) decreases (increases) its information ratio.

> Reasoning: Scaling upward a portfolio's active weights scales its tracking error up proportionally but scales its expected continuous return less than proportionally. This reduces the information ratio.

In this paper, SPT optimum active portfolios are mean-variance efficient portfolios in continuous relative return space. The optimum active portfolio at a given level of risk is the active portfolio with the maximum reward.

The weights of all optimum active portfolios are in the same proportion to each other, i.e., their active weight ratios are the same. The weights of any optimum active portfolio can be obtained by scaling up or down the weights of any other optimum active portfolio. For example, if one optimum portfolio's active weights for stock 1 and stock 2 are 5% and -2.5%, with a ratio of -2.0, then the ratio of the stocks' active weights is -2.0 for all optimum portfolios. A more aggressive portfolio might have active weights for the two stocks of 10% and -5%, with the same ratio of -2.0.

An optimum active portfolio's weights, reward, and risk do depend on the benchmark's weights.

An optimum active portfolio's reward is not proportional to its risk. As risk increases from 0, the optimum active portfolio's reward increases, reaches a maximum, and then decreases and becomes negative. Too much risk leads to a negative long-term relative return.

---

[10] The continuous time processes for arithmetic and continuous returns have the same standard deviation rate. The standard deviation rates are approximately equal in discrete time.

The optimum active portfolio's information ratio depends on its risk. An increase in risk decreases the information ratio. Too much risk leads to a negative information ratio. The maximum information ratio is achieved by the optimum active portfolio with the smallest risk.

An optimum active portfolio's information ratio cannot exceed the Sharpe ratio maximizing portfolio's Sharpe ratio unless the minimum variance portfolio's growth rate materially exceeds the benchmark's growth rate, i.e., unless the benchmark is grossly inefficient.[11] *This is unlikely.*

Short positions are not a prerequisite for achieving a high information ratio. *For every long/short extension portfolio with a given information ratio, there are less risky long-only portfolios with higher information ratios.*

> Reasoning: The long/short extension portfolio has active weights. It has short positions because some of its negative active weights are large enough to offset the benchmark's positive weights. Scaling down a portfolio's active weights increases the portfolio's information ratio. Scale down the long/short extension portfolio's active weights sufficiently to eliminate its short positions. The resulting portfolio is a long-only portfolio with a higher information ratio than the long/short extension portfolio's.

A long/short extension portfolio's information ratio is a declining function of its leverage (the X in (1+X)/X). For every long/short extension portfolio of the form (1+X)/X with a given information ratio, there is a long/short extension portfolio with less leverage, i.e., of the form (1+Y)/Y where Y is less than X, with a higher information ratio. Too much leverage leads to a negative information ratio.

> Reasoning: The long/short extension portfolio has active weights. Scaling the long/short extension portfolio's active weights downward decreases the magnitude of its short positions, hence decreases its leverage (the X in its (1+X)/X). Scaling a portfolio's active weights downward increases a portfolio's information ratio.

The CAPM relation is that a stock's expected excess arithmetic return is equal to its beta times the market's expected excess arithmetic return plus one-half the market's variance of arithmetic return times the excess of the stock's beta over 1. Relative to the MPT version of the CAPM, high beta stocks have higher expected excess arithmetic returns and low beta stocks have lower expected excess arithmetic returns.

---

[11] "growth rate" denotes instantaneous expected continuous return. For most purposes, a portfolio's growth rate can be thought of as its expected continuous return or average continuous return over time.

The MPT analysis

The portfolio's expected arithmetic relative return is its active portfolio's expected arithmetic return.

$$\alpha_\delta = \sum_i R_i \delta_i \tag{1}$$

$\alpha_\delta \equiv$ The active portfolio's expected arithmetic return. This is also the active portfolio's and the portfolio's expected arithmetic relative return.

$R_i \equiv$ Stock i's expected arithmetic return.

$\delta_i \equiv$ Stock i's active portfolio weight, i.e., the difference between the stock's weight in the portfolio and its weight in the benchmark.

The portfolio's variance of arithmetic relative return is its active portfolio's variance of arithmetic return.

$$\sigma_\delta^2 = \sum_{ij} \delta_i \delta_j \sigma_{ij} \tag{2}$$

$\sigma_\delta^2 \equiv$ The active portfolio's variance of arithmetic return. This is also the active portfolio's and the portfolio's variance of arithmetic relative return.

$\sigma_{ij} \equiv$ The covariance of arithmetic return between stock i and stock j.

The portfolio's information ratio is the ratio of its expected arithmetic relative return to its standard deviation of arithmetic relative return.

$$IR_\delta = \frac{\alpha_\delta}{\sigma_\delta} = \frac{\sum_i R_i \delta_i}{\sqrt{\sum_{ij} \delta_i \delta_j \sigma_{ij}}} \tag{3}$$

Equations (1), (2), and (3) show that:

- A portfolio's reward, risk, and information ratio depend on its active weights and do not depend on the benchmark's weights.
- Multiplying a portfolio's active weights by a scale factor multiplies its expected arithmetic relative return and standard deviation of arithmetic relative return by the scale factor, and does not change its information ratio.

The MPT optimization analysis finds the portfolio that maximizes expected arithmetic relative return at a specified standard deviation of arithmetic relative return.

The mathematical statement of the MPT problem is:

$$Max\ \alpha_\delta = \sum_i R_i \delta_i \quad \text{subject to} \quad \sum_{ij} \delta_i \delta_j \sigma_{ij} = \sigma_\alpha^2, \quad \sum_i \delta_i = 0 \tag{4}$$

$\sigma_\alpha^2 \equiv$ The optimum active portfolio's variance of arithmetic return. This is also the active portfolio's and the portfolio's variance of arithmetic relative return.

It is shown in the appendix (available from the authors) that the optimum active portfolio's weights are:

$$\delta_{\alpha i} = \sigma_\alpha \delta_{\alpha 1 i} \tag{5}$$

$\delta_{\alpha i} \equiv$ Stock i's weight in the optimum active portfolio.

$\sigma_\alpha \equiv$ The optimum active portfolio's standard deviation of arithmetic relative return (tracking error).

$\delta_{\alpha 1 i} \equiv$ Stock i's weight in an optimum active portfolio with a standard deviation of arithmetic relative return (tracking error) of 1.

Equation (5) shows that optimum active portfolios have the following properties.

- Given one optimum active portfolio's weights, all other optimum active portfolios' weights are obtained by scaling the first optimum active portfolio's weights up or down.
- The magnitudes of an optimum active portfolio's weights are proportional to its standard deviation of arithmetic relative return.

- All optimum active portfolios have the same weight ratios, i.e., the ratio of any two stocks' weights for one optimum active portfolio is the same as for any other optimum active portfolio.

The optimum active portfolio's weight ratios determine its attractiveness.

Define two portfolios, π_R and π_σ. The portfolio π_R maximizes the Sharpe ratio, the ratio of expected arithmetic return to standard deviation of arithmetic return. The Sharpe ratio is an absolute return information ratio. Denote this portfolio's Sharpe ratio by $IR_{R\pi_R}$. The portfolio π_σ minimizes the variance of arithmetic return. Denote its Sharpe ratio by $IR_{R\pi_\sigma}$. Denote the minimum variance portfolio's expected arithmetic return by R_{π_σ} and its variance of arithmetic return by $\sigma^2_{\pi_\sigma}$.

It is shown in the appendix that stock i's active weight, $\delta_{\alpha i}$, does not depend on the benchmark's weights and that:

$$\delta_{\alpha i} = \sigma_\alpha \delta_{\alpha 1 i} = \sigma_\alpha \left(\frac{R_{\pi_\sigma} \big/ \sigma^2_{\pi_\sigma}}{\sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}}} \right) \left(\pi_{Ri} - \pi_{\sigma i} \right) \tag{6}$$

$\pi_{Ri} \equiv$ Stock i's weight in the Sharpe ratio maximizing portfolio.

$\pi_{\sigma i} \equiv$ Stock i's weight in the minimum variance portfolio.

Equation (6) shows that stock i's weight in the optimum active portfolio is proportional to the difference between its weight in the Sharpe ratio maximizing portfolio and its weight in the minimum variance portfolio, $\left(\pi_{Ri} - \pi_{\sigma i}\right)$. This implies that an optimum active portfolio's weight ratios depend only on these differences. Denote an optimum active portfolio's weight ratio for stock i and stock j by $w_{i/j}$. Then:

$$w_{i/j} = \frac{\left(\pi_{Ri} - \pi_{\sigma i}\right)}{\left(\pi_{Rj} - \pi_{\sigma j}\right)} \tag{7}$$

It is also shown in the appendix, that the optimum active portfolio's expected arithmetic relative return, α_δ, is:

$$\alpha_\delta = \sigma_\alpha \sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}} \tag{8}$$

Equation (8) shows that the optimum active portfolio's expected arithmetic relative return is proportional to its standard deviation of arithmetic relative return and does not depend on the benchmark's weights. Active reward is proportional to active risk.

The constant of proportionality, $\sqrt{IR_{R\pi_R}^2 - IR_{R\pi_\sigma}^2}$, is the square root of the difference between the squared Sharpe ratios of the Sharpe ratio maximizing portfolio and the minimum variance portfolio. The first information ratio is unlikely to exceed 0.5. The second is unlikely to be less than 0.1. This suggests a constant of proportionality of about $\sqrt{0.25-.01} = 0.49$ or less. If so, then a standard deviation of arithmetic relative return of about 10% would correspond to an expected arithmetic relative return of about 4.9% or less.

The optimum active portfolio's standard deviation of arithmetic relative return is σ_α. Consequently, its information ratio, IR_α , is:

$$IR_\alpha = \frac{\alpha_\delta}{\sigma_\alpha} = \sqrt{IR_{R\pi_R}^2 - IR_{R\pi_\sigma}^2} \tag{9}$$

Equation (9) shows that the optimum active portfolio's information ratio is a constant, independent of its standard deviation of arithmetic relative return and that it cannot exceed the Sharpe ratio of the Sharpe ratio maximizing portfolio. It does not depend on the benchmark's weights.

Continuing the numeric example, the optimum active portfolio's information ratio is unlikely to exceed about 0.49.

Since all optimum portfolios share the same information ratio, no level of active risk is inherently better than another is. No optimum active portfolio is inherently better than another is. No long-only or long/short extension portfolio is inherently better than another is.

Given these facts, why do some academics and professional investors claim that higher information ratios can be achieved by going short? The answer is that it is not going short that provides the higher information ratio, it is, effectively, changing the investment process.

An active portfolio's weight ratios are the ratios of each stock's active weight to a chosen stock's active weight. Scaling active weights leaves weight ratios unchanged.

The MPT analysis shows that an optimum active portfolio's information ratio is a property of its weight ratios, that all optimum active portfolios have the same weight ratios, and that all optimum active portfolios have the same information ratio.

Finding the optimal active portfolio is equivalent to finding the optimum active weight ratios.

The only way to improve a portfolio's information ratio is to move its active portfolio from an inferior set of weight ratios to a superior set of weight ratios.

The optimum set of active weight ratios is determined by stocks' expected arithmetic returns and their variances and covariances of arithmetic return. Estimates of these quantities stem from an investment process. Consequently, it makes sense to associate investment processes with the sets of active weight ratios they imply. In this context, two investment processes that lead to the same active weight ratios are equivalent and can be defined as the same.

With this definition of an investment process, the only way to improve a portfolio's information ratio is to move from an inferior investment process to a superior investment process.

The above analysis shows that there is an optimal set of active weight ratios that maximizes the active portfolio's information ratio. A conservative active portfolio has small active weight magnitudes, hence adding the benchmark's weights results in a long-only portfolio with the same information ratio. As the active portfolio is made more aggressive, its weight magnitudes increase and, eventually, some of the negative active weights more than offset the corresponding positive benchmark weights. At this point, short positions are required to maintain the optimal active weight ratios. If short positions are not allowed, the increased aggressiveness requires moving to an inferior set of active weight ratios. An inferior set of active weight ratios is, by definition, an inferior investment process.

Managers can improve their information ratios by going short only if their current investment process is inferior, at best due to excessive aggressiveness, at worst due to poor fundamental analysis, etc.

The arithmetic stock price and portfolio value processes

In the standard stock price process, a stock's arithmetic return over a short time interval is a drift rate multiplied by the time interval, plus a random disturbance. The drift rate can be thought of as an instantaneous expected arithmetic return rate (expressed in continuous form). The random disturbance is the product of a standard deviation of return rate multiplied by a draw from a normal distribution with mean zero and stand deviation equal to the square root of the time interval.

The mathematical statement of the stock's arithmetic return process is:

$$\frac{dX_i}{X_i} = r_i dt + \sigma_i dW_i \tag{10}$$

$X_i \equiv$ Stock i's price or capitalization.

$r_i \equiv$ Stock i's arithmetic return drift rate. This corresponds to the stock's expected arithmetic return. The relation between r and R is $r = \ln(1+R)$, so the correspondence is $r_i = \ln(1+E(R_i))$.

$dt \equiv$ A short time interval.

$\sigma_i \equiv$ Stock i's standard deviation of arithmetic return rate.

$dW_i \equiv$ A draw from a normal distribution with mean 0 and standard deviation $\sqrt{dt}$.

Denote a portfolio's value by Z. Let π denote the portfolio and its weights. Then the portfolio's arithmetic return process over a short time interval is the weighted average of its stocks' arithmetic returns over the short time interval.

$$\frac{dZ_\pi}{Z_\pi} = \left(\sum_i \pi_i r_i\right) dt + \sum_i \pi_i \sigma_i dW_i = r_\pi dt + \sum_i \pi_i \sigma_i dW_i \tag{11}$$

Equation (11) shows that a portfolio's arithmetic return process is just like a stock's, with an arithmetic return drift rate and a disturbance term.[12] The portfolio's arithmetic return drift rate is the weighted average of its stocks' arithmetic return drift rates. This is the familiar formula for a portfolio's arithmetic return and expected arithmetic return.

Let μ denote the benchmark portfolio and its weights. Since the benchmark is a portfolio, the same relationships apply. Consequently, the portfolio's arithmetic relative return process is obtained by subtracting the benchmark's arithmetic return process from the portfolio's.

$$\frac{dZ_\pi}{Z_\pi} - \frac{dZ_\mu}{Z_\mu} = \left(\sum_i (\pi_i - \mu_i) r_i\right) dt + \sum_i (\pi_i - \mu_i) \sigma_i dW_i \tag{12}$$

[12] The disturbance term can be written as $\sum_i \pi_i \sigma_i dW_i = \sigma_\pi dW_\pi$, where $\sigma_\pi^2 = \sum_{i,j} \pi_i \pi_j \sigma_{ij}$.

Equation (12) shows that the portfolio's arithmetic relative return process also has the same form as a stock's arithmetic return process.

Recall that the portfolio's active weight in stock i, δ_i, is $(\pi_i - \mu_i)$ and rewrite Equation (12) in terms of the portfolio's active weights.

$$\frac{dZ_\pi}{Z_\pi} - \frac{dZ_\mu}{Z_\mu} = \left(\sum_i \delta_i r_i\right) dt + \sum_i \delta_i \sigma_i dW_i \tag{13}$$

Equation (13) shows that the MPT analysis carries over to the stock price process context by using r in place of R. The solution has exactly the same result and the optimum active portfolio has the same characteristics as a function of the specified standard deviation of arithmetic relative return. All the conclusions of the MPT analysis remain the same when carried over to the arithmetic return process context.

The continuous return stock price and portfolio value processes

A stock's continuous return over a short time interval is the change in the logarithm of its price. It takes the same form as the stock's arithmetic return over a short time interval, consisting of the sum of a drift rate and a random disturbance. Fernholz (2002) shows that Equation (10) implies that the stock's continuous return process is:

$$d\log(X_i) = \gamma_i dt + \sigma_i dW_i \tag{14}$$

The standard deviation rate of continuous return in Equation (14) is the same as the standard deviation rate of arithmetic return in Equation (10).

The stock's continuous return drift rate, γ_i, is called the stock's growth rate. It is the stock's instantaneous expected continuous return rate and corresponds to the stock's long-term arithmetic return. Fernholz (2002) shows that it is:

$$\gamma_i = \left(r_i - \frac{\sigma_i^2}{2}\right) \tag{15}$$

According to Equation(15), a stock's growth rate is less than its arithmetic return drift rate by one-half of its variance. This is equivalent to saying that a stock's long-term arithmetic return is less than its expected arithmetic return by approximately one-half its variance of arithmetic return. A stock's expected arithmetic return is a misleading guide to its long-term arithmetic return.

A portfolio's continuous return process takes the same form as the stock's continuous return process, consisting of the sum of a drift rate and a random disturbance (Fernholz 2002):

$$d\log\left(Z_\pi\right) = \gamma_\pi dt + \sum_i \pi_i \sigma_i dW_i \tag{16}$$

A portfolio's continuous return drift rate, called its growth rate, is (Fernholz (2002)):

$$\gamma_\pi = \sum_i \pi_i \gamma_i + \frac{1}{2}\left(\sum_i \pi_i \sigma_i^2 - \sigma_\pi^2\right) = \sum_i \pi_i r_i - \frac{\sigma_\pi^2}{2} = r_\pi - \frac{\sigma_\pi^2}{2} \tag{17}$$

The middle term of Equation (17) shows that a portfolio's growth rate is its weighted average stock growth rate plus an excess growth rate equal to one-half the difference between the weighted average variance of the portfolio's stocks and the portfolio's variance. The right side term of Equation (17) shows that the portfolio's growth rate is equal to the weighted average arithmetic return drift rate of its stocks less one-half the portfolio's variance of continuous return. This is a significant departure from the MPT analysis, where a portfolio's arithmetic return drift rate is the weighted average of its stocks' arithmetic return drift rates.

Equation (17) shows that a portfolio's long-term arithmetic return is less than its expected arithmetic return by approximately one-half its variance of arithmetic return. A portfolio's expected arithmetic return is a misleading guide to its long-term arithmetic return.

The portfolio's continuous relative return process is obtained by subtracting the benchmark's continuous relative return process from the portfolio's.

$$d\log\left(Z_\pi / Z_\mu\right) = \left[\sum_i \left(\pi_i - \mu_i\right) r_i - \frac{1}{2}\left(\sum_{ij} \pi_i \pi_j \sigma_{ij} - \sum_{ij} \mu_i \mu_j \sigma_{ij}\right)\right] dt + \sum_i \left(\pi_i - \mu_i\right)\sigma_i dW_i \tag{18}$$

This process also is of the same form as a stock's continuous return process, with a drift rate plus a random disturbance. The drift rate is the portfolio's instantaneous expected continuous relative return rate, which is termed the portfolio's relative growth rate, denoted by a_δ.

$$a_\delta = \sum_i \left(\pi_i - \mu_i\right) r_i - \frac{1}{2}\left(\sum_{ij} \pi_i \pi_j \sigma_{ij} - \sum_{ij} \mu_i \mu_j \sigma_{ij}\right) \tag{19}$$

The portfolio's relative growth rate can be expressed in terms of its active weights as follows.

$$a_\delta = \sum_i \delta_i r_i - \sum_{ij} \mu_i \delta_j \sigma_{ij} - \frac{1}{2} \sum_{ij} \delta_i \delta_j \sigma_{ij} \tag{20}$$

An important difference between the MPT and Stochastic Portfolio Theory analysis is that the portfolio's relative growth rate is significantly different from its relative arithmetic return drift rate. It is the relative arithmetic return drift rate less the covariance rate between the benchmark and active portfolio returns and less one-half the active portfolio's variance of continuous return rate.[13][14]

The portfolio's active variance of continuous return rate is its active portfolio's variance rate, σ_a^2.

$$\sigma_a^2 = \sum_{ij} \delta_i \delta_j \sigma_{ij} \tag{21}$$

The portfolio's information ratio, IR_a, is the ratio of its active portfolio's continuous drift rate to its standard deviation of continuous return rate.

$$IR_a = \frac{a_\delta}{\sigma_a} = \frac{\sum_i \delta_i r_i - \sum_{ij} \mu_i \sigma_{ij} \delta_j - \frac{1}{2} \sum_{ij} \delta_i \delta_j \sigma_{ij}}{\sqrt{\sum_{ij} \delta_i \delta_j \sigma_{ij}}} \tag{22}$$

Equations (20), (21), and (22) show that:

- A portfolio's risk depends only on its active weights.
- A portfolio's reward and information ratio depend on its active weights and the benchmark's weights.
- Multiplying a portfolio's active weights by a scale factor:
  - Multiplies its standard deviation of continuous relative return by the same scale factor.

---

13 The active portfolio's variance of return is also its variance of relative return.

14 The covariance rate between the benchmark and active portfolio returns is likely to be close to zero in some applications. If so, then the portfolio's relative growth rate is approximately the sum of its active portfolio's weighted average stock growth rate and its active portfolio's Excess Growth Rate.

- Beginning with a scale factor of 0, increasing the scale factor initially increases the active portfolio's reward but subsequently decreases it.[15] A large enough scale factor makes the reward negative.
- Beginning with a scale factor of 0, increasing the scale factor decreases the active portfolio's information ratio.

The SPT analysis

The Stochastic Portfolio Theory analysis finds the portfolio that maximizes the continuous relative return drift rate, a_δ, at a specified standard deviation of continuous relative return rate, σ_α.

The mathematical statement of the Stochastic Portfolio Theory problem is:

$$Max\ a_\delta = \sum_i \delta_i r_i - \sum_{ij} \mu_i \sigma_{ij} \delta_j - \frac{1}{2} \sum_{ij} \delta_i \delta_j \sigma_{ij} \quad \text{subject to} \quad \sum_{ij} \delta_i \delta_j \sigma_{ij} = \sigma_\alpha^2, \quad \sum_i \delta_i = 0 \tag{23}$$

The solution (available from the authors) shows that the optimum active portfolio's weights are:

$$\delta_{ai} = \sigma_a \delta_{a1i} \tag{24}$$

$\delta_{ai} \equiv$	Stock i's weight in the optimum active portfolio with standard deviation rate of continuous relative return of σ_a.
$\sigma_a \equiv$	The optimum active portfolio's standard deviation of continuous relative return (rate), i.e., its tracking error (rate).
$\delta_{a1i} \equiv$	Stock i's weight in an optimum active portfolio with a standard deviation of continuous relative return of 1.

Equation (24) shows that the optimum active portfolios' weights have the same properties as in the traditional analysis, specifically:

---

15 It is assumed that the active portfolio is attractive, i.e., that $\sum_i \delta_i r_i - \sum_{ij} \mu_i \delta_j \sigma_{ij} > 0$ for a scale factor close enough to zero.

- Given one optimum active portfolio's weights, all other optimum active portfolios' weights are obtained by scaling the first optimum active portfolio's weights up or down.
- The magnitudes of an optimum active portfolio's weights are proportional to its standard deviation of continuous relative return.
- All optimum active portfolios have the same weight ratios, i.e., the ratio of any two stocks' weights for one optimum active portfolio is the same as for any other optimum active portfolio.

Define a portfolio, π_r, that maximizes the Sharpe ratio, defined as the ratio of the portfolio's arithmetic return rate, $r_\pi = \sum_i \pi_i r_i$, to its standard deviation of arithmetic return rate, σ_{π_r}.

Define Sharpe ratios, $IR_{r\pi_r}$ and $IR_{r\pi_\sigma}$, for the π_r and π_σ portfolios as the ratio of their arithmetic return rates, r_π, to their standard deviation of arithmetic return rates, $\sigma_\pi = \sqrt{\sum_{ij} \pi_i \pi_j \sigma_{ij}}$.

Define two active portfolios, $\underline{\delta}_{r\sigma}$ and $\underline{\delta}_{\sigma\mu}$ as follows.

$$\underline{\delta}_{r\sigma} = \underline{\pi}_r - \underline{\pi}_\sigma \tag{25}$$

$$\underline{\delta}_{\sigma\mu} = \underline{\pi}_\sigma - \underline{\mu} \tag{26}$$

Denote the minimum variance portfolio's expected arithmetic return rate by r_{π_σ}. It is shown in the appendix (available from the authors) that stock i's active weight, $\delta_{\alpha i}$ is:

$$\delta_{ai} = \sigma_a \delta_{a1i} = \sigma_a \frac{\left(r_{\pi_\sigma} / \sigma_{\pi_\sigma}^2\right)\delta_{r\sigma i} + \delta_{\sigma\mu i}}{\sqrt{\left(IR_{r\pi_r}^2 - IR_{r\pi_\sigma}^2\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}} \tag{27}$$

Unlike the MPT case, the optimum active weights do depend on the benchmark's weights.

The appendix shows that the optimum active portfolio's relative growth rate is:

$$a_\delta = \sigma_a \sqrt{\left(IR_{r\pi_r}^2 - IR_{r\pi_\sigma}^2\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)} - \frac{\sigma_a^2}{2} \tag{28}$$

This also depends on the benchmark's weights.

Equation (28) shows that the optimum active portfolio's relative growth rate is not proportional to its standard deviation of continuous relative return rate. Active reward is not proportional to active risk. Active reward first rises with risk and then declines, eventually becoming negative.[16]

The optimum active portfolio's information ratio is obtained by dividing Equation (28) by σ_a.

$$IR_a = \frac{a_\delta}{\sigma_a} = \sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)} - \frac{\sigma_a}{2} \tag{29}$$

Equation (29) shows that the optimum active portfolio's information ratio is a linear monotonically declining function of its standard deviation of continuous relative return rate. Portfolios that are more aggressive are less attractive than less aggressive portfolios. In addition, the optimum active portfolio's information ratio is unlikely to exceed the Sharpe ratio maximizing portfolio's Sharpe ratio because of the one-half tracking error deduction and the fact that it is unlikely that the minimum variance portfolio's growth rate exceeds the benchmark's growth rate and even less likely that it does so by one-half the minimum variance portfolio's Sharpe ratio.

The standard deviation of continuous relative return rate that maximizes the optimum active portfolio's relative growth rate is obtained by setting the derivative of Equation (28) to 0 and solving for σ_a.

$$\sigma_a = \sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)} \tag{30}$$

The corresponding maximum relative growth rate is found by substituting from Equation (30) back into Equation (28).

$$a_\delta = \frac{1}{2}\left[\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)\right] \tag{31}$$

Continuing the previous numeric example, assume that the growth rates of the minimum variance portfolio and the benchmark are 10% and 8%, respectively. Then the standard

---

16 In principle, active reward could decline with risk from the start if the difference between the growth rates of the minimum variance and benchmark portfolios is sufficiently negative.

deviation of continuous relative return rate that maximizes the optimum active portfolio's relative growth rate is 52.9%.

$$\sigma_a = \sqrt{\left(0.50^2 - 0.10^2\right) + 2\left(0.10 - 0.08\right)} = 0.529$$

The numeric example's maximum relative growth rate is about 14%.

$$a_\delta = \frac{1}{2}\left[\left(0.50^2 - 0.10^2\right) + 2\left(0.10 - 0.08\right)\right] = 0.14$$

Contrast this with the expected arithmetic relative return at a standard deviation of continuous relative return rate of 52.9% given by the MPT analysis of 25.9%.

$$\alpha_\delta = \sigma_\alpha \sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}} = 0.529\sqrt{0.50^2 - 0.10^2} = 0.259$$

The standard deviation of continuous relative return rate beyond which the active portfolio's relative growth rate is negative is obtained by setting Equation (28) to 0 and solving for σ_a to obtain:

$$\sigma_a = 2\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)} \tag{32}$$

The numeric example's breakeven standard deviation of relative return rate is 105.8%. Beyond this tracking error, the portfolio's long-term arithmetic return is negative.[17]

The implications for the Capital Asset Pricing Model

The standard CAPM relation is easiest to obtain by finding the condition that the Sharpe ratio maximizing portfolio must satisfy. This is the approach taken in the following two sections. The MPT and SPT versions of CAPM stem from doing this in terms of arithmetic and continuous returns, respectively.

The MPT version of the CAPM

With arithmetic returns, a portfolio's expected excess return, R_π, variance of return, σ^2_π, and squared Sharpe ratio, $IR^2_{R\pi}$, are:

---

[17] While the SPT analysis shows that the reward for aggressiveness implied by the MPT analysis is an illusion, the SPT analysis suggests that even considerable aggressiveness pays. However, trying to realize this potential is likely to lead to a disaster because the inputs to the analysis are estimates and the estimation errors lead to an excessively rosy picture of the incremental reward provided by incremental risk, thereby encouraging too much aggressiveness.

$$R_\pi = \sum_i \pi_i R_i \tag{33}$$

$$\sigma_\pi^2 = \sum_{ij} \pi_i \pi_j \sigma_{ij} \tag{34}$$

$$IR_{R\pi}^2 = \frac{\left(\sum_i \pi_i R_i\right)^2}{\sum_{ij} \pi_i \pi_j \sigma_{ij}} \tag{35}$$

The CAPM derives from taking the derivative of $IR_{R\pi}^2$ with respect to π_i, setting it equal to 0, and rearranging terms. The result is:

$$R_i = \beta_i R_\pi \tag{36}$$

Given the normal CAPM assumptions, π is the market portfolio.

Stocks' expected arithmetic excess returns are equal to their beta times the market's expected arithmetic excess return. Reward is proportional to systematic risk. No other risk matters.

The SPT version of the CAPM

In arithmetic return format, a portfolio's expected continuous return, γ_π, variance of return, σ_π^2, and squared Sharpe ratio, $IR_{r\pi}^2$, are:

$$\gamma_\pi = \sum_i \pi_i r_i - \frac{1}{2} \sum_{ij} \pi_i \pi_j \sigma_{ij} \tag{37}$$

$$\sigma_\pi^2 = \sum_{ij} \pi_i \pi_j \sigma_{ij} \tag{38}$$

$$IR_{r\pi}^2 = \frac{\left(\sum_i \pi_i r_i - \frac{1}{2} \sum_{ij} \pi_i \pi_j \sigma_{ij}\right)^2}{\underline{\pi}' \underline{\underline{V}} \underline{\pi}} \tag{39}$$

As before, the CAPM derives from taking the derivative of $IR_{r\pi}^2$ with respect to π_i with the added constraint that the sum of the weights must be 1, setting the derivative equal to 0, and rearranging terms. The SPT version of the CAPM in terms of arithmetic return drift rates is:

$$r_i = \beta_i r_\pi + \frac{\sigma_\pi^2}{2}\left(\beta_i - 1\right) \tag{40}$$

A stock's expected arithmetic excess return rate is equal to its beta times the market's expected arithmetic excess return rate plus one-half the market's variance rate of arithmetic return rate times the difference between the stock's beta and 1.

Using Equations (15), (17), and (40), the SPT version of CAPM can be restated in terms of expected continuous return rates as follows.

$$\gamma_i = \beta_i \gamma_\pi + \sigma_\pi^2\left(\beta_i - \frac{1}{2}\right) - \frac{\sigma_i^2}{2} \tag{41}$$

$$\gamma_i = \beta_i \gamma_\pi - \frac{1}{2}\left(\sigma_i^2 - \beta_i \sigma_\pi^2\right) \tag{42}$$

A stock's growth rate is equal to its beta times the market's growth rate less one-half of the difference between the stock's variance and the product of the stock's beta and the market's variance.

In contrast to the MPT version of CAPM, a stock's non-systematic risk is priced to the extent it differs from its systematic risk.

Conclusion

Long-only and long/short extension portfolios are characterized by their expected relative returns and information ratios expressed as functions of their standard deviations of relative return (tracking error). The characterization, derived from the MPT analysis, is misleading because the MPT definition of expected relative return is misleading with respect to what can be achieved over time. A definition of expected relative return that characterizes long-term results accurately leads to dramatically different conclusions. In particular:

Myth: An optimum portfolio's reward is proportional to its risk.

Fact: An optimum portfolio's reward is not proportional to its risk. As risk increases from 0, the optimum portfolio's reward rises, reaches a maximum, and then declines and eventually becomes negative.

Myth: The optimum portfolio's information ratio is independent of its risk.

Fact: The optimum portfolio's information ratio is a linearly declining function of its risk.

Myth: Short positions are a prerequisite for achieving a high information ratio.

Fact: Short positions are not a prerequisite for achieving a high information ratio.

Myth: There is an optimum leverage (the X in (1+X)/X) for an optimum long/short extension portfolio that maximizes the information ratio.

Fact: An optimum long/short extension portfolio's information ratio is a declining function of its leverage (the X in (1+X)/X).

The analysis also showed that the CAPM based on SPT differs from that based on MPT. According to the MPT analysis, a stock's expected excess arithmetic return is equal to its beta times the market's expected excess arithmetic return. The SPT analysis shows that a stock's expected excess continuous return is equal to its beta times the market's expected excess continuous return plus one-half the market's variance of continuous return times the excess of the stock's beta over 1. Compared to the MPT version of CAPM, the SPT version of CAPM showed that stocks with betas greater than 1 offer more expected excess return and stocks with betas less than 1 offer less expected excess return.

References

Bernstein, William J. and David Wilkinson. 1997. "Diversification, Rebalancing, and the Geometric Mean Frontier." *Unpublished Manuscript*.

Clarke, Roger, Harindra de Silva, Steven Sapra, and Steven Thorley. 2008. "Long-Short Extensions: How Much Is Enough?" *Financial Analysts Journal*, Vol. 64, No. 1 (January/February): 16-30.

Clarke, Roger, Harindra de Silva, Steven Sapra, and Steven Thorley. 2006. "The Fundamental Law of Active Portfolio Management." *Journal of Investment Management*, Vol. 4, No. 3 (Third Quarter): 54-72.

Fernholz, E Robert. 2002. *Stochastic Portfolio Theory*. New York, NY: Springer-Verlag.

Fernholz, E Robert, and Brian Shay. 1982. "Stochastic Portfolio Theory and Stock Market Equilibrium. "*Journal of Finance*, Vol. 37, No. 2 (May): 615-624.

Grinold, Richard. 2006. "A Dynamic Model of Portfolio Management." *Journal of Investment Management*, Vol. 4, No. 2: 5-22.

Hakansson, Nils H. 1971. "Capital Growth and the Mean-Variance Approach to Portfolio Selection." *Journal of Finance and Quantitative Analysis*, Vol VI, No. 1:517-558.

Jacobs, Bruce and Kenneth Levy. 1996. "20 Myths about Long-Short." *Financial Analysts Journal*, Vol. 52, No. 5 (September): 81-85.

Latane, Henry A. 1959. "Criteria For Choice Among Risky Ventures". *Journal of Political Economy*, April.

Leibowitz, Martin and Anthony Bova. 2007. "Active 130/30 Extensions: Alpha Hunting at the Fund Level." *Journal of Investment Management*, Vol. 5, No. 3 (Third Quarter).

Merton, Robert C. 1990. "Continuous Time Finance." Blackwell Publishers.

Appendix (available on request from the authors)

The MPT analysis

The MPT analysis finds the active portfolio that maximizes expected arithmetic relative return at a specified standard deviation of arithmetic relative return (tracking error).

The mathematical statement of the MPT problem is:

$$Max\ \alpha_\delta = \underline{R}'\underline{\delta} \quad \text{subject to} \quad \underline{\delta}'\underline{\underline{V}}\underline{\delta} = \sigma_\alpha^2$$
$$\underline{1}'\underline{\delta} = 0$$
(43)

$\alpha_\delta \equiv$ The active portfolio's expected arithmetic return. This is also the active portfolio's expected arithmetic relative return.

$\underline{R} \equiv$ A column vector of stocks' expected arithmetic returns.

$\sigma_\alpha^2 \equiv$ The active portfolio's variance of arithmetic return. This is also the active portfolio's variance of arithmetic relative return.

$\underline{\underline{V}} \equiv$ Stocks' covariance matrix of arithmetic returns.

$\underline{\delta} \equiv$ A column vector of the active portfolio's weights.

Form the Lagrangian, L.

$$L = \underline{R}'\underline{\delta} - \lambda_1\left(\underline{\delta}'\underline{\underline{V}}\underline{\delta} - \sigma_\alpha^2\right) - \lambda_2\left(\underline{1}'\underline{\delta}\right)$$
(44)

Take the derivative of the Lagrangian, set it equal to 0, and solve for the optimum active weights.

$$\frac{dL}{d\underline{\underline{\delta}}} = \underline{R} - 2\lambda_1\underline{\underline{V}}\underline{\delta} - \lambda_2\underline{1} = 0$$
(45)

$$\underline{\delta} = \left(\frac{1}{2\lambda_1}\right)\left(\underline{\underline{V}}^{-1}\underline{R} - \lambda_2\underline{\underline{V}}^{-1}\underline{1}\right)$$

(46)

Use the second constraint to solve for λ_2.

$$\underline{1}'\underline{\delta} = \left(\frac{1}{2\lambda_1}\right)\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{R} - \lambda_2\underline{1}'\underline{\underline{V}}^{-1}\underline{1}\right) = 0$$

(47)

$$\lambda_2 = \frac{\underline{1}'\underline{\underline{V}}^{-1}\underline{R}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}}$$

(48)

Equation (48) holds as long as $\underline{1}'\underline{\underline{V}}^{-1}\underline{1} \neq 0$, which is true, since the inverse of the variance matrix is positive definite.

$$\underline{\delta} = \left(\frac{1}{2\lambda_1}\right)\left(\underline{\underline{V}}^{-1}\underline{R} - \left(\frac{\underline{1}'\underline{\underline{V}}^{-1}\underline{R}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}}\right)\underline{\underline{V}}^{-1}\underline{1}\right)$$

(49)

Define two new portfolios, $\underline{\pi}_R$ and $\underline{\pi}_\sigma$.

$$\underline{\pi}_R = \frac{\underline{\underline{V}}^{-1}\underline{R}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{R}}$$

(50)

$$\underline{\pi}_\sigma = \frac{\underline{\underline{V}}^{-1}\underline{1}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}}$$

(51)

The portfolio $\underline{\pi}_R$ maximizes the Sharpe ratio. The portfolio $\underline{\pi}_R$ is defined if $\underline{1}'\underline{\underline{V}}^{-1}\underline{R} \neq 0$, which is almost surely true. Its weights sum to 1.[18]

The portfolio $\underline{\pi}_\sigma$ minimizes the variance of arithmetic return. Its weights sum to 1.

---

18 If $\underline{1}'\underline{\underline{V}}^{-1}\underline{R} = 0$, then the unconstrained portfolio that maximizes the ratio of expected arithmetic return to standard deviation of arithmetic return (the Sharpe ratio) is an active (market-neutral) portfolio.

Substitute from Equations (50) and (51) into Equation (49).

$$\underline{\delta} = \left(\frac{\underline{1}' \underline{\underline{V}}^{-1} \underline{R}}{2\lambda_1} \right) \left(\underline{\pi}_R - \underline{\pi}_\sigma \right)$$

(52)

Use the first constraint to solve for λ_1.

$$\underline{\delta}' \underline{\underline{V}} \underline{\delta} = \left(\frac{\underline{1}' \underline{\underline{V}}^{-1} \underline{R}}{2\lambda_1} \right)^2 \left(\underline{\pi}_R - \underline{\pi}_\sigma \right)' \underline{\underline{V}} \left(\underline{\pi}_R - \underline{\pi}_\sigma \right) = \sigma_\alpha^2$$

(53)

$$\underline{\pi}_R{}' \underline{\underline{V}} \underline{\pi}_R = \frac{\underline{R}' \underline{\underline{V}}^{-1} \underline{R}}{\left(\underline{1}' \underline{\underline{V}}^{-1} \underline{R} \right)^2}$$

(54)

$$\underline{\pi}_\sigma{}' \underline{\underline{V}} \underline{\pi}_\sigma = \frac{1}{\underline{1}' \underline{\underline{V}}^{-1} \underline{1}}$$

(55)

$$\underline{\pi}_R{}' \underline{\underline{V}} \underline{\pi}_\sigma = \frac{1}{\underline{1}' \underline{\underline{V}}^{-1} \underline{1}}$$

(56)

$$\left(\frac{\underline{1}' \underline{\underline{V}}^{-1} \underline{R}}{2\lambda_1} \right)^2 \left(\frac{\underline{R}' \underline{\underline{V}}^{-1} \underline{R}}{\left(\underline{1}' \underline{\underline{V}}^{-1} \underline{R} \right)^2} - \frac{1}{\underline{1}' \underline{\underline{V}}^{-1} \underline{1}} \right) = \sigma_\alpha^2$$

(57)

$$\left(\frac{1}{2\lambda_1} \right)^2 \left(\underline{R}' \underline{\underline{V}}^{-1} \underline{R} - \frac{\left(\underline{1}' \underline{\underline{V}}^{-1} \underline{R} \right)^2}{\underline{1}' \underline{\underline{V}}^{-1} \underline{1}} \right) = \sigma_\alpha^2$$

(58)

$$\left(\frac{1}{2\lambda_1}\right)=\frac{\sigma_\alpha}{\sqrt{\underline{R}'\underline{\underline{V}}^{-1}\underline{R}-\frac{\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{R}\right)^2}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}}}} \quad (59)$$

The optimum active portfolio, $\underline{\delta}_\alpha$, for a standard deviation of arithmetic relative return of σ_α is:

$$\underline{\delta}_\alpha=\sigma_\alpha\left(\frac{\underline{1}'\underline{\underline{V}}^{-1}\underline{R}}{\sqrt{\underline{R}'\underline{\underline{V}}^{-1}\underline{R}-\frac{\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{R}\right)^2}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}}}}\right)\left(\underline{\pi}_R-\underline{\pi}_\sigma\right) \quad (60)$$

Denote the Sharpe ratios of the $\underline{\pi}_R$ and $\underline{\pi}_\sigma$ portfolios by $IR_{R\pi_R}$ and $IR_{R\pi_\sigma}$, respectively. These information ratios are expected arithmetic return divided by standard deviation of arithmetic return. It can be shown that:

$$IR^2_{R\pi_R}=\left(\frac{R_{\pi_R}}{\sigma_{\pi_R}}\right)^2=\underline{R}'\underline{\underline{V}}^{-1}\underline{R} \quad (61)$$

$$IR^2_{R\pi_\sigma}=\left(\frac{R_{\pi_\sigma}}{\sigma_{\pi_\sigma}}\right)^2=\frac{\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{R}\right)^2}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}} \quad (62)$$

$$\underline{1}'\underline{\underline{V}}^{-1}\underline{R}=\frac{R_{\pi_\sigma}}{\sigma^2_{\pi_\sigma}} \quad (63)$$

Substitute from Equations (61), (62) and (63) into Equation (60).

$$\underline{\delta}_\alpha = \sigma_\alpha \left(\frac{R_{\pi_\sigma} / \sigma^2_{\pi_\sigma}}{\sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}}} \right) \left(\underline{\pi}_R - \underline{\pi}_\sigma \right)$$

(64)

Define a reference optimum active portfolio, $\underline{\delta}_{\alpha 1}$, as the optimum active portfolio with a standard deviation of arithmetic relative return of 1. Then:

$$\underline{\delta}_{\alpha 1} = \left(\frac{R_{\pi_\sigma} / \sigma^2_{\pi_\sigma}}{\sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}}} \right) \left(\underline{\pi}_R - \underline{\pi}_\sigma \right)$$

(65)

$$\underline{\delta}_\alpha = \sigma_\alpha \underline{\delta}_{\alpha 1}$$

(66)

Denote the column vector of the benchmark's weights by $\underline{\mu}$. The requirement that the benchmark's weights sum to 1 means that they correspond to a point on the hyperplane $\underline{\mu}'\underline{1} = 1$. A portfolio whose weights are the sum of the benchmark's weights and a set of active weights also corresponds to a point on this hyperplane. Therefore, the active weight vector corresponds to a vector lying in this hyperplane. The active weight vector is a segment of a ray originating at the benchmark point. Scaling an active portfolio's weights corresponds to moving along this ray. Every optimum active portfolio is a segment of the same ray from the benchmark point. The segment is longer or shorter as the specified standard deviation of arithmetic relative return is larger or smaller.

The optimum active portfolio's expected arithmetic relative return, α_δ, is:

$$\alpha_\delta = \underline{R}' \underline{\delta}_\alpha = \sigma_\alpha \left(\frac{\underline{R}' \underline{\underline{V}}^{-1} \underline{R} - \left(\frac{\underline{1}' \underline{\underline{V}}^{-1} \underline{R}}{\underline{1}' \underline{\underline{V}}^{-1} \underline{1}} \right) \underline{R}' \underline{\underline{V}}^{-1} \underline{1}}{\sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}}} \right) = \sigma_\alpha \frac{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}}{\sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}}}$$

(67)

$$\alpha_\delta = \sigma_\alpha \sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}}$$

(68)

Equation (68) implies that the efficient frontier in arithmetic relative return mean-variance space is a parabola, just as it is in arithmetic return mean-variance space. However, in this case, the parabola is symmetric with respect to the risk axis and is tangent to the expected arithmetic relative return axis at the origin.

The optimum active portfolio's standard deviation of arithmetic relative return is σ_α. Consequently, its information ratio, IR_α, is:

$$IR_\alpha = \frac{\alpha_\delta}{\sigma_\alpha} = \sqrt{IR^2_{R\pi_R} - IR^2_{R\pi_\sigma}} \tag{69}$$

The MPT analysis shows that an optimum active portfolio's information ratio is a property of its active weight ratios, that all optimum active portfolios lie along the same ray from the benchmark, and that all optimum active portfolios have the same information ratio. A similar analysis shows that scaling any active weights, not just optimum active weights, leaves the information ratio unchanged. All active portfolio's that lie on the same ray from the benchmark have the same information ratio.

Finding the optimal active portfolio is equivalent to finding the optimum ray from the benchmark.

The only way to improve a portfolio's information ratio is to move its active portfolio from an inferior ray to a superior ray.

Equation (65) shows that the optimum ray from the benchmark is determined by stocks' expected arithmetic returns and their variances and covariances of arithmetic return. Estimates of these quantities stem from an investment process. Consequently, it makes sense to associate investment processes with the rays from the benchmark they imply. In this context, two investment processes that lead to the same ray from the benchmark are equivalent.

As the above analysis shows, there is an optimal ray from the benchmark that maximizes the active portfolio's information ratio. A conservative active portfolio on this ray has small active weight magnitudes, hence adding the benchmark's weights results in a long-only portfolio with the same information ratio. As the active portfolio is made more aggressive, it moves out along the optimal ray and the magnitudes of the negative active weights increase until, eventually, some more than offset the corresponding positive benchmark weights. At this point, short positions are required to stay on the optimal ray. If short positions are not allowed, the increased aggressiveness requires moving to an inferior ray. An inferior ray is, by definition, an inferior investment process.

The arithmetic stock price and portfolio value processes

The mathematical statement of a stock's arithmetic return process is:

$$\frac{dX_i(t)}{X_i(t)} = r_i(t)\,dt + \sigma_i(t)\,dW_i(t)$$

(70)

$X_i(t) \equiv$	Stock i's price or capitalization at time t.
$r_i(t) \equiv$	Stock i's expected arithmetic return rate. The relation between $\underline{r}$ and $\underline{R}$ is $\underline{r} = \underline{\ln(\underline{1} + \underline{R})}$.
$\sigma_i(t) \equiv$	Stock i's standard deviation of arithmetic return rate.
$dW_i(t) \equiv$	A draw from a normal distribution with mean 0 and standard deviation $\sqrt{dt}$.

In what follows, the time dependence of the parameters and variables is presumed but the notation is dropped for expositional ease.

Denote a portfolio's value by Z. Let $\underline{\pi}$ be a column vector of portfolio weights. Then the portfolio's arithmetic return process over a short time interval is the weighted average of its stocks' arithmetic returns over the short time interval.

$$\frac{dZ_\pi}{Z_\pi} = \left(\sum_i \pi_i r_i\right) dt + \sum_i \pi_i \sigma_i dW_i$$

(71)

Taking into account that the disturbances are correlated, with covariances σ_{ij}, this can be rewritten as:

$$\frac{dZ_\pi}{Z_\pi} = r_\pi dt + \sigma_\pi dW_\pi$$

(72)

$$r_\pi = \sum_i \pi_i r_i$$

(73)

$$\sigma_\pi^2 = \sum_{i,j} \pi_i \pi_j \sigma_{ij}$$

(74)

The benchmark is a portfolio with weights $\underline{\mu}$. Its arithmetic return process is:

$$\frac{dZ_{\mu}}{Z_{\mu}} = r_{\mu}dt + \sigma_{\mu}dW_{\mu} \tag{75}$$

$$r_{\mu} = \sum_{i} \mu_i r_i \tag{76}$$

$$\sigma_{\mu}^2 = \sum_{ij} \mu_i \mu_j \sigma_{ij} \tag{77}$$

The portfolio's arithmetic relative return process is:

$$\frac{dZ_{\pi}}{Z_{\pi}} - \frac{dZ_{\mu}}{Z_{\mu}} = \left(\sum_{i} (\pi_i - \mu_i) r_i \right) dt + \sum_{i} (\pi_i - \mu_i) \sigma_i dW_i \tag{78}$$

$$\frac{dZ_{\pi}}{Z_{\pi}} - \frac{dZ_{\mu}}{Z_{\mu}} = \left(\sum_{i} \delta_i r_i \right) dt + \sum_{i} \delta_i \sigma_i dW_i \tag{79}$$

$\delta_i \equiv$ The portfolio's active weight for stock i.

Equation (79) shows that portfolio's arithmetic relative return process is of the same form as for the MPT analysis except that $\underline{r}$ replaces $\underline{R}$. Therefore, all the conclusions of the MPT analysis remain the same.

The logarithmic stock price and portfolio value processes

A stock's continuous return over a time interval is the change in the logarithm of its price. Over a short time interval, this is (Fernholz (2002)):

$$d\log(X_i) = \gamma_i dt + \sigma_i dW_i \tag{80}$$

The stock's growth rate, γ_i, is:

$$\gamma_i = \left(r_i - \frac{\sigma_i^2}{2} \right) \tag{81}$$

Fernholz (2002) shows that, under reasonable assumptions, the stock's long-term time average continuous return is almost surely its long-term time average growth rate, γ_i.

The portfolio's logarithmic process is (Fernholz 2002):

$$d\log\left(Z_\pi\right) = \gamma_\pi dt + \sum_i \pi_i \sigma_i dW_i \tag{82}$$

$$\gamma_\pi = \sum_i \pi_i \gamma_i + \gamma_\pi^* \tag{83}$$

$$\gamma_\pi^* = \frac{1}{2}\left(\sum_i \pi_i \sigma_{ii} - \sum_{ij} \pi_i \pi_j \sigma_{ij}\right) \tag{84}$$

γ_π is the portfolio's growth rate. Fernholz (2002) terms γ_π^* the portfolio's excess growth rate.

Equation (82) can be rewritten as:

$$d\log\left(Z_\pi\right) = \left(\sum_i \pi_i r_i - \frac{\sum_{ij} \pi_i \pi_j \sigma_{ij}}{2}\right) dt + \sum_i \pi_i \sigma_i dW_i \tag{85}$$

A similar equation holds for the benchmark.

$$d\log\left(Z_\mu\right) = \left(\sum_i \mu_i r_i - \frac{\sum_{ij} \mu_i \mu_j \sigma_{ij}}{2}\right) dt + \sum_i \mu_i \sigma_i dW_i \tag{86}$$

Combining Equations (85) and (86), the portfolio's continuous relative return process is:

$$d\log\left(Z_\pi / Z_\mu\right) = \sum_i \delta_i r_i - \frac{1}{2}\left(\sum_{ij} \pi_i \pi_j \sigma_{ij} - \sum_{ij} \mu_i \mu_j \sigma_{ij}\right) + \sum_i \delta_i \sigma_i dW_i \tag{87}$$

The portfolio's relative growth rate, a_δ, is

$$a_\delta = \sum_i \delta_i r_i - \frac{1}{2}\left(\sum_{ij} \pi_i \pi_j \sigma_{ij} - \sum_{ij} \mu_i \mu_j \sigma_{ij}\right) = \underline{r}'\,\underline{\delta} - \frac{1}{2}\left(\underline{\pi}'\underline{\underline{V}}\,\underline{\pi} - \underline{\mu}'\underline{\underline{V}}\,\underline{\mu}\right) \tag{88}$$

Express the portfolio's weights in terms of the benchmark's and active portfolio's weights.

$$a_\delta = \underline{r}'\,\underline{\delta} - \frac{1}{2}\left(\left(\underline{\mu} + \underline{\delta}\right)'\underline{\underline{V}}\left(\underline{\mu} + \underline{\delta}\right) - \underline{\mu}'\underline{\underline{V}}\,\underline{\mu}\right) = \underline{r}'\,\underline{\delta} - \underline{\mu}'\underline{\underline{V}}\,\underline{\delta} - \frac{\underline{\delta}'\underline{\underline{V}}\,\underline{\delta}}{2} \tag{89}$$

$$a_\delta = \left(\underline{r} - \underline{\underline{V}}\,\underline{\mu}\right)'\underline{\delta} - \frac{\underline{\delta}'\underline{\underline{V}}\,\underline{\delta}}{2} \tag{90}$$

The active portfolio's variance of continuous (relative) return rate, σ_a^2, is:

$$\sigma_a^2 = \sum_{ij} \delta_i \delta_j \sigma_{ij} = \underline{\delta}'\underline{\underline{V}}\,\underline{\delta} \tag{91}$$

The continuous relative return process's information ratio, IR_a, is the ratio of its growth rate to its standard deviation of continuous (relative) return rate.

$$IR_a = \frac{a_\delta}{\sigma_a} = \frac{\left(\underline{r} - \underline{\underline{V}}\,\underline{\mu}\right)'\underline{\delta} - \frac{\underline{\delta}'\underline{\underline{V}}\,\underline{\delta}}{2}}{\sqrt{\underline{\delta}'\underline{\underline{V}}\,\underline{\delta}}} \tag{92}$$

The SPT analysis

The mathematical statement of the SPT version of the problem is:

$$Max\ a_\delta = \left(\underline{r} - \underline{\underline{V}}\,\underline{\mu}\right)'\underline{\delta} - \frac{\underline{\delta}'\underline{\underline{V}}\,\underline{\delta}}{2} \quad \text{subject to} \quad \begin{array}{l} \underline{\delta}'\underline{\underline{V}}\,\underline{\delta} = \sigma_\alpha^2 \\ \underline{1}'\,\underline{\delta} = 0 \end{array} \tag{93}$$

Form the Lagrangian, L.

$$L = \left(\underline{r} - \underline{\underline{V}}\underline{\mu}\right)' \underline{\delta} - \frac{\underline{\delta}' \underline{\underline{V}} \underline{\delta}}{2} - \lambda_1 \left(\underline{\delta}' \underline{\underline{V}} \underline{\delta} - \sigma_\alpha^2\right) - \lambda_2 \left(\underline{1}' \underline{\delta}\right)$$

(94)

Take the derivative of the Lagrangian, set it to 0, and solve for the optimum active weights.

$$\underline{\left(\frac{dL}{d\underline{\delta}}\right)} = \left(\underline{r} - \underline{\underline{V}}\underline{\mu}\right) - \underline{\underline{V}}\underline{\delta} - 2\lambda_1 \underline{\underline{V}}\underline{\delta} - \lambda_2 \underline{1} = 0$$

(95)

$$\underline{\delta} = \left(\frac{1}{1+2\lambda_1}\right)\left[\underline{\underline{V}}^{-1}\left(\underline{r} - \underline{\underline{V}}\underline{\mu}\right) - \lambda_2 \underline{\underline{V}}^{-1}\underline{1}\right]$$

(96)

Define a portfolio, $\underline{\pi}_r$, that maximizes the absolute continuous information ratio defined as the ratio of the portfolio's arithmetic return rate, $\underline{\pi}_r{}'\underline{r}$ to its standard deviation of arithmetic return rate. It can be shown that:

$$\underline{\pi}_r = \frac{\underline{\underline{V}}^{-1}\underline{r}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}}$$

(97)

The portfolio $\underline{\pi}_r$ is defined as long as $\underline{1}'\underline{\underline{V}}^{-1}\underline{r} \neq 0$, which is almost surely true.

$$\underline{\delta} = \left(\frac{1}{1+2\lambda_1}\right)\left[\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\underline{\pi}_r - \underline{\mu} - \lambda_2\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{1}\right)\underline{\pi}_\sigma\right]$$

(98)

Use the second constraint to evaluate λ_2.

$$\underline{1}'\underline{\delta} = \left(\frac{1}{1+2\lambda_1}\right)\left[\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right) - 1 - \lambda_2\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{1}\right)\right] = 0$$

(99)

$$\lambda_2 = \left(\frac{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}} - \frac{1}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}}\right)$$

(100)

Substitute in Equation (98).

$$\underline{\delta} = \left(\frac{1}{1+2\lambda_1}\right)\left[\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\underline{\pi}_r - \underline{\mu} - \left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\underline{\pi}_\sigma + \underline{\pi}_\sigma\right] \tag{101}$$

Rearrange Equation (101) using the two active portfolios $\underline{\delta}_{r\sigma}$ and $\underline{\delta}_{\sigma\mu}$.

$$\underline{\delta} = \left(\frac{1}{1+2\lambda_1}\right)\left[\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\underline{\delta}_{r\sigma} + \underline{\delta}_{\sigma\mu}\right] \tag{102}$$

Now, evaluate $\left(\frac{1}{1+2\lambda_1}\right)$ using the first constraint.

$$\underline{\delta}'\underline{\underline{V}}\underline{\delta} = \left(\frac{1}{1+2\lambda_1}\right)^2\left[\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)^2 \underline{\delta}_{r\sigma}'\underline{\underline{V}}\underline{\delta}_{r\sigma} + 2\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\underline{\delta}_{r\sigma}'\underline{\underline{V}}\underline{\delta}_{\sigma\mu} + \underline{\delta}_{\sigma\mu}'\underline{\underline{V}}\underline{\delta}_{\sigma\mu}\right] = \sigma_a^2 \tag{103}$$

Evaluate the terms in Equation (103) and substitute.

$$\underline{\delta}_{r\sigma}'\underline{\underline{V}}\underline{\delta}_{r\sigma} = \left(\underline{\pi}_r - \underline{\pi}_\sigma\right)'\underline{\underline{V}}\left(\underline{\pi}_r - \underline{\pi}_\sigma\right) = \sigma_{\pi_r}^2 - 2\sigma_{\pi_r\pi_\sigma} + \sigma_{\pi_\sigma}^2 \tag{104}$$

$$\sigma_{\pi_r\pi_\sigma} = \frac{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}}{\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{1}\right)} = \frac{1}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}} = \sigma_{\pi_\sigma}^2 \tag{105}$$

$$\underline{\delta}_{r\sigma}'\underline{\underline{V}}\underline{\delta}_{r\sigma} = \sigma_{\pi_r}^2 - \sigma_{\pi_\sigma}^2 \tag{106}$$

$$\underline{\delta}_{\sigma\mu}'\underline{\underline{V}}\underline{\delta}_{\sigma\mu} = \left(\underline{\pi}_\sigma - \underline{\mu}\right)'\underline{\underline{V}}\left(\underline{\pi}_\sigma - \underline{\mu}\right) = \sigma_{\pi_\sigma}^2 - 2\sigma_{\pi_\sigma\mu} + \sigma_\mu^2 \tag{107}$$

$$\sigma_{\pi_\sigma\mu} = \frac{\underline{1}'\underline{\underline{V}}^{-1}\underline{\underline{V}}\underline{\mu}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}} = \frac{1}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}} = \sigma_{\pi_\sigma}^2$$

(108)

$$\underline{\delta}_{\sigma\mu}{}'\underline{\underline{V}}\underline{\delta}_{\sigma\mu} = -\left(\sigma^2_{\pi_\sigma} - \sigma^2_{\mu}\right)$$

(109)

$$\underline{\delta}_{r\sigma}{}'\underline{\underline{V}}\underline{\delta}_{\sigma\mu} = \left(\underline{\pi}_r - \underline{\pi}_\sigma\right)'\underline{\underline{V}}\left(\underline{\pi}_\sigma - \underline{\mu}\right) = \sigma_{\pi_r\pi_\sigma} - \sigma_{\pi_r\mu} - \sigma^2_{\pi_\sigma} + \sigma_{\pi_\sigma\mu}$$

(110)

$$\sigma_{\pi_r\mu} = \frac{\underline{r}'\underline{\underline{V}}^{-1}\underline{\underline{V}}\underline{\mu}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}} = \frac{r_\mu}{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}}$$

(111)

$$\underline{\delta}_{r\sigma}{}'\underline{\underline{V}}\underline{\delta}_{\sigma\mu} = \sigma^2_{\pi_\sigma} - \frac{r_\mu}{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}} - \sigma^2_{\pi_\sigma} + \sigma^2_{\pi_\sigma} = \sigma^2_{\pi_\sigma} - \frac{r_\mu}{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}}$$

(112)

$$\underline{\delta}'\underline{\underline{V}}\underline{\delta} = \left(\frac{1}{1+2\lambda_1}\right)^2\left[\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)^2\left(\sigma^2_{\pi_r} - \sigma^2_{\pi_\sigma}\right) + 2\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\left(\sigma^2_{\pi_\sigma} - \frac{r_\mu}{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}}\right) - \left(\sigma^2_{\pi_\sigma} - \sigma^2_{\mu}\right)\right] = \sigma^2_a$$

(113)

Define arithmetic return rate information ratios, $IR_{r\pi_r}$ and $IR_{r\pi_\sigma}$, for the $\underline{\pi}_r$ and $\underline{\pi}_\sigma$ portfolios as their arithmetic return rate, $r = \underline{r}'\underline{\pi}$, divided by their standard deviation of arithmetic return rate, $\sigma_\pi = \sqrt{\underline{\pi}'\underline{\underline{V}}\underline{\pi}}$. Then it can be shown that:

$$IR^2_{r\pi_r} = \underline{r}'\underline{\underline{V}}^{-1}\underline{r} = \left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)^2\sigma^2_{\pi_r}$$

(114)

$$IR^2_{r\pi_\sigma} = \frac{\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)^2}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}} = \left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)^2\sigma^2_{\pi_\sigma}$$

(115)

Also note that:

$$\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\sigma_{\pi_\sigma}^2 = \frac{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}}{\underline{1}'\underline{\underline{V}}^{-1}\underline{1}} = r_{\pi_\sigma}$$

(116)

Substitute from Equations (114), (115), and (116) into (113).

$$\left(\frac{1}{1+2\lambda_1}\right)^2\left[\left(IR_{r\pi_r}^2 - IR_{r\pi_\sigma}^2\right) + 2\left(r_{\pi_\sigma} - r_\mu\right) - \left(\sigma_{\pi_\sigma}^2 - \sigma_\mu^2\right)\right] = \sigma_a^2$$

(117)

Equations (83) and(84) imply that:

$$\gamma_\pi = r_\pi - \frac{\sigma_\pi^2}{2}$$

(118)

Therefore:

$$\left(\frac{1}{1+2\lambda_1}\right)^2\left[\left(IR_{r\pi_r}^2 - IR_{r\pi_\sigma}^2\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)\right] = \sigma_a^2$$

(119)

$$\left(\frac{1}{1+2\lambda_1}\right)^2 = \frac{\sigma_a^2}{\left(IR_{r\pi_r}^2 - IR_{r\pi_\sigma}^2\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}$$

(120)

Substitute from Equation (120) into Equation (102) to obtain the optimum active portfo-lio.

$$\underline{\delta}_a = \sigma_a \frac{\left(\underline{1}'\underline{\underline{V}}^{-1}\underline{r}\right)\underline{\delta}_{r\sigma} + \underline{\delta}_{\sigma\mu}}{\sqrt{\left(IR_{r\pi_r}^2 - IR_{r\pi_\sigma}^2\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}}$$

(121)

Equation (116) implies that:

$$\underline{1}'\underline{\underline{V}}^{-1}\underline{r} = \frac{r_{\pi_\sigma}}{\sigma_{\pi_\sigma}^2}$$

(122)

$$\underline{\delta}_a = \sigma_a \frac{\left(r_{\pi_\sigma} / \sigma^2_{\pi_\sigma}\right)\underline{\delta}_{r\sigma} + \underline{\delta}_{\sigma\mu}}{\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}}$$

(123)

Define a reference optimum active portfolio, $\underline{\delta}_{a1}$, as the optimum active portfolio for a standard deviation of continuous (relative) return rate of 1. Then:

$$\underline{\delta}_{a1} = \frac{\left(r_{\pi_\sigma} / \sigma^2_{\pi_\sigma}\right)\underline{\delta}_{r\sigma} + \underline{\delta}_{\sigma\mu}}{\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}}$$

(124)

$$\underline{\delta}_a = \sigma_a \underline{\delta}_{a1}$$

(125)

As with the MPT analysis, the optimum active weight vector is a segment of a ray originating at the benchmark point, every optimum active portfolio is a segment of the same ray from the benchmark point, and the segment is longer or shorter as the specified standard deviation of continuous (relative) return rate is larger or smaller.

The optimum active portfolio's relative growth rate, a_δ, is found by substituting from Equation (125) into Equation(90).

$$a_\delta = \left[\left(\underline{r} - \underline{\underline{V}}\,\underline{\mu}\right)' \underline{\delta}_{a1}\right]\sigma_a - \left(\frac{\underline{\delta}_{a1}{}'\,\underline{\underline{V}}\,\underline{\delta}_{a1}}{2}\right)\sigma_a^2$$

(126)

Equation (126) can be expressed in terms of the foregoing information ratios and continuous drift rates as follows.

$$\underline{r}'\underline{\delta}_{a1} = \underline{r}' \frac{\left(r_{\pi_\sigma} / \sigma^2_{\pi_\sigma}\right)\underline{\delta}_{r\sigma} + \underline{\delta}_{\sigma\mu}}{\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}} = \frac{\left(r_{\pi_\sigma} / \sigma^2_{\pi_\sigma}\right)\left(r_{\pi_r} - r_{\pi_\sigma}\right) + \left(r_{\pi_\sigma} - r_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}}$$

(127)

$$\left(\underline{\underline{V}}\,\underline{\mu}\right)' \underline{\delta}_{a1} = \underline{\mu}'\underline{\underline{V}} \frac{\left(r_{\pi_\sigma} / \sigma^2_{\pi_\sigma}\right)\underline{\delta}_{r\sigma} + \underline{\delta}_{\sigma\mu}}{\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}} = \frac{\left(r_{\pi_\sigma} / \sigma^2_{\pi_\sigma}\right)\left(\sigma_{\pi_r\mu} - \sigma_{\pi_\sigma\mu}\right) + \left(\sigma_{\pi_\sigma\mu} - \sigma^2_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}}$$

(128)

Substitute from Equations (108) and (111) into Equation(128).

$$\left(\underline{\underline{V}}\,\underline{\mu}\right)'\underline{\delta}_{a1}=\frac{\left(r_{\pi_\sigma}/\sigma^2_{\pi_\sigma}\right)\left(\frac{r_\mu}{\underline{1}'\underline{\underline{V}}^{-1}\underline{r}}-\sigma^2_{\pi_\sigma}\right)+\left(\sigma^2_{\pi_\sigma}-\sigma^2_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r}-IR^2_{r\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}}=\frac{\left(r_\mu-\left(r_{\pi_\sigma}/\sigma^2_{\pi_\sigma}\right)\sigma^2_{\pi_\sigma}\right)+\left(\sigma^2_{\pi_\sigma}-\sigma^2_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r}-IR^2_{r\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}} \tag{129}$$

$$\left(\underline{\underline{V}}\,\underline{\mu}\right)'\underline{\delta}_{a1}=\frac{\left(r_\mu-r_{\pi_\sigma}\right)+\left(\sigma^2_{\pi_\sigma}-\sigma^2_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r}-IR^2_{r\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}} \tag{130}$$

$$\left(\underline{r}-\underline{\underline{V}}\,\underline{\mu}\right)'\underline{\delta}_{a1}=\frac{\left(r_{\pi_\sigma}/\sigma^2_{\pi_\sigma}\right)\left(r_{\pi_r}-r_{\pi_\sigma}\right)+\left(r_{\pi_\sigma}-r_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r}-IR^2_{r\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}}-\frac{\left(r_\mu-r_{\pi_\sigma}\right)+\left(\sigma^2_{\pi_\sigma}-\sigma^2_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r}-IR^2_{r\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}} \tag{131}$$

$$\left(\underline{r}-\underline{\underline{V}}\,\underline{\mu}\right)'\underline{\delta}_{a1}=\frac{\left(r_{\pi_\sigma}/\sigma^2_{\pi_\sigma}\right)\left(r_{\pi_r}-r_{\pi_\sigma}\right)+2\left(r_{\pi_\sigma}-r_\mu\right)-\left(\sigma^2_{\pi_\sigma}-\sigma^2_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r}-IR^2_{r\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}} \tag{132}$$

Substitute from Equation (118) into Equation (132).

$$\left(\underline{r}-\underline{\underline{V}}\,\underline{\mu}\right)'\underline{\delta}_{a1}=\frac{\left(r_{\pi_\sigma}/\sigma^2_{\pi_\sigma}\right)\left(r_{\pi_r}-r_{\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}{\sqrt{\left(IR^2_{r\pi_r}-IR^2_{r\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}} \tag{133}$$

It can be shown that:

$$IR^2_{r\pi_r}=\left(r_{\pi_\sigma}/\sigma^2_{\pi_\sigma}\right)r_{\pi_r} \tag{134}$$

Substitute from Equation (134) into Equation (133) and thence into Equation(126).

$$a_\delta=\sigma_a\sqrt{\left(IR^2_{r\pi_r}-IR^2_{r\pi_\sigma}\right)+2\left(\gamma_{\pi_\sigma}-\gamma_\mu\right)}-\frac{\sigma^2_a}{2} \tag{135}$$

Equation (135) shows that the optimum active portfolio's relative growth rate is not proportional to its standard deviation of continuous (relative) return rate. Active reward is not proportional to active risk. Active reward first rises with risk and then declines, eventually becoming negative.

The standard deviation of continuous (relative) return rate that maximizes the optimum active portfolio's relative growth rate is found as follows.

$$\frac{da_\delta}{d\sigma_a} = \sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)} - \sigma_a = 0$$

(136)

$$\sigma_a = \sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}$$

(137)

The corresponding maximum relative growth rate is:

$$a_\delta = \sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)} - \frac{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}{2}$$

(138)

$$a_\delta = \frac{1}{2}\left[\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)\right]$$

(139)

The standard deviation of continuous (relative) return rate beyond which the active portfolio's relative growth rate is negative is obtained by setting Equation (135) to 0 and solving for σ_a to obtain:

$$\sigma_a = 2\sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)}$$

(140)

The active portfolio's information ratio is obtained by dividing Equation (135) by σ_a.

$$IR_a = \frac{a_\delta}{\sigma_a} = \sqrt{\left(IR^2_{r\pi_r} - IR^2_{r\pi_\sigma}\right) + 2\left(\gamma_{\pi_\sigma} - \gamma_\mu\right)} - \frac{\sigma_a}{2}$$

(141)

Equation (141) shows that the optimum active portfolio's information ratio is a linear monotonically declining function of its standard deviation of continuous (relative) return rate. More aggressive portfolios are less attractive than less aggressive portfolios.

The MPT version of CAPM

With arithmetic returns, a portfolio's expected excess return, R_π, and variance of return, σ_π^2, are:

$$R_\pi = \underline{R}'\underline{\pi} \tag{142}$$

$$\sigma_\pi^2 = \underline{\pi}'\underline{\underline{V}}\underline{\pi} \tag{143}$$

The squared Sharpe ratio, $IR_{R\pi}^2$, is:

$$IR_{R\pi}^2 = \frac{\left(\underline{R}'\underline{\pi}\right)^2}{\underline{\pi}'\underline{\underline{V}}\underline{\pi}} \tag{144}$$

The condition the Sharpe ratio maximizing portfolio must satisfy is found by taking the derivative of the squared Sharpe ratio and setting it equal to zero.[19]

$$\frac{d\left(IR_{R\pi}^2\right)}{d\underline{\pi}} = \frac{2\left(\underline{R}'\underline{\pi}\right)\underline{R}}{\underline{\pi}'\underline{\underline{V}}\underline{\pi}} - \frac{2\left(\underline{R}'\underline{\pi}\right)^2\underline{\underline{V}}\underline{\pi}}{\left(\underline{\pi}'\underline{\underline{V}}\underline{\pi}\right)^2} = 0 \tag{145}$$

Simplify Equation (145) to obtain the MPT version of the CAPM.

$$\underline{R} = \left(\frac{\underline{\underline{V}}\underline{\pi}}{\underline{\pi}'\underline{\underline{V}}\underline{\pi}}\right)\left(\underline{R}'\underline{\pi}\right) = \underline{\beta}R_\pi \tag{146}$$

The SPT version of CAPM

With continuous returns, a portfolio's growth rate, γ_π, and variance of continuous return, σ_π^2, are:

---

[19] The constraint that the sum of the weights must equal 1 is not necessary since the squared Sharpe ratio is not changed by multiplying the portfolio weights by a scale factor.

$$\gamma_\pi = \underline{r}'\underline{\pi} - \frac{1}{2}\underline{\pi}'\underline{\underline{V}}\underline{\pi}$$

(147)

$$\sigma_\pi^2 = \underline{\pi}'\underline{\underline{V}}\underline{\pi}$$

(148)

The squared Sharpe ratio, $IR_{r\pi}^2$, is:

$$IR_{r\pi}^2 = \frac{\gamma_\pi^2}{\sigma_\pi^2} = \frac{\left(\underline{r}'\underline{\pi} - \frac{1}{2}\underline{\pi}'\underline{\underline{V}}\underline{\pi}\right)^2}{\underline{\pi}'\underline{\underline{V}}\underline{\pi}}$$

(149)

The condition the Sharpe ratio maximizing portfolio must satisfy is found by taking the derivative of the squared Sharpe ratio, subject to the constraint that the sum of the weights is 1 and setting it equal to zero.

Form the Lagrangian, take the derivative, and simplify.

$$L = \frac{\gamma_\pi^2}{\sigma_\pi^2} - \lambda\left(\underline{1}'\underline{\pi} - 1\right)$$

(150)

$$\frac{dL}{d\underline{\underline{\pi}}} = \frac{2\gamma_\pi}{\sigma_\pi^2}\left(\underline{r} - \underline{\underline{V}}\underline{\pi}\right) - 2\frac{\gamma_\pi^2}{\sigma_\pi^4}\underline{\underline{V}}\underline{\pi} - \lambda\underline{1} = 0$$

(151)

Multiply on the left by $\underline{\pi}'$ and solve for λ.

$$\lambda = \frac{2\gamma_\pi}{\sigma_\pi^2}\left(r_\pi - \sigma_\pi^2\right) - 2\frac{\gamma_\pi^2}{\sigma_\pi^2} = 0$$

(152)

Substitute for λ in Equation (151).

$$\frac{2\gamma_\pi}{\sigma_\pi^2}\left(\underline{r} - \underline{\underline{V}}\underline{\pi}\right) - 2\frac{\gamma_\pi^2}{\sigma_\pi^4}\underline{\underline{V}}\underline{\pi} - \left(\frac{2\gamma_\pi}{\sigma_\pi^2}\left(r_\pi - \sigma_\pi^2\right) - 2\frac{\gamma_\pi^2}{\sigma_\pi^2}\right)\underline{1} = 0$$

(153)

$$\left(\underline{r} - \underline{\underline{V}}\,\underline{\pi}\right) - \gamma_{\pi}\underline{\beta} - \left(\left(r_{\pi} - \sigma_{\pi}^{2}\right) - \gamma_{\pi}\right)\underline{1} = 0 \qquad (154)$$

$$\left(\underline{r} - \sigma_{\pi}^{2}\underline{\beta}\right) - \left(r_{\pi} - \frac{\sigma_{\pi}^{2}}{2}\right)\underline{\beta} - \left(\left(r_{\pi} - \sigma_{\pi}^{2}\right) - \left(r_{\pi} - \frac{\sigma_{\pi}^{2}}{2}\right)\right)\underline{1} = 0 \qquad (155)$$

$$\left(\underline{r} - \sigma_{\pi}^{2}\underline{\beta}\right) - \left(r_{\pi} - \frac{\sigma_{\pi}^{2}}{2}\right)\underline{\beta} + \frac{\sigma_{\pi}^{2}}{2}\underline{1} = 0 \qquad (156)$$

Rearranging terms yields the SPT version of the CAPM.

$$r_{i} = \beta_{i} r_{\pi} + \frac{\sigma_{\pi}^{2}}{2}\left(\beta_{i} - 1\right) \qquad (157)$$

The SPT version of CAPM can be restated in terms of growth rates as follows.

$$\gamma_{i} = \beta_{i}\gamma_{\pi} + \sigma_{\pi}^{2}\left(\beta_{i} - \frac{1}{2}\right) - \frac{\sigma_{i}^{2}}{2} = \beta_{i}\gamma_{\pi} - \frac{1}{2}\left(\sigma_{i}^{2} - \beta_{i}\sigma_{\pi}^{2}\right) \qquad (158)$$

In contrast to the MPT version of CAPM, a stock's non-systematic risk is priced to the extent it differs from its systematic risk.

For infinitesimal time intervals, the transformation of the MPT version of the CAPM to the continuous return format results in:

$$\gamma_{i} = \beta_{i}\gamma_{\pi} - \frac{1}{2}\left(\sigma_{i}^{2} - \beta_{i}\sigma_{\pi}^{2}\right) \qquad (159)$$

Equation (159) is the same as Equation (158), showing that the MPT version of the CAPM approaches the SPT version of the CAPM as the length of the time interval approaches zero.