SEMIMARTINGALES ON RAYS, WALSH DIFFUSIONS, AND RELATED PROBLEMS OF CONTROL AND STOPPING *

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Abstract

We introduce a class of continuous planar processes, called "semimartingales on rays", and develop for them a change-of-variable formula involving quite general classes of functions. Special cases of such planar processes are diffusions which choose, once they reach the origin, the rays for their subsequent voyage according to a fixed probability measure in the manner of Walsh (1978). We develop existence and uniqueness results up to an explosion time for these "Walsh diffusions", study their asymptotic behavior, and develop tests for explosions in finite time. We use these results to find an optimal strategy, in a problem of control with discretionary stopping involving Walsh diffusions.

Key Words: Semimartingales on rays, tree-topology, Walsh semimartingales and diffusions, Skorokhod reflection, local time, stochastic calculus, explosion times, Feller's test, stochastic control, optimal stopping.

1 Introduction and Summary

A pathwise construction was given recently in [12] for so-called WALSH *semimartingales* on the plane. A typical such process is a two-dimensional continuous semimartingale, whose motion away from the origin follows a scalar "driver semimartingale" $U(\cdot)$ along rays emanating from the origin. Once at the origin, the process chooses a new ray for its voyage randomly, according to a given probability measure on angles. When the driver $U(\cdot)$ is a Brownian motion, this WALSH semimartingale becomes the renowned WALSH *Brownian motion,* a process introduced by WALSH (1978) in the epilogue of [22] and studied by BARLOW, PITMAN & YOR (1989) in [1], as well as by many other authors after them. The recent work [12] established stochastic integral equations that the so-constructed WALSH semimartingales satisfy, as well as additional features of their singular nature at the origin. Taken together, these equations and properties gave a FREIDLIN-SHEU-type change-of-variable formula for any processes satisfying them. Previous results in this regard include also [10] for diffusion processes on graphs, [18] for semimartingales on trees, and [9], [11] for WALSH's Brownian motion. Local martingale problems for WALSH *diffusions,* where the driver $U(\cdot)$ is an ITÔ diffusion, were also considered in [12].

We introduce in this paper a novel class of planar processes, called *"semimartingales on rays"*, which generalizes considerably the class of WALSH semimartingales. We develop in Theorem 2.13 a FREIDLIN-SHEU-type stochastic calculus for these processes, and use their continuity in the so-called "tree-topology"

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to characterize the property of moving alongs rays. [1] Roughly speaking, semimartingales on rays also move along rays emanating from the origin; but when at the origin, these processes choose a new ray not necessarily according to a fixed measure. Every WALSH semimartingale is then a semimartingale on rays with some fixed measure on angles, specified through a "partition of local time at the origin" condition. Theorem 2.13 also generalizes the stochastic calculus for WALSH semimartingales to a broader class of functions, with the help of Lemmas 2.10 and 2.11; and a further generalization is found in Theorem 5.3.

This new treatment is important for our discussion in Sections 3 and 4 of WALSH diffusions with angular dependence, up to an exit or "explosion" time, from a given set which is open in the tree-topology. A preliminary idea in this direction appears in Section 8 of [12]; here it is developed and studied in further detail, enabling us to obtain results on existence, uniqueness, asymptotic behavior, and explosion tests for WALSH diffusions (Theorems 3.7, 3.13, 4.5 and 4.9). These results are analogous to those for one-dimensional diffusions (cf. Section 5.5 of [14]; see also [6]-[8]), in the framework of the ENGELBERT-SCHMIDT and FELLER tests.

The power of the approach and of the calculus developed in the present paper, is illustrated in Section 5. We study there an optimization problem involving both *control* and *stopping* of a WALSH diffusion on the unit disc with an absorbing boundary, and for which a "reward" function is specified. This can be seen as the analogue in the WALSH setting of the problem studied in [15]. We find that several interesting new aspects arise from the roundhouse singularity at the origin. Answers to the "pure" optimal stopping problem and, quite a bit more surprisingly, to the "mixed" stochastic control problem with discretionary stopping, are given in Theorems 5.8 and 5.16, respectively, under very mild assumptions. The underlying dynamic programming equations are specified in Subsection 5.4, though they are not used explicitly in the development. Proofs of selected technical results can be found in the Appendix, Section 6.

2 A Stochastic Calculus for Semimartingales on Rays

Throughout this work, whenever a function f is defined on a subset of $\mathbb{R}^2$, we will write "$f(r,\theta)$" (or sometimes "$f_\theta(r)$") to mean its expression in polar coordinates; we have for example $f(r,\theta) = f(x)$, where $x = (r\cos\theta, r\sin\theta)$ in Euclidean coordinates. We also note that the polar coordinates $(0,\theta),\ \theta \in [0,2\pi)$ are identical and identified with $\mathbf{0} \in \mathbb{R}^2$. Therefore, whenever we define a function f via polar coordinates as $f(r,\theta)$, we must make sure $f(0,\theta) \equiv f(\mathbf{0})$ is constant.

We shall write $\arg(x) \in [0,2\pi)$ for the argument of a generic vector $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

2.1 Semimartingales on Rays

We shall introduce in this section a class of processes called "semimartingales on rays"; this class includes the WALSH semimartingales that will be studied later.

Indispensable in the study of such semimartingales is the so-called "tree-metric" on the Euclidean plane.

Definition 2.1. We define the *tree-metric* (cf. [9], [11]) on the plane as follows:

$$\varrho(x_1,x_2) := (r_1+r_2)\,\mathbf{1}_{\{\theta_1\neq\theta_2\}} + |r_1-r_2|\,\mathbf{1}_{\{\theta_1=\theta_2\}}\,, \qquad x_1,\, x_2 \in \mathbb{R}^2, \tag{2.1}$$

where (r_1,θ_1), (r_2,θ_2) are the expressions in polar coordinates of x_1 and x_2, respectively.

We shall call *tree-topology* the topology on the plane induced by this metric.

[1] This continuity property is implied by the stochastic integral equations in [12], and is considerably less restrictive than those equations are. However, the scalar process $R^A(\cdot)$ in Theorem 2.1 of [12] is *shown* here to be a semimartingale, if the planar process $X(\cdot)$ is a semimartingale on rays – rather than assumed to be one, as was done in Theorem 4.1 of [12].

Remark 2.2. It is checked that the recipe of (2.1) defines a metric on the plane. The distance in the tree-metric between two points on the plane, is the shortest distance of going from one point to the other along rays emanating from the origin. Thus, *the tree-topology is stronger than the usual topology on the plane.*

Proposition 2.3. *Assume that a function* $x : [0, \infty) \to \mathbb{R}^2$ *is continuous in the tree-topology. Then, whenever* $\|x(t)\| \neq 0$ *holds for all* $t \in [t_1, t_2]$, *the mapping* $t \mapsto \arg(x(t))$ *is constant on* $[t_1, t_2]$.

Proof. Clearly, showing that $t \mapsto \arg(x(t))$ is constant is equivalent to showing that $t \mapsto \frac{x(t)}{\|x(t)\|}$ is constant. By way of contradiction, let us assume that $\|x(t)\| \neq 0$ holds for all $t \in [t_1, t_2]$ but the ratio $\frac{x(t)}{\|x(t)\|}$ is *not* constant on the interval $[t_1, t_2]$. From Remark 2.2, the function $x : [0, \infty) \to \mathbb{R}^2$ is also continuous in the usual sense. Thus the mapping $t \mapsto \frac{x(t)}{\|x(t)\|}$ is continuous on $[t_1, t_2]$ in the usual sense, so we have

$$\frac{x(t_3)}{\|x(t_3)\|} = \frac{x(t_1)}{\|x(t_1)\|} \qquad \text{for} \qquad t_3 := \inf\left\{ t \geq t_1 : \frac{x(t)}{\|x(t)\|} \neq \frac{x(t_1)}{\|x(t_1)\|} \right\} < t_2 \,.$$

It follows that there exists $\{t_{(n)}\}_{n=1}^{\infty} \subseteq (t_3, t_2]$ with $t_{(n)} \downarrow t_3$ and $\frac{x(t_{(n)})}{\|x(t_{(n)})\|} \neq \frac{x(t_1)}{\|x(t_1)\|} = \frac{x(t_3)}{\|x(t_3)\|}$, therefore also $\arg(x(t_{(n)})) \neq \arg(x(t_3))$. We have then

$$\varrho(x(t_{(n)}), x(t_3)) \,=\, \|x(t_{(n)})\| + \|x(t_3)\| \geq \|x(t_3)\| > 0\,, \qquad \forall \; n \in \mathbb{N}\,,$$

contradicting the continuity of $x(\cdot)$ in the tree-topology. ☐

Proposition 2.3 shows that any process, which is continuous in the tree-topology, does not change the ray along which it travels when away from the origin; any such change to a new ray can happen only when the process is at the origin.

Definition 2.4. Semimartingales on Rays: We place ourselves on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ that satisfies the "usual conditions", i.e., $\mathbb{F}$ is right-continuous and $\mathcal{F}(0)$ contains every $\mathbb{P}-$negligible event. On this space, we are given a continuous scalar semimartigale $U(\cdot)$.

We say that a two-dimensional process $X(\cdot)$ is a *semimartingale on rays driven by* $U(\cdot)$, if:

(i) It is adapted, and is continuous in the tree-topology.

(ii) Its radial part $\|X(\cdot)\|$ is the SKOROKHOD reflection (cf. Section 3.6.C in [14]) of $U(\cdot)$, i.e.,

$$\|X(t)\| \,=\, U(t) + \Lambda(t)\,, \qquad \text{where} \quad \Lambda(t) = \max_{0 \leq s \leq t} \big(- U(s) \big)^+\,, \qquad 0 \leq t < \infty\,. \tag{2.2}$$

Remark 2.5. *Terminology:* We do not assume explicitly in Definition 2.4, that $X(\cdot)$ is a two-dimensional semimartingale; only its radial part $\|X(\cdot)\|$ is clearly seen from (2.2) to be a semimartingale.

But $X(\cdot)$ will indeed turn out to be a semimartingale, thanks to the assumption that it is continuous in the tree-topology. This fact is implied by the general result of Theorem 2.13 below. In light of this theorem we use the terminology "semimartingale on rays" here, leaving it somewhat unjustified for the moment.

Remark 2.6. *Generalization:* The WALSH Semimartingales, defined as in Theorem 2.1 in [12] (see also Definition 2.14 below), are also semimartingales on rays in the context of the present work; cf. the discussion at the beginning of Section 3 of [12].

The converse is not necessarily true; cf. Remark 9.4 of [12]. In fact, Definition 2.4 right above, gives no indication regarding the behavior of $X(\cdot)$ at the origin — i.e., about the manner in which $X(\cdot)$ chooses the next ray for its voyage, when it tries to "extricate itself" from the origin.

2.2 A Generalized Change-of-Variable Formula

The property of "moving along rays" given by Proposition 2.3, suggests considering functions on the plane that have good properties only along every ray emanating from the origin (see also Section 4 of [12]). In this vein, we develop a generalized FREIDLIN-SHEU-type change of variable formula that extends the result of Theorem 4.1 in [12].

Definition 2.7. Let $\mathfrak{D}$ be the class of BOREL-measurable functions $g : \mathbb{R}^2 \to \mathbb{R}$, such that

(i) for every $\theta \in [0, 2\pi)$, the function $r \mapsto g_\theta(r) := g(r,\theta)$ is differentiable on $[0,\infty)$, and the derivative $r \mapsto g'_\theta(r)$ is absolutely continuous on $[0,\infty)$;

(ii) the function $\theta \mapsto g'_\theta(0+)$ is bounded; and

(iii) there exist a real number $\eta > 0$ and a LEBESGUE-integrable function $c : (0,\eta] \to [0,\infty)$ such that, for all $\theta \in [0,2\pi)$ and $r \in (0,\eta]$, we have $|g''_\theta(r)| \le c(r)$.

Definition 2.8. For every given function $g : \mathbb{R}^2 \to \mathbb{R}$ in the class $\mathfrak{D}$, we set for every $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$:

$$g'(x) := g'_\theta(r), \qquad g''(x) := g''_\theta(r), \qquad \text{where } (r,\theta) \text{ is the expression of } x \text{ in polar coordinates.}$$

Proposition 2.9. *Every function $g \in \mathfrak{D}$ has the following properties:*

(i) *The mappings $(r,\theta) \mapsto g'_\theta(r)$ and $(r,\theta) \mapsto g''_\theta(r)$ are* BOREL*-measurable on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, and the mapping $\theta \mapsto g'_\theta(0+)$ is* BOREL*-measurable on $[0,2\pi)$. Also, for the constant $\eta > 0$ that appears in Definition 2.7 (iii) for g, we have*

$$\sup_{\substack{0<r\le\eta \\ \theta\in[0,2\pi)}} |g'_\theta(r)| < \infty .$$

(ii) *The function g is continuous in the tree-topology.*

Proof: **(i)** The first claim is a consequence of the measurability of the function g and of Definition 2.7(i), while the second comes from the requirements (ii) and (iii) in Definition 2.7.

(ii) By the second claim of **(i)**, the function g is continuous at the origin in the tree-topology. The continuity at other points is implied by (i) of Definition 2.7. □

The class $\mathfrak{D}$ includes the functions in Definition 4.1 of [12]. In contrast to that definition, which assumes the derivatives to be locally bounded, *here we only assume some boundedness near the origin.* The reason why this will suffice for the development of a stochastic calculus for semimartingales on rays, is provided by the following two lemmas; these supply the keys to the main result of this section, Theorem 2.13.

Lemma 2.10. *Let $f : \mathbb{R}^2 \to [0,\infty)$ be a* BOREL*-measurable function with the following properties:*

(i) For every $\theta \in [0,2\pi)$, the function $r \mapsto f(r,\theta)$ is locally integrable on $[0,\infty)$.

(ii) There exist a real number $\eta > 0$ and a LEBESGUE*-integrable function $c : [0,\eta] \to [0,\infty)$ such that, for all $\theta \in [0,2\pi)$ and $r \in [0,\eta]$, we have $f(r,\theta) \le c(r)$.*

Then for any semimartingale on rays $X(\cdot)$ in the context of Definition 2.4, we have

$$\mathbb{P}\bigg(\int_0^T f(X(t))\, \mathrm{d}\langle U\rangle(t) < \infty, \quad \forall\ 0 \le T < \infty \bigg) = 1 .$$

Lemma 2.11. *In the context of Definitions 2.4 and 2.7 we have, for every semimartingale $X(\cdot)$ on rays, and for every function $g \in \mathfrak{D}$, the properties*

(i) $$\mathbb{P}\bigg(\sup_{0\le t\le T,\ X(t)\ne\mathbf{0}} \big|g'\big(X(t)\big)\big| < \infty, \quad \forall\ 0 \le T < \infty \bigg) = 1 ,$$

(ii) $$\mathbb{P}\bigg(\int_0^T \mathbf{1}_{\{X(t)\ne\mathbf{0}\}} \big|g''\big(X(t)\big)\big|\, \mathrm{d}\langle U\rangle(t) < \infty, \quad \forall\ 0 \le T < \infty \bigg) = 1 .$$

To prove Lemma 2.10, we recall the *(right) local time* $L^{\Xi}(T,a)$ accumulated at the site $a \in \mathbb{R}$ during the time-interval $[0,T]$ by a generic one-dimensional continuous semimartingale $\Xi(\cdot)$, namely

$$L^{\Xi}(T,a) \,:=\, \lim_{\varepsilon \downarrow 0} \frac{1}{2\,\varepsilon} \int_0^T \mathbf{1}_{\{a \le \Xi(t) < a+\varepsilon\}} \,\mathrm{d}\langle \Xi \rangle(t)\,, \qquad 0 \le T < \infty\,, \quad a \in \mathbb{R}\,. \tag{2.3}$$

From the theory of semimartingale local time (e.g., section 3.7 in [14]), for every BOREL-measurable function $k : \mathbb{R} \to [0,\infty)$ the identity

$$\int_0^T k\big(\Xi(t)\big)\,\mathrm{d}\langle \Xi \rangle(t) \,=\, 2\int_{-\infty}^{\infty} k(a)\, L^{\Xi}(T,a)\,\mathrm{d}a\,, \qquad 0 \le T < \infty \tag{2.4}$$

holds a.e. on the underlying probability space. If, in addition, the continuous semimartingale $\Xi(\cdot)$ is *nonnegative*, then this local time admits the representation

$$L^{\Xi}(\cdot\,,0) \,=\, \int_0^{\cdot} \mathbf{1}_{\{\Xi(t)\,=\,0\}}\,\mathrm{d}\Xi(t)\,. \tag{2.5}$$

From now on, we will always write "$L^{\Xi}(\cdot)$" to denote the semimartingale local time $L^{\Xi}(\cdot\,,0)$ at the origin.

Proof of Lemma 2.10: By condition (ii) of Lemma 2.10, we have $f(x) \le c(\|x\|)$ whenever $x \in \mathbb{R}^2$ and $\|x\| \le \eta$. In conjunction with (2.4) and (2.2), we have on the one hand

$$\int_0^T f\big(X(t)\big)\,\mathbf{1}_{\{\|X(t)\| \le \eta\}}\,\mathrm{d}\langle U \rangle(t) \le \int_0^T c(\|X(t)\|)\,\mathbf{1}_{\{\|X(t)\| \le \eta\}}\,\mathrm{d}\langle \|X\| \rangle(t) = 2\int_0^{\eta} c(r) L^{\|X\|}(T,r)\mathrm{d}r\,.$$

By the theory of semimartingale local time (e.g., section 3.7 in [14]), the mapping $r \mapsto L^{\|X\|}(T,r,\omega)$ is RCLL (right-continuous with left-limits), hence bounded on $[0,\eta]$, for $\mathbb{P}-$a.e. $\omega \in \Omega$. Thus, the integrability of c gives the $\mathbb{P}-$a.e. finiteness of the last expression above.

Let us define for every $\varepsilon > 0$ the stopping times $\tau_{-1}^{\varepsilon} \equiv 0$, $\tau_0^{\varepsilon} := \inf\big\{t \ge 0 : \|X(t)\| = 0\big\}$ and

$$\tau_{2\ell+1}^{\varepsilon} \,:=\, \inf\big\{t > \tau_{2\ell}^{\varepsilon} : \|X(t)\| \ge \varepsilon\big\}\,, \qquad \tau_{2\ell+2}^{\varepsilon} \,:=\, \inf\big\{t > \tau_{2\ell+1}^{\varepsilon} : \|X(t)\| \,=\, 0\big\} \tag{2.6}$$

recursively, for $\ell \in \mathbb{N}_0$. With $\Theta(\cdot) := \arg(X(\cdot))$, we have on the other hand

$$\int_0^T f\big(X(t)\big)\,\mathbf{1}_{\{\|X(t)\| > \eta\}}\,\mathrm{d}\langle U \rangle(t) \,\le\, \int_0^T f\big(X(t)\big)\Big(\sum_{\ell \in \mathbb{N}_0 \cup \{-1\}} \mathbf{1}_{(\tau_{2\ell+1}^{\eta},\,\tau_{2\ell+2}^{\eta})}(t)\Big)\,\mathrm{d}\langle U \rangle(t)$$

$$= \sum_{\{\ell:\ \tau_{2\ell+1}^{\eta} < T\}} \int_{T \wedge \tau_{2\ell+1}^{\eta}}^{T \wedge \tau_{2\ell+2}^{\eta}} f\big(\|X(t)\|, \Theta(t)\big)\,\mathrm{d}\langle U \rangle(t) = \sum_{\{\ell:\ \tau_{2\ell+1}^{\eta} < T\}} \int_{T \wedge \tau_{2\ell+1}^{\eta}}^{T \wedge \tau_{2\ell+2}^{\eta}} f\big(\|X(t)\|, \Theta(\tau_{2\ell+1}^{\eta})\big)\,\mathrm{d}\langle \|X\| \rangle(t)$$

$$\le \sum_{\{\ell:\ \tau_{2\ell+1}^{\eta} < T\}} \int_0^T f\big(\|X(t)\|, \Theta(\tau_{2\ell+1}^{\eta})\big)\,\mathrm{d}\langle \|X\| \rangle(t) \,=\, 2 \sum_{\{\ell:\ \tau_{2\ell+1}^{\eta} < T\}} \int_0^{\infty} f\big(r, \Theta(\tau_{2\ell+1}^{\eta})\big) L^{\|X\|}(T,r)\,\mathrm{d}r$$

$$=\, 2 \sum_{\{\ell:\ \tau_{2\ell+1}^{\eta} < T\}} \int_0^{M(T)} f\big(r, \Theta(\tau_{2\ell+1}^{\eta})\big) L^{\|X\|}(T,r)\,\mathrm{d}r\,, \qquad \text{where} \quad M(T) := \max_{0 \le t \le T} \|X(t)\|\,.$$

We have used (2.2) and Proposition 2.3 for the second equality, and (2.4) for the third. The last equality follows from the theory of semimartingale local time.

We claim that the last expression above is a.e. finite. Indeed, $r \mapsto L^{\|X\|}(T,r,\omega)$ is a.e. bounded on $[0, M(T,\omega)]$, just as before; thus, by condition (i) of Lemma 2.10, each integral in the last expression is a.e. finite. Moreover, the set $\{\ell : \tau_{2\ell+1}^{\eta} < T\}$ is a.e. finite; for otherwise the continuity of the path $t \mapsto \|X(t,\omega)\|$ would be violated. The validity of this finiteness claim follows.

With all the considerations above, Lemma 2.10 is seen to have been established. □

Remark 2.12. Lemma 2.10 can be thought of as an analogue of the ENGELBERT-SCHMIDT 0-1 law (cf. [6] and Section 3.6.E of [14]), as it gives a condition guaranteeing the finiteness of some integral functional of the process $X(\cdot)$. In contrast to the necessary and sufficient condition of local integrability considered in the ENGELBERT-SCHMIDT 0-1 law, the condition here is only sufficient, due to the difficulty in dealing with the "roundhouse singularity" at the origin.

Proof of Lemma 2.11: The claim (ii) is a direct consequence of Lemma 2.10 and of Definition 2.7. For the claim (i), we observe by Definition 2.7 and Proposition 2.3 that

$$\sup_{0\le t\le T,\ 0<\|X(t)\|\le\eta}\big|g'\big(X(t)\big)\big| \,\le\, \Big(\sup_{\theta\in[0,2\pi)} g'_\theta(0+)\Big) \,+\, \int_0^\eta c(r)\,\mathrm{d}r \,<\, \infty\,, \quad \text{a.e.,} \quad \text{and}$$

$$\sup_{0\le t\le T,\ \|X(t)\|>\eta}\big|g'\big(X(t)\big)\big| \,\le\, \sup_{\{\ell:\ \tau^\eta_{2\ell+1}<T\}}\bigg(\sup_{t\in[\tau^\eta_{2\ell+1},\,\tau^\eta_{2\ell+2}\wedge T]}\Big|g'_{\Theta(\tau^\eta_{2\ell+1})}\big(\|X(t)\|\big)\Big|\bigg) \,<\, \infty\,, \quad \text{a.e.,}$$

thanks also to the fact that the set $\big\{\ell : \tau^\eta_{2\ell+1} < T\big\}$ is a.e. finite. The claim (i) follows then. ☐

Now we can state and prove the main result of this section, a generalized FREIDLIN-SHEU-type identity.

Theorem 2.13. A Generalized Change-of-Variable Formula: *Let* $X(\cdot)$ *be a semimartingale on rays with driver* $U(\cdot)$*, in the context of Definition 2.4.*

(i) *Then for every function* $g\in\mathfrak{D}$*, the process* $g(X(\cdot))$ *is a continuous semimartingale and satisfies*

$$g\big(X(\cdot)\big) = g\big(X(0)\big) \,+\, \int_0^{\cdot} \mathbf{1}_{\{X(t)\neq\mathbf{0}\}}\bigg(g'\big(X(t)\big)\,\mathrm{d}U(t) + \frac{1}{2}\,g''\big(X(t)\big)\,\mathrm{d}\langle U\rangle(t)\bigg) + V^X_g(\cdot)\,. \tag{2.7}$$

Here $V^X_g(\cdot)$ *is a continuous process of finite variation on compact intervals, with*

$$\big|V^X_g(t_2) - V^X_g(t_1)\big| \,\le\, \Big(\sup_{\theta\in[0,2\pi)}|g'_\theta(0+)|\Big)\big(L^{\|X\|}(t_2) - L^{\|X\|}(t_1)\big)\,, \qquad \forall\ 0\le t_1<t_2<\infty\,. \tag{2.8}$$

In particular, for every set $A\in\mathcal{B}([0,2\pi))$ *and with the recipe* $g_A(r,\theta):=r\cdot\mathbf{1}_A(\theta)$*, we have* $g_A\in\mathfrak{D}$*; therefore, the process*

$$R^X_A(\cdot) := g_A\big(X(\cdot)\big) = \|X(\cdot)\|\cdot\mathbf{1}_A\big(\mathit{arg}\big(X(\cdot)\big)\big) \tag{2.9}$$

is a continuous semimartingale.

(ii) *Assume that there exists a probability measure* $\boldsymbol{\nu}$ *on* $\mathcal{B}([0,2\pi))$ *such that, for every set* $A\in\mathcal{B}([0,2\pi))$*, the semimartingale local time at the origin for the process* $R^X_A(\cdot)$ *in (2.9) has the "partition property"*

$$L^{R^X_A}(\cdot) \,\equiv\, \boldsymbol{\nu}(A)\,L^{\|X\|}(\cdot)\,. \tag{2.10}$$

Then for every function $g\in\mathfrak{D}$*, the decomposition (2.7) holds with*

$$V^X_g(\cdot) = \Big(\int_0^{2\pi} g'_\theta(0+)\,\boldsymbol{\nu}(\mathrm{d}\theta)\Big)\,L^{\|X\|}(\cdot)\,. \tag{2.11}$$

Proof: **(i)** We employ a method similar to that used in the proof of Theorem 4.1 in [12]. With $\mathbb{N}_{-1}:=\mathbb{N}_0\cup\{-1\}$ and the sequence of stopping times $\{\tau^\varepsilon_k\}_{k\in\mathbb{N}_{-1}}$ defined as in (2.6), we have the decomposition

$$g\big(X(T)\big) \,=\, g\big(X(0)\big) + \sum_{\ell\in\mathbb{N}_{-1}}\Big(g\big(X(T\wedge\tau^\varepsilon_{2\ell+2})\big) - g\big(X(T\wedge\tau^\varepsilon_{2\ell+1})\big)\Big)$$
$$+\sum_{\ell\in\mathbb{N}_0}\Big(g\big(X(T\wedge\tau^\varepsilon_{2\ell+1})\big) - g\big(X(T\wedge\tau^\varepsilon_{2\ell})\big)\Big)\,. \tag{2.12}$$

Recalling the notation $\Theta(\cdot) = \arg(X(\cdot))$, we can write the first summation above as

$$\sum_{\ell\in\mathbb{N}_{-1}} \Big(g\big(X(T\wedge\tau^{\varepsilon}_{2\ell+2})\big) - g\big(X(T\wedge\tau^{\varepsilon}_{2\ell+1})\big)\Big) = \sum_{\ell\in\mathbb{N}_{-1}} \Big(g_\theta(\|X(T\wedge\tau^{\varepsilon}_{2\ell+2})\|) - g_\theta(\|X(T\wedge\tau^{\varepsilon}_{2\ell+1})\|)\Big)\Big|_{\theta=\Theta(T\wedge\tau^{\varepsilon}_{2\ell+1})}$$

$$= \sum_{\ell\in\mathbb{N}_{-1}} \int_{T\wedge\tau^{\varepsilon}_{2\ell+1}}^{T\wedge\tau^{\varepsilon}_{2\ell+2}} \Big(g'_\theta(\|X(t)\|)\,\mathrm{d}\|X(t)\| + \frac{1}{2}\, g''_\theta(\|X(t)\|)\,\mathrm{d}\langle\|X\|\rangle(t)\Big)\Big|_{\theta=\Theta(t)}$$

$$= \int_0^T \Big(\sum_{\ell\in\mathbb{N}_{-1}} \mathbf{1}_{(\tau^{\varepsilon}_{2\ell+1},\,\tau^{\varepsilon}_{2\ell+2})}(t)\Big)\Big(g'\big(X(t)\big)\,\mathrm{d}U(t) + \frac{1}{2}\, g''\big(X(t)\big)\,\mathrm{d}\langle U\rangle(t)\Big).$$

For the second equality of this string, we have used Proposition 2.3 and the generalized ITÔ's rule (Problem 3.7.3 in [14]; although $\theta = \Theta(T\wedge\tau^{\varepsilon}_{2\ell+1})$ is random, a careful look into the proof of the generalized ITÔ's rule will justify the application here). The third equality is valid because of (2.2), and of the fact that the process $\Lambda(\cdot)$ that appears there is flat off the set $\{0\le t<\infty : \|X(t)\| = 0\}$.

Now with the help of Lemma 2.11, we let $\varepsilon\downarrow 0$ and obtain the convergence in probability

$$\sum_{\ell\in\mathbb{N}_{-1}} \Big(g\big(X(T\wedge\tau^{\varepsilon}_{2\ell+2})\big) - g\big(X(T\wedge\tau^{\varepsilon}_{2\ell+1})\big)\Big) \xrightarrow[\varepsilon\downarrow 0]{} \int_0^T \mathbf{1}_{\{X(t)\neq\mathbf{0}\}} \Big(g'\big(X(t)\big)\,\mathrm{d}U(t) + \frac{1}{2}\, g''\big(X(t)\big)\,\mathrm{d}\langle U\rangle(t)\Big). \tag{2.13}$$

By Definition 2.7 and Proposition 2.9, the process $g(X(\cdot))$ is adapted and continuous. Thus the process

$$V^X_g(\cdot) := g\big(X(\cdot)\big) - g\big(X(0)\big) - \int_0^{\cdot} \mathbf{1}_{\{X(t)\neq\mathbf{0}\}} \Big(g'\big(X(t)\big)\,\mathrm{d}U(t) + \frac{1}{2}\, g''\big(X(t)\big)\,\mathrm{d}\langle U\rangle(t)\Big)$$

is also adapted and continuous, and we have from (2.12), (2.13) the convergence in probability

$$\sum_{\ell\in\mathbb{N}_0} \Big(g\big(X(T\wedge\tau^{\varepsilon}_{2\ell+1})\big) - g\big(X(T\wedge\tau^{\varepsilon}_{2\ell})\big)\Big) \xrightarrow[\varepsilon\downarrow 0]{} V^X_g(T)\,. \tag{2.14}$$

Let us concentrate now on the summation on the left-hand side of the above display. We recall the constant $\eta>0$ and the function c in Definition 2.7(iv). We have for $0<\varepsilon\le\eta$ the decompositions

$$\sum_{\ell\in\mathbb{N}_0} \Big(g\big(X(T\wedge\tau^{\varepsilon}_{2\ell+1})\big) - g\big(X(T\wedge\tau^{\varepsilon}_{2\ell})\big)\Big) = \sum_{\{\ell\,:\,\tau^{\varepsilon}_{2\ell+1}<T\}} \big(g_\theta(\varepsilon) - g_\theta(0)\big)\Big|_{\theta=\Theta(\tau^{\varepsilon}_{2\ell+1})} + O(\varepsilon)$$

$$= \sum_{\{\ell\,:\,\tau^{\varepsilon}_{2\ell+1}<T\}} \Big(\varepsilon g'_\theta(0+) + \int_0^{\varepsilon} (\varepsilon - r) g''_\theta(r)\mathrm{d}r\Big)\Big|_{\theta=\Theta(\tau^{\varepsilon}_{2\ell+1})} + O(\varepsilon), \tag{2.15}$$

where we have used Proposition 2.9(i) to obtain the term $O(\varepsilon)$. We also have

$$\Bigg| \sum_{\{\ell:\,\tau^{\varepsilon}_{2\ell+1}<T\}} \Big(\int_0^{\varepsilon} (\varepsilon - r)\, g''_\theta(r)\mathrm{d}r\Big)\Big|_{\theta=\Theta(\tau^{\varepsilon}_{2\ell+1})}\Bigg| \le \sum_{\{\ell:\,\tau^{\varepsilon}_{2\ell+1}<T\}} \varepsilon \int_0^{\varepsilon} c(r)\mathrm{d}r = \big(\varepsilon N(T,\varepsilon)\big) \int_0^{\varepsilon} c(r)\mathrm{d}r \xrightarrow[\varepsilon\downarrow 0]{} 0$$

in probability, where we set

$$N(T,\varepsilon) := \sharp\big\{\ell\in\mathbb{N}_{-1} : \tau^{\varepsilon}_{2\ell+1} < T\big\}$$

and use Theorem VI.1.10 in [19] for the convergence $\varepsilon N(T,\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{} L^{\|X\|}(T)$ in probability (the "down-crossings" representation of local time). This, in conjunction with (2.14) and (2.15), gives the convergence

$$\sum_{\{\ell\,:\,t_1\le\tau^{\varepsilon}_{2\ell+1}<t_2\}} \Big(\varepsilon g'_\theta(0+)\Big)\Big|_{\theta=\Theta(\tau^{\varepsilon}_{2\ell+1})} \xrightarrow[\varepsilon\downarrow 0]{} V^X_g(t_2) - V^X_g(t_1) \qquad \text{in probability,} \tag{2.16}$$

for fixed $0 \le t_1 < t_2 < \infty$. On the other hand, with $C_g := \sup_{\theta \in [0,2\pi)} |g'_\theta(0+)|$, we have again

$$\sum_{\{\ell \,:\, t_1 \le \tau^\varepsilon_{2\ell+1} < t_2\}} \Big(\varepsilon |g'_\theta(0+)| \Big)\Big|_{\theta = \Theta(\tau^\varepsilon_{2\ell+1})} \le C_g \cdot \varepsilon \big(N(t_2, \varepsilon) - N(t_1, \varepsilon) \big) \xrightarrow[\varepsilon \downarrow 0]{} C_g \big(L^{\|X\|}(t_2) - L^{\|X\|}(t_1) \big) ,$$

in probability. Together with (2.16), this last convergence in probability leads to the estimate (2.8), which in turn implies that the process $V^X_g(\cdot)$ is of finite variation on compact intervals. Thus the process $g(X(\cdot))$ is a continuous semimartingale. The last claim of **(i)** follows.

(ii) Finally, we need to argue that the "partition of local time" property (2.10) leads to the representation (2.11). By virtue of (2.16), it suffices to show

$$\sum_{\{\ell \,:\, \tau^\varepsilon_{2\ell+1} < T\}} \Big(\varepsilon g'_\theta(0+) \Big)\Big|_{\theta = \Theta(\tau^\varepsilon_{2\ell+1})} \xrightarrow[\varepsilon \downarrow 0]{} \Big(\int_0^{2\pi} g'_\theta(0+) \, \boldsymbol{\nu}(\mathrm{d}\theta) \Big) L^{\|X\|}(\cdot) .$$

This can be done in exactly the same manner as in the last part of the proof of Theorem 4.1 in [12], so we refer to that proof for this part. □

Definition 2.14. WALSH Semimartingales: A given semimartingale on rays $X(\cdot)$ as in Definition 2.4, which satisfies the "partition of local time" property (2.10) for some probability measure $\boldsymbol{\nu}$ on $\mathcal{B}([0, 2\pi))$, will be called WALSH *semimartingale with driver* $U(\cdot)$ *and angular measure* $\boldsymbol{\nu}$.

Remark 2.15. *The Planar Semimartingale Property;* WALSH *Semimartingales:* Theorem 2.13 generalizes Theorem 4.1 of [12] to a larger class of functions, namely, the class $\mathfrak{D}$ of Definition 2.7; its part (i) provides results on a larger class of processes, for which the "partition of local time" property (2.10) may not hold.

With $g_1(r, \theta) = r \cos \theta,\ g_2(r, \theta) = r \sin \theta$, we deduce from Theorem 2.13(i) that a process $X(\cdot)$ as in Definition 2.4 is indeed a two-dimensional semimartingale. If this semimartingale $X(\cdot)$ satisfies also the "partition of local time" property (2.10) for some probability measure $\boldsymbol{\nu}$ on $\mathcal{B}([0, 2\pi))$, then

$$g_1\big(X(\cdot)\big) = g_1\big(X(0)\big) + \int_0^\cdot \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} \cos\big(\arg(X(t))\big) \, \mathrm{d}U(t) + \gamma_1 L^{\|X\|}(\cdot) , \tag{2.17}$$

$$g_2\big(X(\cdot)\big) = g_2\big(X(0)\big) + \int_0^\cdot \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} \sin\big(\arg(X(t))\big) \, \mathrm{d}U(t) + \gamma_2 L^{\|X\|}(\cdot) , \tag{2.18}$$

hold by virtue of Theorem 2.13(ii), where

$$\gamma_1 := \int_0^{2\pi} \cos(\theta) \, \boldsymbol{\nu}(\mathrm{d}\theta) \quad \text{and} \quad \gamma_2 := \int_0^{2\pi} \sin(\theta) \, \boldsymbol{\nu}(\mathrm{d}\theta) .$$

The equations (2.17), (2.18) are equivalent to the stochastic integral equations in Theorem 2.1 of [12], after some slight adaptation in notation. Thus, Definition 2.14 above is consistent with the terminology in [12].

By virtue of (2.10), the probability measure $\boldsymbol{\nu}$ captures the "intensity of excursions of $X(\cdot)$ away from the origin", and along the rays in any given set of angles. Thus, under the property (2.10), when the process $X(\cdot)$ finds itself at the origin it chooses the next ray for its voyage according to this "angular measure" $\boldsymbol{\nu}$.

3 A Study of Walsh Diffusions with Angular Dependence

In this section we provide conditions, under which existence and uniqueness in distribution hold for processes that we call WALSH *diffusions with angular dependence,* up to an explosion time. In the three subsections that follow we discuss, respectively, the basic setting, the driftless case, and the case with drift.

In the rest of this work, we consider an arbitrary but fixed probability measure $\boldsymbol{\nu}$ on the space $([0, 2\pi),\ \mathcal{B}([0, 2\pi)))$, which will always be the "angular measure" of our WALSH diffusions.

3.1 Walsh Diffusions with Angular Dependence; Explosions

We will consider WALSH diffusions with values in a BOREL set *which is open in the tree-topology and contains the origin.* More precisely, we fix a measurable function $\ell : [0, 2\pi) \to (0, \infty]$ which is bounded away from zero, and consider the set

$$I \,:=\, \big\{ x \in \mathbb{R}^2 : \, 0 < \|x\| < \ell(\text{arg}(x)) \text{ or } x = \mathbf{0} \big\} \,=\, \big\{ (r, \theta) : \, 0 \leq r < \ell(\theta), \; 0 \leq \theta < 2\pi \big\}$$

expressed in Euclidean and polar coordinates, respectively. We consider also the punctured set $\check{I} := I \setminus \{\mathbf{0}\}$, as well as the closure $\overline{I}$ of I under the tree-topology in the collection of all the "extended rays"; that is, even when $\ell(\theta) = \infty$ holds for some θ's, we set

$$\overline{I} \,=\, \big\{ (r, \theta) : \, 0 \leq r \leq \ell(\theta), \; 0 \leq \theta < 2\pi \big\} \,.$$

Finally, we consider a strictly increasing sequence of measurable functions $\{\ell_n\}_{n=1}^{\infty}$, where each $\ell_n : [0, 2\pi) \to (0, \infty)$ is bounded away from zero, and such that $\ell_n(\theta) \uparrow \ell(\theta)$ as $n \uparrow \infty$, $\forall\, \theta \in [0, 2\pi)$. We set

$$I_n \,:=\, \big\{ (r, \theta) : \, 0 \leq r < \ell_n(\theta), \; 0 \leq \theta < 2\pi \big\}, \quad \forall \; n \in \mathbb{N}.$$

By the generalized ITÔ rule (Theorem 3.7.1 in [14]), we see that (2.2) implies

$$\|X(\cdot)\| = \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\| > 0\}} \, \mathrm{d}U(t) \, + L^{\|X\|}(\cdot) \,. \tag{3.1}$$

Now let us fix BOREL-measurable functions $\boldsymbol{b} : \check{I} \to \mathbb{R}$ and $\boldsymbol{s} : \check{I} \to \mathbb{R}$ and consider finding a planar process $X(\cdot)$, continuous in the tree-topology and satisfying (3.1), with $U(\cdot)$ an ITÔ process whose instantaneous drift and dispersion depend at any given time t on the current position $X(t)$ through the functions $\boldsymbol{b}$ and $\boldsymbol{s}$. With this dispensation, the equation (3.1) becomes

$$\|X(\cdot)\| = \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\| > 0\}} \big[\boldsymbol{b}\big(X(t)\big) \, \mathrm{d}t + \boldsymbol{s}\big(X(t)\big) \, \mathrm{d}W(t) \big] \, + L^{\|X\|}(\cdot) \,. \tag{3.2}$$

Furthermore, with the measure $\boldsymbol{\nu}$ as specified at the beginning of this section and with the notation of (2.9), we impose also the requirements

$$\int_0^{\cdot} \mathbf{1}_{\{X(t) = \mathbf{0}\}} \, \mathrm{d}t \,\equiv\, 0 \qquad \text{and} \qquad L^{R_A^X}(\cdot) \,\equiv\, \boldsymbol{\nu}(A) \, L^{\|X\|}(\cdot) \,, \quad \forall \; A \in \mathcal{B}([0, 2\pi)). \tag{3.3}$$

It is important to note that (3.2) represents the radial part $\|X(\cdot)\|$ as a reflected ITÔ process, whose local characteristics depend at each time t on the full position $X(t)$, not just the radial part $\|X(t)\|$. We note also that the second requirement of (3.3) is just (2.10), which implies that $\boldsymbol{\nu}$ is the angular measure. Since a process $X(\cdot)$ satisfying only (3.2) can spend an arbitrary amount of time at the origin, we impose the "non-stickiness" condition in the first requirement of (3.3) in order to guarantee uniqueness.

Definition 3.1. WALSH Diffusion: *A* WALSH *diffusion with state-space* I, *associated with the triple* $(\boldsymbol{b}, \boldsymbol{s}, \boldsymbol{\nu})$ *and defined up until an explosion time,* is a triple (X, W), $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$, such that:

(i) $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ is a filtered probability space satisfying the usual conditions.

(ii) The process $\{X(t), \, \mathcal{F}(t); \, 0 \leq t < \infty\}$ is adapted, $\overline{I}-$valued, and continuous in the tree-topology with $X(0) \in I$ a.s.; and $\{W(t), \, \mathcal{F}(t); \, 0 \leq t < \infty\}$ is a standard one-dimensional Brownian motion.

(iii) With $S_n := \inf\big\{ t \geq 0 : \, \|X(t)\| \geq \ell_n\big(\arg(X(t))\big)\big\} = \inf\big\{ t \geq 0 : \, X(t) \notin I_n \big\}$, we have

$$\mathbb{P}\bigg(\int_0^{T \wedge S_n} \mathbf{1}_{\{\|X(t)\|>0\}} \Big(\big|\boldsymbol{b}\big(X(t)\big)\big| + \boldsymbol{s}^2\big(X(t)\big)\Big) \, \mathrm{d}t < \infty \,, \quad 0 \leq T < \infty \bigg) \, = \, 1 \,.$$

(iv) For every $n \in \mathbb{N}$, the process $\|X(\cdot \wedge S_n)\|$ is a semimartingale that satisfies

$$\mathbb{P}\bigg(\|X(T \wedge S_n)\| = \|X(0)\| + \int_0^{T \wedge S_n} \mathbf{1}_{\{\|X(t)\|>0\}} \big[\boldsymbol{b}\big(X(t)\big) \, \mathrm{d}t \, + \, \boldsymbol{s}\big(X(t)\big) \, \mathrm{d}W(t) \big]$$

$$+ L^{\|X(\cdot \wedge S_n)\|}(T), \quad 0 \leq T < \infty \bigg) \, = \, 1 \,.$$

(v) We have that $\int_0^{\cdot} \mathbf{1}_{\{X(t)\,=\,\mathbf{0}\}} \, \mathrm{d}t \, \equiv \, 0$, and for every $n \in \mathbb{N}$ the "partition of local time property"

$$L^{R_A^X(\cdot \wedge S_n)}(T) \, = \, \boldsymbol{\nu}(A) \, L^{\|X(\cdot \wedge S_n)\|}(T) \,, \qquad \forall \; A \in \mathcal{B}([0, 2\pi)) \,, \quad \mathbb{P}-\text{a.s.} \qquad \square$$

Abusing terminology slightly, we shall also call the state-process $X(\cdot)$ a WALSH *diffusion*, omitting the underlying probability space and Brownian motion. We shall refer to

$$S := \lim_{n \to \infty} S_n \tag{3.4}$$

as the *explosion time* of $X(\cdot)$ from I, and stipulate that $X(t) = X(S)\,, \; S \leq t < \infty$. We note that the assumption of continuity of $X(\cdot)$ on $\overline{I}$, in the topology induced by the tree-metric, implies that

$$S = \inf\{ t \geq 0 : \, X(t) \notin I \} \qquad \text{and} \qquad X(S) \in \big\{ (r, \theta) : r = \ell(\theta), \; 0 \leq \theta < 2\pi \big\} \; \text{ a.e. on } \; \{ S < \infty \}. \tag{3.5}$$

Remark 3.2. By Theorem 2.13(i), the processes $R_A^X(\cdot \wedge S_n)\,, \, A \in \mathcal{B}([0, 2\pi)), \, n \in \mathbb{N}$ are continuous semimartingales. Moreover, the sets $I_n, \, n \in \mathbb{N}$ and I are open in the topology induced by the tree-metric; thus, the continuity of $X(\cdot)$ in the above topology implies that $S_n, \, n \in \mathbb{N}$ and S are stopping times. It should be noted that we do not assume the continuity up to time ∞, thus $X(S)$ may not be defined on the event $\{ S = \infty \}$.

3.2 The Driftless Case: Method of Time-Change

In this subsection, we study WALSH diffusions with drift $\boldsymbol{b} \equiv 0$ and state space $I = \mathbb{R}^2 = \{ (r, \theta) : 0 \leq r < \infty, \; 0 \leq \theta < 2\pi \}$. To employ the method of time-change, we shall first establish in our setting results analogous to the DAMBIS-DUBINS-SCHWARZ representation for local martingales, and to the non-explosion property (Problem 5.5.3 in [14]).

Definition 3.3. WALSH Brownian Motion: A WALSH semimartingale $X(\cdot)$ will be called WALSH *Brownian motion,* if its driver $U(\cdot) \equiv B(\cdot)$ is a Brownian motion; see Definitions 2.14 and 2.4.

This terminology is consistent with the construction of the WALSH Brownian motion in [1]; this is thanks to Proposition 7.2 in [12], and to Remark 2.15 here. We note at this point that a WALSH Brownian motion is also a WALSH diffusion with state-space $I = \mathbb{R}^2\,, \; \boldsymbol{b} \equiv 0\,, \; \boldsymbol{\sigma} \equiv 1\,$, and $\mathbb{P}(S = \infty) = 1\,$.

Proposition 3.4. A DAMBIS-DUBINS-SCHWARZ-Type Representation: *Let $X(\cdot)$ be a* WALSH *semimartingale driven by a continuous local martingale $U(\cdot)$, and with angular measure $\boldsymbol{\nu}$.*

There exists then, on a possibly extended probability space, a WALSH *Brownian motion $Z(\cdot)$ with the same angular measure and with the property $X(\cdot) = Z\big(\langle U \rangle(\cdot)\big)$.*

Proof. Let us assume first that $\langle U\rangle(\infty)=\infty$. Define

$$T(s) := \inf\{t\ge 0 : \langle U\rangle(t)>s\}\,,\quad Z(s):=X\big(T(s)\big)\,,\quad \mathcal{G}(s):=\mathcal{F}\big(T(s)\big)\,,\quad 0\le s<\infty\,. \tag{3.6}$$

Recall that $U(\cdot)$ is a continuous local martingale. Thus, by the proof of Theorem 3.4.6 in [14], we have:
(i) With $B(\cdot):=U\big(T(\cdot)\big)$, the process $\{B(s),\mathcal{G}(s),\,0\le s<\infty\}$ is Brownian motion, and $U(t)=B\big(\langle U\rangle(t)\big)\,,\ 0\le t<\infty$.
(ii) There exists $\Omega^*\in\mathcal{F}$ with $\mathbb{P}(\Omega^*)=1$, such that for every $\omega\in\Omega^*$, we have

$$\langle U\rangle(t_1,\omega)=\langle U\rangle(t_2,\omega)\ \text{ for some } 0\le t_1<t_2<\infty \quad\Rightarrow\quad t\mapsto U(t,\omega)\ \text{ is constant on } [t_1,t_2]. \tag{3.7}$$

Since $X(\cdot)$ is continuous in the tree-topology, we see from Proposition 2.3 that the constancy of $X(\cdot)$ on some interval $[t_1,t_2]$ is implied by the constancy of $\|X(\cdot)\|$ on $[t_1,t_2]$, which by (2.2) can be implied by the same constancy of $U(\cdot)$. Thus the above property (ii) is still true if we replace (3.7) by

$$\langle U\rangle(t_1,\omega)=\langle U\rangle(t_2,\omega)\ \text{ for some } 0\le t_1<t_2<\infty \quad\Rightarrow\quad t\mapsto X(t,\omega)\ \text{ is constant on } [t_1,t_2]. \tag{3.8}$$

In the spirit of Problem 3.4.5(iv) in [14], this implies the continuity in the tree-topology of the process $Z(\cdot):=X(T(\cdot))$. Moreover, we observe from (2.2) that

$$\|Z(s)\| = U\big(T(s)\big)+\max_{0\le t\le T(s)}\big(-U(t)\big)^+ = U\big(T(s)\big)+\max_{0\le t\le s}\Big(-U\big(T(t)\big)\Big)^+ = B(s)+\max_{0\le t\le s}\big(-B(t)\big)^+.$$

Here we have used for the second equality the fact that $U(\cdot)$ is constant on $[T(t-),T(t)]$ for every t, which is implied by (3.7).

Finally, we claim that *the "partition of local time" property (2.10) of $X(\cdot)$ is inherited by $Z(\cdot)$*. Indeed,

$$R^Z_A(\cdot)=\|Z(\cdot)\|\cdot\mathbf{1}_A\big(\arg\big(Z(\cdot)\big)\big)=R^X_A\big(T(\cdot)\big)$$

is continuous, in the same way $Z(\cdot)=X(T(\cdot))$ is. Then by Theorem 2.13 and time-change (Proposition 3.4.8 in [14]), we obtain that $R^Z_A(\cdot)$ is a continuous semimartingale of the filtration $\{\mathcal{G}(s)\}$, and that $\langle R^Z_A\rangle(\cdot)\equiv\langle R^X_A\rangle\big(T(\cdot)\big)$. Now it is easy to use (2.3) to obtain $L^{R^Z_A}(\cdot)\equiv L^{R^X_A}\big(T(\cdot)\big)$; in particular, $L^{\|Z\|}(\cdot)\equiv L^{\|X\|}\big(T(\cdot)\big)$. Thus (2.10) implies $L^{R^Z_A}(\cdot)\equiv\boldsymbol{\nu}(A)\,L^{\|Z\|}(\cdot)$, which is the claim.

It is clear at this point that $Z(\cdot)$ is a WALSH Brownian motion with the same angular measure $\boldsymbol{\nu}$ as $X(\cdot)$, and that $X(\cdot)=Z\big(\langle U\rangle(\cdot)\big)$ holds, thanks to (3.8).

• Next, we consider the case $\mathbb{P}\big(\langle U\rangle(\infty)<\infty\big)>0$. We shall argue this case heuristically, as a rigorous argument is straightforward but laborious. On the event $\{\langle U\rangle(\infty)<\infty\}$, the limit $\lim_{t\to\infty}U(t)$ exists; therefore, so do the limits $\lim_{t\to\infty}\|X(t)\|$ and $\lim_{t\to\infty}X(t)$, thanks to (2.2) and the continuity of $X(\cdot)$ in the tree-topology. It follows that (3.6) is still well-defined; the only problem is that $Z(\cdot)$ need not be a WALSH Brownian motion anymore: it "runs out of gas" from the time $\langle U\rangle(\infty)$ onwards, as does $B(\cdot)$.

We deal with this issue as follows: On the event $\{\langle U\rangle(\infty)<\infty\}$, we keep $Z(\cdot)$ running on the time interval $[\langle U\rangle(\infty),\infty)$, by first redefining $B(\cdot)$ on $[\langle U\rangle(\infty),\infty)$ to make it a Brownian motion, as described in Problem 3.4.7 of [14]; then following the "folding and unfolding" scheme in the proof of Theorem 2.1 in [12], to construct pathwise a WALSH Brownian motion $Z(\langle U\rangle(\infty)+\cdot)$ with angular measure $\boldsymbol{\nu}$ and driven by $B(\langle U\rangle(\infty)+\cdot)$. This "continued" process $Z(\cdot)$ satisfies all the required properties. □

We also have the following result, regarding the absence of explosions for WALSH diffusions with $\boldsymbol{b}\equiv 0$ and state-space $I=\mathbb{R}^2$. Its proof is in the Appendix, Section 6.

Proposition 3.5. *Suppose $X(\cdot)$ is a* WALSH *diffusion associated with the triple $(\mathbf{0},\boldsymbol{s},\boldsymbol{\nu})$ on the Euclidean plane $\mathbb{R}^2=\{(r,\theta):\,0\le r<\infty,\ 0\le\theta<2\pi\}$ and defined up to an explosion time S. Then $S=\infty$ a.e.*

Now we can state the existence-and-uniqueness result for a WALSH diffusion without drift. As in the scalar case, we recall Remark 2.12 and define the sets

$$\mathcal{I}(\boldsymbol{s}) := \Big\{ (r,\theta) \in \check{\mathbb{R}}^2 : \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{d}y}{\boldsymbol{s}^2(r+y,\theta)} = \infty,\ \forall\, \varepsilon \in (0,r) \Big\}, \quad \mathcal{Z}(\boldsymbol{s}) := \big\{ x \in \check{\mathbb{R}}^2 : \boldsymbol{s}(x) = 0 \big\}. \tag{3.9}$$

Since the ENGELBERT-SCHMIDT 0-1 law is critical for establishing the one-dimensional existence-and-uniqueness result, we need to impose the following additional condition, in order to ensure that the above two sets are both bounded away from the origin, and that the integral process $T(\cdot)$ in the proof Theorem 3.7 does not explode when the WALSH Brownian motion considered there is near the origin.

Condition 3.6. *There exist an $\eta > 0$ and an integrable function $c : (0,\eta] \to [0,\infty)$, such that*

$$\frac{1}{\boldsymbol{s}^2(r,\theta)} \le c(r) \qquad \textit{holds for all} \quad \theta \in [0,2\pi)\,, \ \ r \in (0,\eta]\,. \qquad \square$$

Under this condition, the following existence-and-uniqueness result, for a WALSH diffusion without drift, is a two-dimensional analogue of Theorem 5.5.4 in [14]; its proof is also in the Appendix.

Theorem 3.7. *Let the function $\boldsymbol{s} : \check{\mathbb{R}}^2 \to \mathbb{R}$ satisfy Condition 3.6. Then, for every given initial distribution $\boldsymbol{\mu}$ on $\mathcal{B}(\mathbb{R}^2)$, there exists a non-explosive and unique-in-distribution* WALSH *diffusion $X(\cdot)$ with values in $\mathbb{R}^2$ and associated with the triple $(\mathbf{0}, \boldsymbol{s}, \boldsymbol{\nu})$, if and only if $\mathcal{I}(\boldsymbol{s}) = \mathcal{Z}(\boldsymbol{s})$.*

Remark 3.8. Assuming $\mathcal{I}(\boldsymbol{s}) = \mathcal{Z}(\boldsymbol{s})$ and Condition 3.6, the WALSH diffusion in Theorem 3.7 becomes motionless once it hits the set $\mathcal{I}(\boldsymbol{s})$, but keeps moving before that time. This can be seen in the same spirit as in Remarks 5.5.6, 5.5.8 of [14].

3.3 The General Case: Removal of Drift by Change of Scale

Let us move now on to the study of WALSH diffusions with drift, via the method of "removal of drift" followed by reduction to the driftless case of the previous subsection.

We recall the set $I := \{(r,\theta) : 0 \le r < \ell(\theta),\ 0 \le \theta < 2\pi\}$, where the function $\ell : [0,2\pi) \to (0,\infty]$ is measurable and bounded away from zero. We recall also the class $\mathfrak{D}$ of functions in Definition 2.7, and adjust it presently to "fit" the domain I.

Definition 3.9. Let $\mathfrak{D}_I$ be the class of BOREL-measurable functions $g : I \to \mathbb{R}$ satisfying:

(i) for every $\theta \in [0,2\pi)$, the function $r \mapsto g_\theta(r) := g(r,\theta)$ is differentiable on $[0,\ell(\theta))$, and the derivative $r \mapsto g_\theta'(r)$ is absolutely continuous on $[0,\ell(\theta))$;

(ii) the function $\theta \mapsto g_\theta'(0+)$ is bounded;

(iii) there exist a constant η with $0 < \eta < \inf_{\theta\in[0,2\pi)} \ell(\theta)$ and a LEBESGUE-integrable function $c : (0,\eta] \to [0,\infty)$, such that for all $\theta \in [0,2\pi)$ and $r \in (0,\eta]$, we have $|g_\theta''(r)| \le c(r)$.

We shall work throughout this subsection in the most general setting of Definition 3.1 for WALSH diffusions, and impose the following condition on the functions $\boldsymbol{b} : \check{I} \to \mathbb{R}$ and $\boldsymbol{s} : \check{I} \to \mathbb{R}$.

Condition 3.10. *(i) We have $\boldsymbol{s}(x) \neq 0$, $\forall\, x \in \check{I}$.*

(ii) For every fixed $\theta \in [0,2\pi)$, both functions below are locally integrable on $(0,\ell(\theta))$:

$$r \longmapsto \frac{\boldsymbol{b}(r,\theta)}{\boldsymbol{s}^2(r,\theta)} \qquad \textit{and} \qquad r \longmapsto \frac{1}{\boldsymbol{s}^2(r,\theta)}\,.$$

(iii) There exists a constant η with $0 < \eta < \inf_{\theta\in[0,2\pi)} \ell(\theta)$, such that

$$\sup_{\substack{0<r\le\eta \\ \theta\in[0,2\pi)}} \left(\frac{1 + |\boldsymbol{b}(r,\theta)|}{\boldsymbol{s}^2(r,\theta)} \right) < \infty\,.$$

With this condition, we define the *radial scale function* $p : I \to [0,\infty)$ by

$$p(r,\theta) = p_\theta(r) := \int_0^r \exp\Big(-2\int_0^y \frac{\boldsymbol{b}(z,\theta)}{\boldsymbol{s}^2(z,\theta)}\mathrm{d}z\Big)\mathrm{d}y\,, \qquad (r,\theta)\in I\,, \tag{3.10}$$

as well as the *scale mapping* $\mathcal{P} : I \to J$, where

$$J := \big\{(r,\theta) : 0 \le r < p_\theta(\ell(\theta)-),\ 0\le\theta<2\pi\big\} \quad \text{and} \quad \mathcal{P}(r,\theta) := \big(p(r,\theta),\theta\big), \quad (r,\theta)\in I\,. \tag{3.11}$$

These are well-defined, as $p(0,\theta)\equiv 0$ and $\mathcal{P}(0,\theta) = (0,\theta)\equiv \mathbf{0}$. Moreover, since the mapping $r\mapsto p(r,\theta)$ is strictly increasing on $[0,\ell(\theta))$ for every $\theta\in[0,2\pi)$, we see that the mapping $\mathcal{P}$ is invertible; we denote by $\mathcal{Q} : J\to I$ its inverse. From (3.11), we have the representation

$$\mathcal{Q}(r,\theta) = \big(q(r,\theta),\theta\big), \qquad (r,\theta)\in J \tag{3.12}$$

where $q : J\to[0,\infty)$ is a function with the property that, for every $\theta\in[0,2\pi)$, the mappings $r\mapsto p_\theta(r)$ and $r\mapsto q_\theta(r) := q(r,\theta)$ are inverses of each other.

We equip both sets I and J with the tree-topology, and consider their closures in the extended rays, as in the beginning of Subsection 3.1. Finally, we extend $\mathcal{P}$ to $\overline{I}$ and $\mathcal{Q}$ to $\overline{J}$ continuously, with the aid of Proposition 3.11(iii) below.

The following result can be checked in a very direct manner; its proof is omitted.

Proposition 3.11. *Assume Condition 3.10 holds for* $\boldsymbol{b} : \check{I}\to\mathbb{R}$ *and* $\boldsymbol{s} : \check{I}\to\mathbb{R}$*. Then:*
(i) *The mapping* $\theta\mapsto p_\theta(\ell(\theta)-)$ *is bounded away from zero, thus* J *is open in the tree-topology.*
(ii) *We have* $p\in\mathfrak{D}_I$, $q\in\mathfrak{D}_J$, $p_\theta(0)\equiv 0\equiv q_\theta(0)$, $p'_\theta(0+)\equiv 1\equiv q'_\theta(0+)$*, and that*

$$p''_\theta(r) = -\frac{2\,\boldsymbol{b}(r,\theta)}{\boldsymbol{s}^2(r,\theta)}p'_\theta(r)\,, \quad q'_\theta(r) = \frac{1}{p'_\theta\big(q_\theta(r)\big)}\,, \quad q''_\theta(r) = \frac{2\,\boldsymbol{b}(q_\theta(r),\theta)}{\boldsymbol{s}^2\big(q_\theta(r),\theta\big)\cdot\big(p'_\theta(q_\theta(r))\big)^2}$$

hold for every $\theta\in[0,2\pi)$ *and a.e.* $r\in(0,\ell(\theta))$.
(iii) *The mappings* $\mathcal{P} : I\to J$ *and* $\mathcal{Q} : J\to I$ *are both continuous in the tree-topology.*

We have then the following "removal-of-drift" result.

Proposition 3.12. *Assume that Condition 3.10 holds, and consider the function* $\widetilde{\boldsymbol{s}} : \check{J}\to\mathbb{R}$ *given by*

$$\widetilde{\boldsymbol{s}}(r,\theta) := p'_\theta\big(q_\theta(r)\big)\,\boldsymbol{s}\big(q_\theta(r),\,\theta\big)\,, \qquad (r,\theta)\in\check{J}. \tag{3.13}$$

If $X(\cdot)$ *is a* WALSH *diffusion with state-space* I, *associated with the triple* $(\boldsymbol{b},\boldsymbol{s},\boldsymbol{\nu})$ *and defined up to an explosion time* S, *then* $Y(\cdot) := \mathcal{P}(X(\cdot))$ *is a* WALSH *diffusion associated with the triple* $(\mathbf{0},\widetilde{\boldsymbol{s}},\boldsymbol{\nu})$ *and defined up to the same explosion time* S, *with state-space* J *and the same underlying probability space and Brownian motion as* $X(\cdot)$; *and vice versa.*

Proof. We prove only the first claim, as the "vice-versa" part can be established in the same way. Assume that $X(\cdot)$ is a WALSH diffusion with state-space I associated with the triple $(\boldsymbol{b},\boldsymbol{s},\boldsymbol{\nu})$ and up to an explosion time S, and let $Y(\cdot)=\mathcal{P}(X(\cdot))$. It follows that $Y(\cdot)$ is $\overline{J}$-valued and continuous in the tree-topology.

Let us recall Definition 2.8. By Definition 3.1, Theorem 2.13(ii) (rather, its obvious generalization to processes valued in I and functions in the class $\mathfrak{D}_I$) and Proposition 3.11, direct calculation gives

$$\|Y(\cdot\wedge S_n)\| = p\big(X(\cdot\wedge S_n)\big) = \|Y(0)\| + \int_0^{\cdot\wedge S_n}\mathbf{1}_{\{X(t)\neq\mathbf{0}\}}\,\widetilde{\boldsymbol{s}}\big(Y(t)\big)\mathrm{d}W(t) \;+\; L^{\|X(\cdot\wedge S_n)\|}(\cdot). \tag{3.14}$$

From $Y(\cdot)=\mathcal{P}(X(\cdot))$ it is clear that the equality $\{t : \|Y(t)\|=0\} = \{t : \|X(t)\|=0\}$ holds pathwise, so by (3.14) and (2.5) we have

$$L^{\|Y(\cdot\wedge S_n)\|}(\cdot) = \int_0^{\cdot}\mathbf{1}_{\{\|Y(t\wedge S_n)\|=0\}}\,\mathrm{d}\|Y(t\wedge S_n)\| \;=\; L^{\|X(\cdot\wedge S_n)\|}(\cdot) \tag{3.15}$$

and (3.14) turns into

$$\|Y(\cdot \wedge S_n)\| = \|Y(0)\| + \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{Y(t) \neq \mathbf{0}\}}\, \widetilde{\boldsymbol{s}}\big(Y(t)\big)\mathrm{d}W(t) \,+\, L^{\|Y(\cdot \wedge S_n)\|}(\cdot)\,.$$

Therefore, it suffices to verify that (v) of Definition 3.1 holds for $Y(\cdot)$.

It is apparent that $\int_0^{\cdot} \mathbf{1}_{\{Y(t) = \mathbf{0}\}}\,\mathrm{d}t \,=\, \int_0^{\cdot} \mathbf{1}_{\{X(t) = \mathbf{0}\}}\,\mathrm{d}t \equiv 0$ holds. On the other hand, since

$$R_A^Y(\cdot) = \|Y(\cdot)\| \cdot \mathbf{1}_A\big(\arg\big(Y(\cdot)\big)\big) = p\big(X(t)\big) \cdot \mathbf{1}_A\big(\arg\big(X(\cdot)\big)\big)\,,$$

we obtain the following, in the same way as in the derivation of (3.14):

$$R_A^Y(\cdot \wedge S_n) = R_A^Y(0) + \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} \cdot \mathbf{1}_A\big(\arg\big(X(\cdot)\big)\big)\, \widetilde{\boldsymbol{s}}\big(Y(t)\big)\mathrm{d}W(t) \,+\, \boldsymbol{\nu}(A)\, L^{\|X(\cdot \wedge S_n)\|}(\cdot)\,.$$

Moreover, we observe $\{t :\, R_A^Y(t) = 0\} = \{t : X(t) = \mathbf{0} \text{ or } \mathbf{1}_A\big(\arg\big(X(t)\big)\big) = 0\}$, thus

$$L^{R_A^Y(\cdot \wedge S_n)}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{R_A^Y(t \wedge S_n) = 0\}}\, \mathrm{d}R_A^Y(t \wedge S_n) \,=\, \boldsymbol{\nu}(A)\, L^{\|X(\cdot \wedge S_n)\|}(\cdot) = \boldsymbol{\nu}(A)\, L^{\|Y(\cdot \wedge S_n)\|}(\cdot)\,;$$

we have used (3.15) for the last equality. Now (v) of Definition 3.1 is seen to hold for $Y(\cdot)$. □

We obtain the following result regarding existence and uniqueness of a general WALSH diffusion.

Theorem 3.13. *Assume that Condition 3.10 holds for* $\boldsymbol{b} : \check{I} \to \mathbb{R}$ *and* $\boldsymbol{s} : \check{I} \to \mathbb{R}$. *Then, for every initial distribution* $\boldsymbol{\mu}$ *on the* BOREL *subsets of* I, *there exists a unique-in-distribution* WALSH *diffusion* $X(\cdot)$ *with state-space* I, *associated with the triple* $(\boldsymbol{b}, \boldsymbol{s}, \boldsymbol{\nu})$ *and defined up to an explosion time* S.

Proof. In light of Proposition 3.12, it suffices to show existence and uniqueness for the WALSH diffusion $Y(\cdot)$ in J associated with the triple $(\mathbf{0}, \widetilde{\boldsymbol{s}}, \boldsymbol{\nu})$, up to an explosion time S, given any initial distribution.

We shall reduce this to Theorem 3.7, which considers the full state space $\mathbb{R}^2$, not J. In addition to (3.13), let us define $\widetilde{\boldsymbol{s}}(r, \theta) := 0$ for $(r, \theta) \in J^c := \big\{(r, \theta) : r \geq p_\theta\big(\ell(\theta)\big),\ 0 \leq \theta < 2\pi\big\}$. It is now straightforward, using Condition 3.10, to check that $\widetilde{\boldsymbol{s}}$ satisfies Condition 3.6 in Section 3.2, and that $\mathcal{I}(\widetilde{\boldsymbol{s}}) = \mathcal{Z}(\widetilde{\boldsymbol{s}}) = J^c$. By Theorem 3.7, there exists a unique-in-distribution, non-explosive WALSH diffusion $Y(\cdot)$ with values in $\mathbb{R}^2$ associated with the triple $(\mathbf{0}, \widetilde{\boldsymbol{s}}, \boldsymbol{\nu})$, given any initial distribution in J. Moreover, by Remark 3.8, $Y(\cdot)$ becomes motionless once it hits $\mathcal{I}(\widetilde{\boldsymbol{s}}) = J^c$, i.e., once it exits from the set J. Thus it is clear by definition that $Y(\cdot)$ is also a WALSH diffusion in J, with explosion time $S := \inf\{t :\, Y(t) \notin J\}$.

On the other hand, assume that $Y(\cdot)$ is a WALSH diffusion with values in J associated with the triple $(\mathbf{0}, \widetilde{\boldsymbol{s}}, \boldsymbol{\nu})$, up to an explosion time $S = \inf\{t :\, Y(t) \notin J\}$. Note that we stipulate $Y(t) = Y(S)$ for $S \leq t < \infty$. Thus by setting $\widetilde{\boldsymbol{s}} \equiv 0$ on J^c as before, we see immediately that $Y(\cdot)$ is also a WALSH diffusion with values in $\mathbb{R}^2$ associated with the triple $(\mathbf{0}, \widetilde{\boldsymbol{s}}, \boldsymbol{\nu})$. By Theorem 3.7, its probability law is uniquely determined, for any given initial distribution. □

4 Explosion Test for Walsh Diffusions with Angular Dependence

Throughout this section, we have for every $x \in I := \{(r, \theta) :\, 0 \leq r < \ell(\theta),\ 0 \leq \theta < 2\pi\}$ a WALSH diffusion (X, W), $(\Omega, \mathcal{F}, \mathbb{P}^x)$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ with values in I, associated with the triple $(\boldsymbol{b}, \boldsymbol{s}, \boldsymbol{\nu})$ and up to an explosion time S, with $X(0) = x$, $\mathbb{P}^x-$a.e. Here $\ell : [0, 2\pi) \to (0, \infty]$ is measurable and bounded away from zero, and the functions $\boldsymbol{b} : \check{I} \to \mathbb{R}$, $\boldsymbol{s} : \check{I} \to \mathbb{R}$ are assumed to satisfy Condition 3.10.

For different initial conditions x, these WALSH diffusions (including the underlying probability space) are different; but we shall use $X(\cdot)$ to denote every one of them. We shall let the measures $\mathbb{P}^x$ distinguish them, since all the conclusions we will draw are about probability distributions.

We develop in this section analogues of all the results in Section 5.5.C of [14]. The two main results are Theorem 4.5, on the asymptotic behavior of $X(\cdot)$; and Theorem 4.9 on the test for explosions in finite time.

4.1 Preliminaries; Explosion in Finite Expected Time

We first note that if $X(\cdot)$ starts at the origin and $A \in \mathcal{B}([0,2\pi))$ satisfies $\boldsymbol{\nu}(A) = 0$, then $X(\cdot)$ never visits any region in the state-space whose rays correspond to angles in A, with positive probability.

Proposition 4.1. *For every $A \in \mathcal{B}([0,2\pi))$ with $\boldsymbol{\nu}(A) = 0$, we have*

$$R_A^X(\cdot) := \|X(\cdot)\| \cdot \mathbf{1}_A\big(\mathit{arg}\big(X(\cdot)\big)\big) \equiv 0\,, \qquad \mathbb{P}^{\mathbf{0}} - a.e.$$

In other words, the set $\{t \geq 0 : \; X(t,\omega) \neq \mathbf{0} \;$ and $\; \mathit{arg}\big(X(t,\omega)\big) \in A\}$ is empty, for $\mathbb{P}^{\mathbf{0}}$−a.e $\omega \in \Omega$.

Proof. From the proofs of Theorem 3.13, Proposition 3.12 and Theorem 3.7, we see that $Y(\cdot) := \mathcal{P}(X(\cdot))$ is a driftless WALSH diffusion, and that it is also a time-changed WALSH Brownian motion with angular measure $\boldsymbol{\nu}$. But a WALSH Brownian motion with angular measure $\boldsymbol{\nu}$ can be constructed as in the proof of Theorem 2.1 in [12], by assigning every excursion of a reflected Brownian motion an angle via a sequence of I.I.D random variables distributed as $\boldsymbol{\nu}$. Therefore, if $Y(\cdot)$ starts at the origin, it almost surely never visits any rays with angles in a set $A \in \mathcal{B}([0,2\pi))$ with $\boldsymbol{\nu}(A) = 0$, because the aforementioned I.I.D. random variables will not be valued in A with any positive probability. This property is inherited by the time-changed WALSH Brownian motion $Y(\cdot)$, and then by the process $X(\cdot) = \mathcal{Q}(Y(\cdot))$. ☐

Next, we note that $X(\cdot)$ has the strong MARKOV property. By Theorem 3.13, the probability

$$\mathfrak{h}(x;\,\Gamma) \;:=\; \mathbb{P}^x\big(X(\cdot) \in \Gamma\big) \tag{4.1}$$

is uniquely determined, for all $x \in I$ and $\Gamma \in \mathcal{B}\big(C(\overline{I})\big)$. Here $C(\overline{I})$ is the collection of all $\overline{I}$-valued functions on $[0,\infty)$ which are continuous in the tree-topology and get absorbed upon hitting the boundary $\partial I := \{(r,\theta) : r = \ell(\theta),\; 0 \leq \theta < 2\pi\}$; the BOREL subsets of this space are generated by its finite-dimensional cylinder sets. Since we constructed $X(\cdot)$ in the last section through scaling and time-change, it is clear that the mapping $x \mapsto \mathfrak{h}(x;\,\Gamma)$ is measurable on I, for every $\Gamma \in \mathcal{B}\big(C(\overline{I})\big)$.

The following result can be proved by connecting to local martingale problems, through a combination of adaptations of Propositions 6.1 and 9.1 in [12], that allow an explosion time; we will omit its proof.

Proposition 4.2. *For every $x \in I$, the process $X(\cdot)$ is time-homogeneous strongly Markovian, in the sense that for every stopping time T of $\mathbb{F}$ and every set $\Gamma \in \mathcal{B}\big(C(\overline{I})\big)$ we have*

$$\mathbb{P}^x\big(X(T+\cdot) \in \Gamma \,\big|\, \mathcal{F}(T)\big) \;=\; \mathfrak{h}(X(T);\,\Gamma)\,, \qquad \mathbb{P}^x - \textit{a.e. on } \{T < S\}\,.$$

Now we recall the radial scale function $p : I \mapsto [0,\infty)$ in (3.10), and observe from (3.14) that p turns $X(\cdot)$ into a reflected local martingale, which is the radial part of the driftless WALSH diffusion $Y = \mathcal{P}(X)$. By analogy with one-dimensional diffusions, we introduce the *speed measure*

$$\boldsymbol{m}_\theta(\mathrm{d}r) := \frac{2\,\mathrm{d}r}{p_\theta'(r)\boldsymbol{s}^2(r,\theta)}\,, \qquad (r,\theta) \in \check{I} \tag{4.2}$$

as well as the FELLER *function*

$$v(r,\theta) = v_\theta(r) := \int_0^r p_\theta'(y)\boldsymbol{m}_\theta([0,y])\mathrm{d}y \;=\; \int_0^r \big(p_\theta(r) - p_\theta(y)\big)\boldsymbol{m}_\theta(\mathrm{d}y)\,, \quad (r,\theta) \in I\,. \tag{4.3}$$

We have the following result regarding the functions p and v.

Proposition 4.3. **(i)** *The function $v : I \to [0,\infty)$ of* (4.3) *is in the class $\mathfrak{D}_I$ (cf. Definition 3.9) with $v_\theta'(0+) \equiv 0$; and for every $\theta \in [0,2\pi)$, we have*

$$\boldsymbol{b}(r,\theta)v_\theta'(r) + \frac{1}{2}\boldsymbol{s}^2(r,\theta)v_\theta''(r) \;=\; 1\,, \qquad \textit{a.e.} \;\; r \in (0,\ell(\theta))\,. \tag{4.4}$$

(ii) *For every* $\theta \in [0, 2\pi)$ *, the function*

$$r \longmapsto \frac{v_\theta(r)}{p_\theta(r)}$$

is strictly increasing on $(0, \ell(\theta))$ *with* $\big(\frac{v_\theta}{p_\theta}\big)(0+) = 0$ *. Thus* $\big(\frac{v_\theta}{p_\theta}\big)(\ell(\theta)-)$ *is well-defined (but may be* ∞*).*
(iii) *We have the implication* $p_\theta(\ell(\theta)-) = \infty \;\Rightarrow\; v_\theta(\ell(\theta)-) = \infty$ *, for every* $\theta \in [0, 2\pi)$ *.*

Proof. The claim **(i)** can be checked in a very direct manner. Moreover, we have by (4.3) that

$$\frac{v_\theta(r)}{p_\theta(r)} = \int_0^r \Big(1 - \frac{p_\theta(y)}{p_\theta(r)}\Big) \boldsymbol{m}_\theta(\mathrm{d}y) \,,$$

and **(ii)** is then immediate from this, and from the fact that $p_\theta(r)$ is positive and strictly increasing on $(0, \ell(\theta))$. Finally, **(iii)** follows clearly from **(ii)**. ☐

Now we give a sufficient condition for $X(\cdot)$ to explode in finite expected time.

Proposition 4.4. *We have* $\mathbb{E}^x[S] < \infty$ *for every* $x \in I$, *if*

$$\boldsymbol{\nu}\big(\{\theta : p_\theta(\ell(\theta)-) < \infty\}\big) > 0 \qquad \textit{and} \qquad \sup_{\theta \in [0,2\pi)} \Big(\frac{v_\theta}{p_\theta}\Big)(\ell(\theta)-) \;<\; \infty \,. \tag{4.5}$$

In particular, we have $\mathbb{E}^x[S] < \infty$ *for every* $x \in I$, *if* $\sup_{\theta \in [0,2\pi)} v_\theta(\ell(\theta)-) \;<\; \infty$ *.*

Proof. Assume that (4.5) holds. Then we can define

$$C_1 := \frac{\int_0^{2\pi} \big(\frac{v_\theta}{p_\theta}\big)(\ell(\theta)-)\, \boldsymbol{\nu}(\mathrm{d}\theta)}{\int_0^{2\pi} \frac{1}{p_\theta(\ell(\theta)-)}\, \boldsymbol{\nu}(\mathrm{d}\theta)} \,, \qquad C_2(\theta) := -\frac{C_1}{p_\theta(\ell(\theta)-)} + \Big(\frac{v_\theta}{p_\theta}\Big)(\ell(\theta)-) \,, \tag{4.6}$$

$$M(r,\theta) \equiv M_\theta(r) := -v_\theta(r) + C_2(\theta)\, p_\theta(r) + C_1 \qquad (r,\theta) \in I \,. \tag{4.7}$$

Note that the expression for $C_2(\theta)$ in (4.6) is meaningful even in the case $p_\theta(\ell(\theta)-) = \infty$.

Now M is a well-defined function on I , as $M(0,\theta) \equiv C_1$. Since $\theta \mapsto p_\theta(\ell(\theta)-)$ is bounded away from zero by Proposition 3.11 (i), we see that $\theta \mapsto C_2(\theta)$ is bounded, and that $M \in \mathfrak{D}_I$, thanks to Proposition 4.3 (i). Moreover, by Propositions 4.3 and 3.11, it is easy to check

$$\int_0^{2\pi} M_\theta'(0+)\, \boldsymbol{\nu}(\mathrm{d}\theta) = 0 \quad \text{ and } \quad \boldsymbol{b}(r,\theta) M_\theta'(r) + \frac{1}{2}\boldsymbol{s}^2(r,\theta) M_\theta''(r) = -1 \,. \tag{4.8}$$

Recalling Definition 2.8, we apply Theorem 2.13 (again, its obvious generalization to processes valued in I and functions in $\mathfrak{D}_I$) and obtain the $\mathbb{P}^x-$a.e. equality

$$M(X(T \wedge S_n)) = M(x) - (T \wedge S_n) + \int_0^{T \wedge S_n} \mathbf{1}_{\{X(t) \neq \mathbf{0}\}}\, M'(X(t))\, \boldsymbol{s}(X(t))\, \mathrm{d}W(t) \,, \quad 0 \le T < \infty \,, \tag{4.9}$$

where S_n is as in Definition 3.1 (iii). With

$$\tau_n := \inf \left\{ t : \int_0^{t \wedge S_n} \mathbf{1}_{\{X(u) \neq \mathbf{0}\}} \big(M'(X(u))\, \boldsymbol{s}(X(u))\big)^2 \mathrm{d}u \;\ge\; n \right\} \wedge S_n \,,$$

taking expectations in (4.9) yields

$$\mathbb{E}^x \big[M(X(\tau_n)) + \tau_n \big] \;=\; M(x) \,, \qquad \forall\, n \in \mathbb{N} \,. \tag{4.10}$$

On the other hand, we have by Proposition 4.3(ii) that

$$M(r,\theta) \;=\; C_1 \left(1 - \frac{p_\theta(r)}{p_\theta(\ell(\theta)-)}\right) \;+\; p_\theta(r) \left[\Big(\frac{v_\theta}{p_\theta}\Big)(\ell(\theta)-) - \Big(\frac{v_\theta}{p_\theta}\Big)(r) \right] \;\ge\; 0 \,, \qquad \forall\, (r,\theta) \in I \,.$$

Thus (4.10) implies $\mathbb{E}^x[\tau_n] \le M(x)$, $\forall\, n \in \mathbb{N}$. Letting $n \to \infty$ we get $\mathbb{E}^x[S] \le M(x) < \infty$.

Finally, we note that $\sup_{\theta \in [0,2\pi)} v_\theta(\ell(\theta)-) \;<\; \infty$ implies (4.5), thanks to Proposition 4.3 (iii) and Proposition 3.11(i). Proposition 4.4 is now proved. ☐

4.2 Asymptotic Behavior Near the Explosion Time

Throughout this subsection and the next one, we use the notation $\Theta(t) := \arg(X(t))$ whenever $X(t) \neq \mathbf{0}$, and recall from (2.9) the process

$$R_A^X(\cdot) := \|X(\cdot)\| \cdot \mathbf{1}_A\big(\Theta(\cdot)\big)\,, \quad \forall\, A \in \mathcal{B}([0, 2\pi))\,.$$

We recall also the functions $\{\ell_n\}_{n=1}^\infty$ and the sets $\{I_n\}_{n=1}^\infty$ at the beginning of Subsection 3.1.

The following main result of this subsection discusses the behavior of $X(t)$ as t approaches S.

Theorem 4.5. *Let $x = \mathbf{0}$ in the context specified at the beginning of this section. With p defined as in (3.10), we distinguish two cases:*

(i) $\boldsymbol{\nu}(\{\theta : p_\theta(\ell(\theta)-) < \infty\}) > 0$.

Then the limit in the tree-topology $\lim_{t\uparrow S} X(t)$ exists $\mathbb{P}^{\mathbf{0}}$-a.e. in the extended rays, and $X(S) := \lim_{t\uparrow S} X(t) \in \partial I := \{(r, \theta) : r = \ell(\theta),\ \theta \in [0, 2\pi)\}$. Moreover, we have in this case

$$\mathbb{P}^{\mathbf{0}}\big(\Theta(S) \in A\big) \;=\; \frac{\int_A \frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)}{\int_0^{2\pi} \frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)}\;, \qquad \forall\, A \in \mathcal{B}([0, 2\pi))\,. \tag{4.11}$$

(ii) $\boldsymbol{\nu}(\{\theta : p_\theta(\ell(\theta)-) < \infty\}) = 0$.

Then $\mathbb{P}^{\mathbf{0}}-$a.e., we have that the limit $\lim_{t\uparrow S} X(t)$ does not exist, and that

$$\boldsymbol{\nu}\bigg(\overline{\Big\{\theta : \sup_{0\le t<S} R^X_{\{\theta\}}(t) \;\ge\; \ell_n(\theta)\Big\}}\bigg) \;=\; 1\,, \qquad \forall\, n \in \mathbb{N} \tag{4.12}$$

holds, where the closure is taken in $[0, 2\pi)$; in particular, $\mathbb{P}^{\mathbf{0}}\big(S = \infty\big) = 1$.

Moreover, whenever $\boldsymbol{\nu}(\{\theta\}) > 0$ holds, we have

$$\sup_{0\le t<S} R^X_{\{\theta\}}(t) \;=\; \ell(\theta)\,, \quad \mathbb{P}^{\mathbf{0}} - \textit{a.e.} \tag{4.13}$$

Remark 4.6. We stipulate $\frac{1}{\infty} = 0$ in (4.11). Since $\theta \mapsto p_\theta(\ell(\theta)-)$ is bounded away from zero, we see that (4.11) makes good sense, provided $\boldsymbol{\nu}(\{\theta : p_\theta(\ell(\theta)-) < \infty\}) > 0$ holds. We make no claim in **(i)** regarding the finiteness of S, and the result holds there regardless of whether S is finite or not. A full discussion regarding the finiteness of S appears in Subsection 4.3.

Remark 4.7. *If we replace "$\ell_n(\theta)$" by "$\ell(\theta)$" in the property (4.12), then this new property no longer holds in general.* Indeed, let $\boldsymbol{\nu}$ be the uniform distribution on $[0, 2\pi)$ in **(ii)**; then (4.13) holds for no $\theta \in [0, 2\pi)$, $\mathbb{P}^{\mathbf{0}}-$a.e. This is because $X(\cdot)$ will be on different rays for any two of its excursions away from the origin.

Proof. We first note that the explosion time S does not depend on the choice of the approximating sequence of functions $\{\ell_n\}_{n=1}^\infty$, because $S = \inf\{t : X(t) \notin I\}$ always holds by (3.5). Thus in the proof of **(i)**, we will assume that

$$p_\theta(\ell_n(\theta)) \le n \qquad \text{and} \qquad v_\theta(\ell_n(\theta)) \le n\,, \qquad \forall\, \theta \in [0, 2\pi)\,, \quad \forall\, n \in \mathbb{N}\,; \tag{4.14}$$

for otherwise, we can define

$$\widetilde{\ell_n}(\theta) \;:=\; \sup\big\{r : 0 \le r \le \ell_n(\theta),\ p_\theta(r) \le n,\ v_\theta(r) \le n\big\}$$

and let the sequence $\{\widetilde{\ell_n}\}_{n=1}^\infty$ play the role of $\{\ell_n\}_{n=1}^\infty$. However, we will not assume (4.14) when proving **(ii)**, because ℓ_n appears explicitly in the conclusion of **(ii)**.

Proof of **(i)**. *Step 1.* We shall prove **(i)** in this step, albeit under the assumptions

$$\sup_{\theta\in[0,2\pi)} p_\theta(\ell(\theta)-) \,<\, \infty \qquad \text{and} \qquad \sup_{\theta\in[0,2\pi)} v_\theta(\ell(\theta)-) \,<\, \infty\,. \tag{4.15}$$

With (4.15), we have $\mathbb{E}^{\mathbf{0}}[S]<\infty$ by Proposition 4.4, thus $\mathbb{P}^{\mathbf{0}}(S<\infty)=1$. Thus, from (3.5) we know that $X(S)=\lim_{t\uparrow S}X(t)$ exists under the tree-topology in $\{(r,\theta):\, r=\ell(\theta),\ 0\le\theta<2\pi\}$, $\mathbb{P}^{\mathbf{0}}$-a.e. It develops that $\Theta(S)$ is a well-defined random variable with values in $[0,2\pi)$; we denote its distribution by $\widetilde{\boldsymbol{\nu}}$, a probability measure on $([0,2\pi),\mathcal{B}([0,2\pi)))$.

Let us define the *scale function associated with a set* $A\in\mathcal{B}([0,2\pi))$ by

$$p^A_\theta(r)=p^A(r,\theta):=\, p(r,\theta)\cdot\big(\boldsymbol{\nu}(A)\cdot\mathbf{1}_{A^c}(\theta)\,-\,\boldsymbol{\nu}(A^c)\cdot\mathbf{1}_A(\theta)\big)\,, \qquad (r,\theta)\in I\,. \tag{4.16}$$

Clearly, we have $p^A\in\mathfrak{D}_I$ and $\int_0^{2\pi}(p^A_\theta)'(0+)\,\boldsymbol{\nu}(\mathrm{d}\theta)\;=\;0$. Now with the help of Proposition 3.11 and Theorem 2.13, we can easily check that $p^A(X(\cdot\wedge S_n))$ is a local martingale – and actually a martingale, because (4.15) gives the boundedness of p^A on I. Then we may let $n\to\infty$ to obtain that $p^A(X(\cdot\wedge S))$ is a bounded martingale. This gives

$$0\,=\,p^A(\mathbf{0})\,=\,\mathbb{E}^{\mathbf{0}}\big[\,p^A\big(X(S)\big)\,\big]\,=\,\mathbb{E}^{\mathbf{0}}\big[\,p^A\big(\ell(\Theta(S)),\Theta(S)\big)\,\big]\,=\,\int_0^{2\pi}p^A\big(\ell(\theta),\theta\big)\,\widetilde{\boldsymbol{\nu}}(\mathrm{d}\theta)$$

$$=\boldsymbol{\nu}(A)\cdot\int_{A^c}p_\theta(\ell(\theta)-)\,\widetilde{\boldsymbol{\nu}}(\mathrm{d}\theta)-\boldsymbol{\nu}(A^c)\cdot\int_A p_\theta(\ell(\theta)-)\,\widetilde{\boldsymbol{\nu}}(\mathrm{d}\theta)\,; \tag{4.17}$$

here we have extended the function p^A to $\overline{I}$ continuously, so that $p^A\big(X(S)\big)$ is well-defined.

From (4.17), we observe that $\widetilde{\boldsymbol{\nu}}(A)=0$ holds whenever $\boldsymbol{\nu}(A)=0$. Thus the measure $\widetilde{\boldsymbol{\nu}}$ is absolutely continuous with respect to $\boldsymbol{\nu}$, and we may assume that $\widetilde{\boldsymbol{\nu}}(\mathrm{d}\theta)\;=\;\psi(\theta)\,\boldsymbol{\nu}(\mathrm{d}\theta)$ for some function $\psi:[0,2\pi)\to[0,\infty)$. Now for (4.11) to hold, we only need to show that

$$\psi(\theta)\;=\;\frac{\frac{1}{p_\theta(\ell(\theta)-)}}{\int_0^{2\pi}\frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)}\;, \qquad \boldsymbol{\nu}\text{-a.e. }\ \theta\in[0,2\pi)\,. \tag{4.18}$$

To this effect, we consider the sets

$$A_1:=\left\{\theta:\ \psi(\theta)>\frac{\frac{1}{p_\theta(\ell(\theta)-)}}{\int_0^{2\pi}\frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)}\right\} \qquad \text{and} \qquad A_2:=\left\{\theta:\ \psi(\theta)<\frac{\frac{1}{p_\theta(\ell(\theta)-)}}{\int_0^{2\pi}\frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)}\right\}.$$

Letting $A=A_1$ in (4.17), it is easy to deduce that either $\boldsymbol{\nu}(A_1)=0$ or $\boldsymbol{\nu}(A_1)=1$ must hold. But the latter cannot happen, for otherwise we would have $\widetilde{\boldsymbol{\nu}}([0,2\pi))\,=\,\int_0^{2\pi}\psi(\theta)\,\boldsymbol{\nu}(\mathrm{d}\theta)>1$. Thus $\boldsymbol{\nu}(A_1)=0$ holds, and we deduce $\boldsymbol{\nu}(A_2)=0$ similarly. This way we get (4.18), and Step 1 is now complete.

Step 2. This step will complete the proof of **(i)**. We first show the existence of $\lim_{t\uparrow S}X(t)$, $\mathbb{P}^{\mathbf{0}}$-a.s.

Case A: $\boldsymbol{\nu}$ concentrates on one angle θ_0. Then $p_{\theta_0}(\ell(\theta_0)-)<\infty$, and $X(\cdot)$ stays a.s. on the ray with angle θ_0, by Proposition 4.1. Thus the process $p(X(\cdot))$ is bounded. But $p(X(\cdot\wedge S_n))$ is a local submartingale (as a reflected local martingale), thus a true submartingale, and so is $p(X(\cdot\wedge S))$. We deduce that $\lim_{t\uparrow S}p(X(t))$ exists $\mathbb{P}^{\mathbf{0}}-$a.e. Since $X(\cdot)$ stays on the same ray, the existence of $\lim_{t\uparrow S}X(t)$ follows.

Case B: $\boldsymbol{\nu}$ does not concentrate on one angle. Since $\boldsymbol{\nu}(\{\theta:\ p_\theta(\ell(\theta)-)<\infty\})>0$, we can choose an $M>0$ and an $A_M\subseteq[0,2\pi)$, such that $p_\theta(\ell(\theta)-)\le M$ for all $\theta\in A_M$, and that $0<\boldsymbol{\nu}(A_M)<1$. Then the function p^{A_M} is bounded from below. Step 1 shows that $p^{A_M}(X(\cdot\wedge S_n))$ is a local martingale for every $n\in\mathbb{N}$, thus a supermartingale, and we may let $n\to\infty$ to obtain by FATOU's lemma that

$p^{A_M}(X(\cdot\wedge S))$ is a bounded from below supermartingale. Therefore, $\lim_{t\uparrow S} p^{A_M}(X(t))$ exists $\mathbb{P}^{\mathbf{0}}-$a.e. Now we set

$$\mathcal{P}^{A_M}(r,\theta) := \big(\big|p^{A_M}(r,\theta)\big|,\theta\big) \qquad \text{for} \qquad (r,\theta)\in I\,,$$

and note $\mathcal{P}^{A_M}(I) = J^{A_M}$ where, thanks to $0<\boldsymbol{\nu}(A_M)<1$, the set

$$J^{A_M} \,:=\, \big\{(r,\theta): 0\le r<|p_\theta^{A_M}(\ell(\theta)-)|,\; 0\le\theta<2\pi\big\}$$

is open in the tree-topology. By the continuity of $X(\cdot)$ in the tree-topology, the existence of the limit $\lim_{t\uparrow S} p^{A_M}(X(t))$ implies the existence of $\lim_{t\uparrow S}\mathcal{P}^{A_M}(X(t))$ in $\overline{J^{A_M}}$, under the tree-topology.

By analogy with Section 3.3, and thanks once again to $0<\boldsymbol{\nu}(A_M)<1$, we can define the inverse mapping $\mathcal{Q}^{A_M}: J^{A_M}\to I$ of $\mathcal{P}^{A_M}$, and both $\mathcal{Q}^{A_M}$ and $\mathcal{P}^{A_M}$ are continuous in the tree-topology. Moreover, we can extend $\mathcal{Q}^{A_M}$ to $\overline{J^{A_M}}$ and $\mathcal{P}^{A_M}$ to $\overline{I}$ continuously. We see then, that the existence of $\lim_{t\uparrow S}\mathcal{P}^{A_M}(X(t))$ in $\overline{J^{A_M}}$ implies the existence of the limit $\lim_{t\uparrow S} X(t)$ in $\overline{I}$.

Next, we turn to the proof of $X(S)\in\{(r,\theta):\, r=\ell(\theta),\; 0\le\theta<2\pi\}$, as well as (4.11). Let us define

$$\ell^A_{n,m}(\theta) :=\, \ell_n(\theta)\cdot\mathbf{1}_A(\theta)+\ell_m(\theta)\cdot\mathbf{1}_{A^c}(\theta)\,, \qquad I^A_{n,m} := \{(r,\theta):\, 0\le r<\ell^A_{n,m}(\theta),\; 0\le\theta<2\pi\}\,, \tag{4.19}$$

$$S^A_{n,m} :=\, \inf\{t\ge 0:\, \|X(t)\|\ge \ell^A_{n,m}(\Theta(t))\} \,=\, \inf\{t\ge 0:\, X(t)\notin I^A_{n,m}\} \tag{4.20}$$

for $A\in\mathcal{B}([0,2\pi))$ and $m,\,n\in\mathbb{N}$ with $m\ge n$. By (4.14), we have $\sup_{\theta\in[0,2\pi)} p_\theta(\ell^A_{n,m}(\theta)) \,\le\, m$ and $\sup_{\theta\in[0,2\pi)} v_\theta(\ell^A_{n,m}(\theta)) \,\le\, m$. Thus Step 1 shows $\mathbb{P}^{\mathbf{0}}(S^A_{n,m}<\infty)=1$, and that

$$\mathbb{P}^{\mathbf{0}}\big(\Theta(S^A_{n,m})\in A\big) \;=\; \frac{\int_A \frac{1}{p_\theta(\ell_n(\theta))}\,\boldsymbol{\nu}(\mathrm{d}\theta)}{\int_A \frac{1}{p_\theta(\ell_n(\theta))}\,\boldsymbol{\nu}(\mathrm{d}\theta)+\int_{A^c}\frac{1}{p_\theta(\ell_m(\theta))}\,\boldsymbol{\nu}(\mathrm{d}\theta)}\;, \qquad \forall\, A\in\mathcal{B}([0,2\pi))\,. \tag{4.21}$$

Note that the events $\{\Theta(S^A_{n,m})\in A\}$ are increasing in m. Setting $K^A_n := \{(r,\theta):\, r\ge\ell_n(\theta),\;\theta\in A\}$, we have then

$$\begin{aligned}\mathbb{P}^{\mathbf{0}}\big(X \text{ hits } K^A_n\big) \;&\ge\; \mathbb{P}^{\mathbf{0}}\big(\Theta(S^A_{n,m})\in A \text{ for some } m\ge n\big)\\ &=\lim_{m\to\infty}\mathbb{P}^{\mathbf{0}}\big(\Theta(S^A_{n,m})\in A\big) \;=\; \frac{\int_A \frac{1}{p_\theta(\ell_n(\theta))}\,\boldsymbol{\nu}(\mathrm{d}\theta)}{\int_A \frac{1}{p_\theta(\ell_n(\theta))}\,\boldsymbol{\nu}(\mathrm{d}\theta)+\int_{A^c}\frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)}\,.\end{aligned} \tag{4.22}$$

Since $X(S):=\lim_{t\uparrow S} X(t)$ exists $\mathbb{P}^{\mathbf{0}}$-a.e., we may let $n\to\infty$ in (4.22) and obtain

$$\mathbb{P}^{\mathbf{0}}\big(X(S)\in\{(r,\theta):\, r=\ell(\theta),\;\theta\in A\}\big) \;=\; \mathbb{P}^{\mathbf{0}}\big(X \text{ hits } K^A_n \text{ for every } n\big) \;\ge\; \frac{\int_A \frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)}{\int_0^{2\pi} \frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)}\,. \tag{4.23}$$

In particular, $\mathbb{P}^{\mathbf{0}}\big(X(S)\in\{(r,\theta):\, r=\ell(\theta),\; 0\le\theta<2\pi\}\big)=1$. Replacing A by A^c in (4.23) and adding this back to (4.23), we find that the inequality sign in (4.23) can be replaced by an equality sign. Thus (4.11) follows, and the proof of Theorem 4.5**(i)** is now complete.

Proof of **(ii)**. Here we cannot assume (4.14), but can use the result of **(i)**. Because $p_\theta(\ell^A_{n,m}(\theta))<\infty$ for every $\theta\in[0,2\pi)$, we recover (4.21) by an application of (4.11). Thus we have

$$\begin{aligned}\mathbb{P}^{\mathbf{0}}\left(\sup_{0\le t<S} R^X_{\{\theta\}}(t) \;\ge\; \ell_n(\theta) \text{ for some } \theta\in A\right) \;&\ge\; \mathbb{P}^{\mathbf{0}}\big(\Theta(S^A_{n,m})\in A \text{ for some } m\ge n\big)\\ =\lim_{m\to\infty}\mathbb{P}^{\mathbf{0}}\big(\Theta(S^A_{n,m})\in A\big) \;&=\; \frac{\int_A \frac{1}{p_\theta(\ell_n(\theta))}\,\boldsymbol{\nu}(\mathrm{d}\theta)}{\int_A \frac{1}{p_\theta(\ell_n(\theta))}\,\boldsymbol{\nu}(\mathrm{d}\theta)+\int_{A^c}\frac{1}{p_\theta(\ell(\theta)-)}\,\boldsymbol{\nu}(\mathrm{d}\theta)} = 1\end{aligned} \tag{4.24}$$

for every $A\in\mathcal{B}([0,2\pi))$ with $\boldsymbol{\nu}(A)>0$, because $\boldsymbol{\nu}(\{\theta:\, p_\theta(\ell(\theta)-)<\infty\})=0$.

Now we can find an event $\Omega^* \in \mathcal{F}$ with $\mathbb{P}^{\mathbf{0}}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$ and every $A = [a, b)$ with $a, b \in \mathbb{Q} \cap [0, 2\pi)$ and $\boldsymbol{\nu}(A) > 0$, we have $\sup_{0 \le t < S} R^X_{\{\theta\}}(t, \omega) \ge \ell_n(\theta)$ for some $\theta \in A$ and all $n \in \mathbb{N}$, so (4.12) is obtained. Moreover, if $\boldsymbol{\nu}(\{\theta\}) > 0$, we can take $A = \{\theta\}$ in (4.24) and see that the inequality $\sup_{0 \le t < S} R^X_{\{\theta\}}(t) \ge \ell_n(\theta)$ holds $\mathbb{P}^{\mathbf{0}}-$a.e., for every $n \in \mathbb{N}$; thus (4.13) follows.

Finally, we show that the nonexistence of $\lim_{t \uparrow S} X(t)$, and the property $S = \infty$, follow directly, thanks to (3.5). To this effect, we set

$$A^p := \big\{ \theta : \, p_\theta(\ell(\theta)-) < \infty \big\}, \qquad I^p := \big\{ (r, \theta) : \, r > 0, \; \theta \in A^p \big\}$$

and $\Gamma^0 := \{\omega_2 \in C(\overline{I}) : \omega_2(t) = \mathbf{0} \text{ for some } t \in [0, \infty)\}$. Recalling (4.1), and using the theory of one-dimensional diffusion (e.g. Propositions 5.5.22, 5.5.32 in [14]), we deduce

$$\mathfrak{h}(x; \Gamma^0) = 1, \qquad \forall \;\; x \in I \setminus I^p. \tag{4.25}$$

With $T_n := S_n \wedge n$, we have $T_n < S$, $\mathbb{P}^{\mathbf{0}}-$a.e. Since $\boldsymbol{\nu}(A^p) = 0$, Proposition 4.1 shows that $X(T_n) \in I \setminus I^p$, $\mathbb{P}^{\mathbf{0}}-$a.e. Now we apply Proposition 4.2 and obtain $\mathbb{P}^{\mathbf{0}}\big(X(T_n + \cdot) \in \Gamma^0\big) = 1$, $\forall n \in \mathbb{N}$. It follows that, $\mathbb{P}^{\mathbf{0}}-$a.e., if $\lim_{t \uparrow S} X(t)$ exists, it must be $\mathbf{0}$. Comparing this fact with (4.12), we see that $\lim_{t \uparrow S} X(t)$ does not exist, $\mathbb{P}^{\mathbf{0}}-$a.e. □

Theorem 4.5 takes the origin $\mathbf{0}$ as the starting point of $X(\cdot)$. For a starting point $x \in \check{I}$, by the strong MARKOV property, we can treat $X(\cdot)$ as a one-dimensional diffusion before it hits the origin, and use Theorem 4.5 afterwards. The following result can be derived in a very direct manner, so we omit its proof.

Corollary 4.8. *In the context specified at the beginning of this section, let $x = (r_0, \theta_0) \in \check{I}$. We distinguish two cases:*

(i) $\boldsymbol{\nu}(\{\theta : \, p_\theta(\ell(\theta)-) < \infty\}) > 0$.
Then $\lim_{t \uparrow S} X(t)$ exists $\mathbb{P}^x-$a.e. in $\{(\ell(\theta), \theta) : \, \theta \in [0, 2\pi)\}$, and for every $A \in \mathcal{B}([0, 2\pi))$ we have

$$\mathbb{P}^x\big(\Theta(S) \in A\big) \; = \; \frac{\int_A \frac{1}{p_\theta(\ell(\theta)-)} \, \boldsymbol{\nu}(\mathrm{d}\theta)}{\int_0^{2\pi} \frac{1}{p_\theta(\ell(\theta)-)} \, \boldsymbol{\nu}(\mathrm{d}\theta)} \cdot \left(1 - \frac{p_{\theta_0}(r_0)}{p_{\theta_0}(\ell(\theta_0)-)} \right) \; + \; \mathbf{1}_A(\theta_0) \cdot \frac{p_{\theta_0}(r_0)}{p_{\theta_0}(\ell(\theta_0)-)} \, . \tag{4.26}$$

(ii) $\boldsymbol{\nu}(\{\theta : \, p_\theta(\ell(\theta)-) < \infty\}) = 0$.
Then we have

$$\mathbb{P}^x\big(\mathcal{L}^x\big) = \frac{p_{\theta_0}(r_0)}{p_{\theta_0}(\ell(\theta_0)-)} \, , \qquad \textit{for} \qquad \mathcal{L}^x := \Big\{ \lim_{t \uparrow S} X(t) = (\ell(\theta_0), \theta_0) \Big\}.$$

On the other hand, $\mathbb{P}^x-$a.e. on $\big(\mathcal{L}^x\big)^c$, we have that $\lim_{t \uparrow S} X(t)$ does not exist, that $S = \infty$, and that

$$\boldsymbol{\nu}\left(\overline{ \Big\{ \theta : \sup_{0 \le t < S} R^X_{\{\theta\}}(t) \, \ge \, \ell_n(\theta) \Big\} } \right) \; = \; 1, \qquad \forall \;\; n \in \mathbb{N}. \tag{4.27}$$

Moreover, whenever $\boldsymbol{\nu}(\{\theta\}) > 0$, we have $\sup_{0 \le t < S} R^X_{\{\theta\}}(t) \; = \; \ell(\theta)$, $\mathbb{P}^x-$*a.e. on* $\big(\mathcal{L}^x\big)^c$.

4.3 Test for Explosions in Finite Time

This subsection provides criteria for the finiteness of the explosion time. These involve the scale and FELLER functions of (3.10), (4.3), and of course the measure $\boldsymbol{\nu}$. The proof of Theorem 4.9 is in the Appendix, Section 6; whereas the proof of Corollary 4.10 is omitted, for the same reason as that of Corollary 4.8.

Theorem 4.9. *Let* $x = \mathbf{0}$ *in the context specified at the beginning of this section. With the functions* p *and* v *defined by (3.10) and (4.3) respectively, we distinguish three cases:*

(i) $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) < \infty\}) = 0$.
Then we have $\mathbb{P}^{\mathbf{0}}\big(S < \infty\big) = 0$.

(ii) $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) < \infty\}) > 0$ *and* $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) = \infty,\, p_\theta(\ell(\theta)-) < \infty\}) = 0$.
Then we have $\mathbb{P}^{\mathbf{0}}\big(S < \infty\big) = 1$.

(iii) $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) < \infty\}) > 0$ *and* $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) = \infty,\, p_\theta(\ell(\theta)-) < \infty\}) > 0$.
Then we have $0 < \mathbb{P}^{\mathbf{0}}\big(S < \infty\big) < 1$.

Corollary 4.10. *In the context specified at the beginning of this section, let* $x = (r_0, \theta_0) \in \check{I}$. *We distinguish three cases:*

(i) $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) < \infty\}) = 0$.
Then we have $\mathbb{P}^{x}\big(S < \infty\big) = 0$ *if* $v_{\theta_0}(\ell(\theta_0)-) = \infty$, *and* $0 < \mathbb{P}^{x}\big(S < \infty\big) < 1$ *otherwise.*

(ii) $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) < \infty\}) > 0$ *and* $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) = \infty,\, p_\theta(\ell(\theta)-) < \infty\}) = 0$.
Then we have $\mathbb{P}^{x}\big(S < \infty\big) = 1$ *if either* $v_{\theta_0}(\ell(\theta_0)-) < \infty$ *or* $p_{\theta_0}(\ell(\theta_0)-) = \infty$ *hold, whereas we have* $0 < \mathbb{P}^{x}\big(S < \infty\big) < 1$ *otherwise.*

(iii) $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) < \infty\}) > 0$ *and* $\boldsymbol{\nu}(\{\theta : v_\theta(\ell(\theta)-) = \infty,\, p_\theta(\ell(\theta)-) < \infty\}) > 0$.
Then we always have $0 < \mathbb{P}^{x}\big(S < \infty\big) < 1$.

5 Optimal Control / Stopping of a WALSH Semimartingale on the Unit Disc

We consider a WALSH semimartingale $X(\cdot)$ as in Definition 2.14, i.e., a semimartingale on rays with the property (2.10) for a fixed measure $\boldsymbol{\nu}$. This process $X(\cdot)$ takes values in the closed unit disc $\overline{B}$ with

$$B := \big\{(r, \theta) : r \in [0,1),\, \theta \in [0, 2\pi)\big\}, \tag{5.1}$$

and is driven by an ITÔ process $U(\cdot)$ whose local drift and dispersion processes $\beta(\cdot), \sigma(\cdot)$ are controlled.

More precisely, we assume now that, for every $\xi \in \check{B} = B \setminus \{\mathbf{0}\}$, there is a nonempty subset $\mathcal{K}(\xi)$ of $\mathbb{R} \times (0, \infty)$, serving as the "control space" at ξ; i.e., the process $(\beta(\cdot), \sigma(\cdot))$ takes value in $\mathcal{K}(\xi)$ at time $t \in [0, \infty)$, whenever the current position is $X(t) = \xi$. We also set $\mathcal{K}(\xi) = \{(0,0)\}$ whenever $\|\xi\| = 1$, meaning that $X(\cdot)$ is absorbed upon reaching the boundary of B. We do *not* assume, however, that there is a control space at the origin; we posit rather that, when at the origin, the process $X(\cdot)$ is "immediately dispatched along some ray", i.e., that $X(\cdot)$ satisfies the first (non-stickiness) requirement in (3.3).

To make all this more precise, consider an adapted, $\overline{B}-$valued semimartingale $X(\cdot)$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ which is continuous in the tree-topology, satisfies (3.3), and

$$\mathrm{d}\|X(t)\| \,=\, \mathbf{1}_{\{X(t) \neq \mathbf{0}\}}\,\big(\beta(t)\,\mathrm{d}t \,+\, \sigma(t)\,\mathrm{d}W(t)\big) \,+\, \mathrm{d}L^{\|X\|}(t)\,, \qquad X(0) = x \in B\,. \tag{5.2}$$

Here $W(\cdot)$ is an $\mathbb{F}-$Brownian motion, and $\beta(\cdot), \sigma(\cdot)$ are $\mathbb{F}-$progressively measurable processes, satisfying almost surely the integrability and consistency conditions

$$\int_0^t \mathbf{1}_{\{X(u) \neq \mathbf{0}\}}\,\big(|\beta(u)| \,+\, \sigma^2(u)\big)\,\mathrm{d}u \,<\infty \quad \text{and} \quad (\beta(t), \sigma(t)) \in \mathcal{K}\big(X(t)\big), \quad \text{for all} \;\; t \in [0, \infty). \tag{5.3}$$

Given an initial position $x \in B$, we denote by $\mathcal{A}(x)$ the collection of all WALSH semimartingales $X(\cdot)$ which can be constructed as above, and are thus "available" to the controller at x.

For every planar semimartingale $X(\cdot) \in \mathcal{A}(x)$, we denote by $\mathcal{J}_X$ the class of all $\mathbb{F}^X-$stopping times, from which the controller can also choose a way to stop the controlled process $X(\cdot)$. We refer to [15] and [17] for similar considerations regarding the collection of all available processes (the "gambling house" in the terminology of DUBINS & SAVAGE [4]).

Problem 5.1. Control and Stopping of a WALSH Semimartingale. *Consider as our "reward function" a bounded, measurable* $U : \overline{B} \to \mathbb{R}$ *, continuous in the tree-topology, . We want to find, for each starting position* $x \in B$ *, a process* $X^*(\cdot) \in \mathcal{A}(x)$ *and a stopping time* $\tau_* \in \mathcal{J}_{X^*}$ *that attain the supremum*

$$V(x) := \sup_{X \in \mathcal{A}(x),\, \tau \in \mathcal{J}_X} \mathbb{E}\big[U(X(\tau))\big] . \tag{5.4}$$

We use here the convention $U(X(\infty)) = \limsup_{t\to\infty} U(X(t))$.

This is a stochastic control problem with discretionary stopping, in the spirit of [2], [13], [15], for a WALSH semimartingale. We shall solve this problem fairly explicitly under some mild additional regularity assumptions and in a manner inspired by [15], which treats a one-dimensional analogue. It is surprising, to us at least, that this problem should admit a very explicit solution; this is given in Theorem 5.16, Subsection 5.3, with the help of the results developed in Section 2-4 and of their refinements in Subsection 5.1.

5.1 A Refined Stochastic Calculus

First, we need to extend the class of functions $\mathfrak{D}$ given in Definition 2.7, as follows.

Definition 5.2. Let $\mathfrak{C}$ be the class of BOREL-measurable functions $g : \mathbb{R}^2 \to \mathbb{R}$, such that:

(i) for every $\theta \in [0, 2\pi)$, the function $r \mapsto g_\theta(r) := g(r,\theta)$ is the difference of two convex and continuous functions on $[0,\infty)$, and thus the left- and right-derivatives $r \mapsto D^{\pm} g_\theta(r)$ exist and are of finite variation on compact subintervals of $(0,\infty)$;

(ii) the function $\theta \mapsto D^+ g_\theta(0)$ is well-defined and bounded;

(iii) there exist a real number $\eta > 0$ and a finite measure μ on $\big((0,\eta),\, \mathcal{B}((0,\eta))\big)$, such that for all $\theta \in [0, 2\pi)$ and $0 < r_1 < r_2 \le \eta$, we have $|D^2 g_\theta([r_1,r_2))| \le \mu\big([r_1,r_2)\big)$. Here we denote by $D^2 g_\theta$ the "second-derivative" measure of g_θ , i.e.,

$$D^2 g_\theta([r_1,r_2)) = D^- g_\theta(r_2) - D^- g_\theta(r_1) \qquad \forall\, 0 < r_1 < r_2 < \infty .$$

For this more general class of functions, we have the following extension of the FREIDLIN-SHEU-type change of variable formula developed in Theorem 2.13; its proof is in the Appendix, Section 6.

Theorem 5.3. *We let* $X(\cdot)$ *be a* WALSH *semimartingale with angular measure* $\boldsymbol{\nu}$ *, and recall the notation* $\Theta(\cdot) = \arg\big(X(\cdot)\big)$*. Then, for any function* $g \in \mathfrak{C}$ *as in Definition 5.2, the process* $g(X(\cdot))$ *is a continuous semimartingale, and satisfies the* FREIDLIN-SHEU*-type identity*

$$g\big(X(\cdot)\big) = g\big(X(0)\big) + \int_0^\cdot \mathbf{1}_{\{X(t)\neq \mathbf{0}\}}\, D^- g_{\Theta(t)}\big(\|X(t)\|\big)\, \mathrm{d}\|X(t)\| \tag{5.5}$$

$$+ \sum_{\theta\in[0,2\pi)} \int_0^\cdot \int_0^\infty \mathbf{1}_{\{X(t)\neq\mathbf{0},\ \Theta(t)=\theta\}}\, D^2 g_\theta(\mathrm{d}r)\, L^{\|X\|}(\mathrm{d}t, r) + \Big(\int_0^{2\pi} D^+ g_\theta(0)\, \boldsymbol{\nu}(\mathrm{d}\theta)\Big)\, L^{\|X\|}(\cdot) .$$

Furthermore, for any function f *as in Lemma 2.10, we have*

$$\int_0^\cdot f(X(t))\, \mathrm{d}\langle \|X\|\rangle(t) = 2 \sum_{\theta\in[0,2\pi)} \int_0^\cdot \int_0^\infty \mathbf{1}_{\{X(t)\neq\mathbf{0},\ \Theta(t)=\theta\}}\, f(r,\theta)\, \mathrm{d}r\, L^{\|X\|}(\mathrm{d}t, r) . \tag{5.6}$$

Remark 5.4. Quite clearly, we may also extend the class $\mathfrak{D}_I$ of functions in Definition 3.9 in the same way, and denote the resulting extended class by $\mathfrak{C}_I$. Then it is also easy to write down a version of the above change-of-variable formula for WALSH semimartingales with values in I, and functions in $\mathfrak{C}_I$. In particular, this stochastic calculus works for any function $g \in \mathfrak{C}_B$ and process $X(\cdot) \in \mathcal{A}(x)$ with $x \in B$.

The summation in (5.5) makes sense, because the summand is nonzero only for countably many θ's; indeed, $\Theta(\cdot)$ is constant on each excursion interval of $\|X(\cdot)\|$ away from the origin, and on each generic path there are at most countably many such intervals.

5.2 Optimal Stopping of a WALSH Diffusion on the Unit Disc

Let $X(\cdot)$ be a WALSH diffusion with values in the unit disc B of (5.1), associated with some given triple $(\boldsymbol{b}, \boldsymbol{s}, \boldsymbol{\nu})$, where the functions $\boldsymbol{b} : \check{B} \to \mathbb{R}$ and $\boldsymbol{s} : \check{B} \to \mathbb{R}$ satisfy Condition 3.10 with $\ell(\theta) \equiv 1$. We recall the radial scale function of (3.10), and assume

$$p_\theta(1-) < \infty\,, \qquad \forall\ \theta \in [0, 2\pi)\,. \tag{5.7}$$

Considering the same function U as in Problem 5.1, we define the value function of the *optimal stopping problem* for $X(\cdot)$ by

$$Q(x) := \sup_{\tau \in \mathcal{J}_X} \mathbb{E}^x\big[U(X(\tau))\big]\,, \quad x \in B\,. \tag{5.8}$$

We are using here the superscript x for the starting position, as in Section 4; we note that there is no superscript in (5.4), as the starting point x is implied through the requirement $X(\cdot) \in \mathcal{A}(x)$. This is a pure optimal stopping problem for the WALSH diffusion process $X(\cdot)$, without any element of control.

In the standard theory of optimal stopping for one-dimensional diffusions on a finite interval, the value function is given by the smallest $\mathcal{S}-$concave majorant of the reward function, where $\mathcal{S}$ is the scale function of the one-dimensional diffusion under consideration. We recall that *a function is said to be $\mathcal{S}-$concave, if and only if it is a concave function of $\mathcal{S}$*. This $\mathcal{S}-$concavity is the precise characterization of all excessive functions for a one-dimensional diffusion; those functions turn the diffusion into a (local) supermartingale. We refer to the works [5], [3] and the references cited there, for treatments of the optimal stopping problem in the context of one-dimensional diffusions, and for some properties of $\mathcal{S}-$concave functions.

For a given WALSH diffusion $X(\cdot)$, a natural guess from the change-of-variable formula of Theorem 5.3, is that an excessive function g for $X(\cdot)$ should have for every θ the $p_\theta-$concavity property along the ray of angle θ, and satisfy the additional requirement

$$\int_0^{2\pi} D^+ g_\theta(0)\, \boldsymbol{\nu}(\mathrm{d}\theta) \ \leq\ 0\,. \tag{5.9}$$

This requirement ensures the supermartingale property of $g(X(\cdot))$, when $X(\cdot)$ passes through the origin.

Condition (5.9) was considered also in [9], where a characterization of all excessive functions for a WALSH Brownian motion was obtained. In the more general setting of a WALSH diffusion as considered here, we cannot obtain such a characterization, due to the angular dependence in the drift and dispersion characteristics that prevents the use of one-dimensional excursion theory. We can, however, use the above idea to describe precisely the value function Q of the pure optimal stopping problem in (5.8), with the help of the FREIDLIN-SHEU-type change-of-variable formula in Theorem 5.3.

Definition 5.5. Concavity: A function $g : \overline{B} \to \mathbb{R}$ is said to be *$p-$concave with angular measure $\boldsymbol{\nu}$*, if
(i) for every $\theta \in [0, 2\pi)$, the function $r \mapsto g_\theta(r)$ is $p_\theta-$concave, i.e., $g_\theta(r) = \widetilde{g}_\theta\big(p_\theta(r)\big)$, $r \in [0, 1]$ holds for some concave function $\widetilde{g}_\theta : [0, p_\theta(1-)] \to \mathbb{R}$, and
(ii) the condition (5.9) is satisfied.

Definition 5.6. Pencil of Least Concave Majorants: For the reward function U of Problem 5.1, and for every constant $c \geq U(\mathbf{0})$, we define the function $U^{(c)} : \overline{B} \to \mathbb{R}$ via

$$U^{(c)}(r, \theta) \equiv U^{(c)}_\theta(r) := \inf\big\{\varphi(r) :\ \varphi(\cdot) \geq U_\theta(\cdot),\ \ \varphi : [0, 1] \to \mathbb{R} \text{ is } p_\theta\text{−concave, } \varphi(0) \geq c\big\}. \tag{5.10}$$

The objects introduced in Definition 5.6 will be seen in Theorem 5.8 to provide the crucial link between the problem of finding the smallest $p-$concave majorant of U with angular measure $\boldsymbol{\nu}$, and the analogous problem along each ray.

The following result provides some useful properties of the function $U^{(c)}$ in (5.10); its proof is in the Appendix, Section 6. Analogues of statement (ii) in Proposition 5.7 have been considered already; see Section III.7 of [5], Section 4 of [20], and Section 3 of [16].

Proposition 5.7. **(i)** *For every real constant* $c \geq U(\mathbf{0})$ *, the function* $U^{(c)}$ *of (5.10) is continuous in the tree topology and satisfies* $U^{(c)}(\mathbf{0}) = c$ *, as well as* $U^{(c)}(1,\theta) = U(1,\theta)$ *for all* $\theta \in [0, 2\pi)$ *.*

(ii) *Whenever* $U_\theta^{(c)}(r) > U_\theta(r)$ *holds for some* θ *and for all* r *in some interval* $(r_1, r_2) \subset [0,1]$ *, the mapping* $r \mapsto U_\theta^{(c)}(r)$ *is an affine transformation of* $r \mapsto p_\theta(r)$ *on* $[r_1, r_2]$ *.*

(iii) *If* $c > U(\mathbf{0})$ *, then the function* $U^{(c)}\big|_B$ *belongs to the class* $\mathfrak{C}_B$ *(cf. Definition 5.2 and Remark 5.4).*

(iv) *The function* $\Phi : [U(\mathbf{0}), \infty) \to \mathbb{R} \bigcup \{\infty\}$ *below is well-defined, continuous, and strictly decreasing:*

$$\Phi(c) := \int_0^{2\pi} D^+ U_\theta^{(c)}(0)\, \boldsymbol{\nu}(\mathrm{d}\theta). \tag{5.11}$$

We have the following crucial result, regarding the problem of optimal stopping in (5.8).

Theorem 5.8. *In the context specified at the beginning of this subsection, the value function* $Q : \overline{B} \to \mathbb{R}$ *of the optimal stopping problem defined as in (5.8) and with* $Q(x) := U(x)$ *for* $\|x\| = 1$ *, is continuous in the tree-topology.*

(i) This function Q *is the smallest* $p-$*concave majorant of* U *with angular measure* $\boldsymbol{\nu}$ *; in particular,* Q *itself is* $p-$*concave with angular measure* $\boldsymbol{\nu}$ *, and can be written as the lower envelope*

$$Q(x) \;=\; \inf\big\{ g(x) :\; g(\cdot) \geq U(\cdot),\;\; g : \overline{B} \to \mathbb{R} \;\text{ is } p\text{-concave with angular measure } \boldsymbol{\nu} \big\} \tag{5.12}$$

of all such functions that dominate U*. Moreover, the stopping time*

$$\tau_\star := \inf\big\{ t \geq 0 :\; U\big(X(t)\big) = Q\big(X(t)\big) \big\} \tag{5.13}$$

belongs to the class $\mathcal{J}_X$ *, and attains the supremum in (5.8).*

(ii) The function Q *can also be cast in the form*

$$Q(x) \;=\; U^{(c_0)}(x)\,, \qquad \text{with} \quad c_0 := \inf\big\{ c \geq U(\mathbf{0}) :\; \Phi(c) \leq 0 \big\}\,; \tag{5.14}$$

here $U^{(c_0)}$ *is as in Definition 5.6, and* Φ *is given by (5.11). Moreover, if* $Q(\mathbf{0}) = c_0 > U(\mathbf{0})$ *, then* Q *has "no concavity at the origin", in the sense that*

$$\int_0^{2\pi} D^+ Q_\theta(0)\, \boldsymbol{\nu}(\mathrm{d}\theta) \;=\; 0\,. \tag{5.15}$$

Remark 5.9. The property (5.15) is the counterpart at the origin, of the property in Proposition 5.7(ii). Taken together, these two properties ensure that the process $Q(X(\cdot))$ "is a martingale before entering the stopping region" $\{x \in \overline{B} \,:\, U(x) = Q(x)\}$; to wit, that $Q\big(X(\cdot \wedge \tau_\star)\big)$ is a martingale. On the other hand, the $p-$concavity with angular measure $\boldsymbol{\nu}$ of the function Q, implies that $Q(X(\cdot))$ is a supermartingale.

Proof of Theorem 5.8: We shall show first that the representations (5.12) and (5.14) are equivalent; then that (5.14) holds, and the stopping time of (5.13) attains the supremum in (5.8). The remaining claims will follow directly from (5.14) and Proposition 5.7.

• From Proposition 5.7, it is clear that the function $U^{(c_0)}$ is $p-$concave with angular measure $\boldsymbol{\nu}$. On the other hand, taking any function $g : \overline{B} \to \mathbb{R}$ that is $p-$concave with angular measure $\boldsymbol{\nu}$ and dominates U, we have $g(\mathbf{0}) = U^{(g(\mathbf{0}))}(\mathbf{0})$ and $g(\cdot) \geq U^{(g(\mathbf{0}))}(\cdot)$, therefore

$$\Phi(g(\mathbf{0})) = \int_0^{2\pi} D^+ U_\theta^{(g(\mathbf{0}))}(0)\, \boldsymbol{\nu}(\mathrm{d}\theta) \leq \int_0^{2\pi} D^+ g_\theta(0)\, \boldsymbol{\nu}(\mathrm{d}\theta) \leq 0\,.$$

It follows that $g(\mathbf{0}) \geq c_0$, and consequently $g(\cdot) \geq U^{(g(\mathbf{0}))}(\cdot) \geq U^{(c_0)}(\cdot)$. We have thus shown that (5.12) and (5.14) are equivalent.

• Next, we show that $U^{(c_0)}(x) = Q(x)$. The main idea lies in the following claim.

Claim 5.10. *The process* $U^{(c_0)}\big(X(\cdot)\big)$ *is a bounded supermartingale; moreover, with*

$$\widetilde{\tau}_\star := \inf\big\{t \geq 0 : U\big(X(t)\big) = U^{(c_0)}\big(X(t)\big)\big\},$$

the stopped process $U^{(c_0)}\big(X(\cdot \wedge \widetilde{\tau}_\star)\big)$ *is a bounded martingale.*

Proof. **(A)** *We assume* $c_0 > U(\mathbf{0})$ *first.* Then $\int_0^{2\pi} D^+ U_\theta^{(c_0)}(0)\,\boldsymbol{\nu}(\mathrm{d}\theta) = 0$ and $U^{(c_0)} \in \mathfrak{C}_B$ hold, thanks to Proposition 5.7 (iii), (iv). We recall the explosion time $S := \inf\{t : \|X(t)\| = 1\}$, and consider the stopping times $S_n := \inf\{t : \|X(t)\| = 1 - (1/n)\}$, $n \in \mathbb{N}$. Now Theorem 5.3 (both (5.5), (5.6)) gives

$$U^{(c_0)}\big(X(\cdot \wedge S_n)\big) = U^{(c_0)}\big(X(0)\big) + \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} D^- U^{(c_0)}_{\Theta(t)}\big(\|X(t)\|\big)\,\big[\boldsymbol{b}\big(X(t)\big)\,\mathrm{d}t + \boldsymbol{s}\big(X(t)\big)\,\mathrm{d}W(t)\big]$$

$$+ \sum_{\theta \in [0,2\pi)} \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t) \neq \mathbf{0},\ \Theta(t) = \theta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r)\, D^2 U_\theta^{(c_0)}(\mathrm{d}r) + \Big(\int_0^{2\pi} D^+ U_\theta^{(c_0)}(0)\,\boldsymbol{\nu}(\mathrm{d}\theta)\Big) L^{\|X\|}(\cdot)$$

$$= U^{(c_0)}\big(X(0)\big) + \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} D^- U^{(c_0)}_{\Theta(t)}\big(\|X(t)\|\big)\, \boldsymbol{s}\big(X(t)\big)\,\mathrm{d}W(t)$$

$$+ \sum_{\theta \in [0,2\pi)} \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t) \neq \mathbf{0},\ \Theta(t) = \theta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r) \left[D^- U_\theta^{(c_0)}(r) \frac{2\boldsymbol{b}(r,\theta)}{\boldsymbol{s}^2(r,\theta)}\mathrm{d}r + D^2 U_\theta^{(c_0)}(\mathrm{d}r)\right]. \tag{5.16}$$

Now let us assume that the function $U_\theta^{(c_0)}$ is of the form $U_\theta^{(c_0)}(\cdot) = \widetilde{U}_\theta^{(c_0)}\big(p_\theta(\cdot)\big)$, with $\widetilde{U}_\theta^{(c_0)} : [0, p_\theta(1-)] \to \mathbb{R}$ concave. We have then

$$\left[D^- U_\theta^{(c_0)}(r) \frac{2\boldsymbol{b}(r,\theta)}{\boldsymbol{s}^2(r,\theta)}\mathrm{d}r + D^2 U_\theta^{(c_0)}(\mathrm{d}r)\right] = D^- \widetilde{U}_\theta^{(c_0)}(p_\theta(r)) p'_\theta(r) \frac{2\boldsymbol{b}(r,\theta)}{\boldsymbol{s}^2(r,\theta)}\mathrm{d}r + \mathrm{d}\big(D^- \widetilde{U}_\theta^{(c_0)}(p_\theta(r)) p'_\theta(r)\big)$$

$$= -D^- \widetilde{U}_\theta^{(c_0)}(p_\theta(r)) p''_\theta(r)\mathrm{d}r + D^- \widetilde{U}_\theta^{(c_0)}(p_\theta(r)) p''_\theta(r)\mathrm{d}r + p'_\theta(r)\,\mathrm{d}\big(D^- \widetilde{U}_\theta^{(c_0)}(p_\theta(r))\big)$$

$$= p'_\theta(r)\,\mathrm{d}\big(D^- \widetilde{U}_\theta^{(c_0)}(p_\theta(r))\big). \tag{5.17}$$

The last expression is nonpositive, since $\widetilde{U}_\theta^{(c_0)}$ is concave; yet it vanishes near r if $U_\theta^{(c_0)}(r) > U_\theta(r)$, thanks to Proposition 5.7(ii). On the other hand, if $U_\theta^{(c_0)}(r) = U_\theta(r)$, then by the definition of $\widetilde{\tau}_\star$ and the nature of local times, the process $L^{\|X\|}(\cdot \wedge \widetilde{\tau}_\star, r)$ does not increase when $X(\cdot)$ is on the ray with angle θ.

Putting these observations together, we see that the right-most side in (5.16) is a local supermartingale; and that if we stop this process at time $\widetilde{\tau}_\star$, we get a local martingale. As it is clear that the function $U^{(c_0)}$ is bounded, we let $n \to \infty$ and obtain the claim.

(B) *Now we consider the case* $c_0 = U(\mathbf{0})$. Then $\Phi(c_0) \leq 0$ holds, and therefore also does $\Phi(c) < 0$ for any $c > c_0$. Thus for any $c > c_0$, we apply Theorem 5.3 as above and show that $U^{(c)}\big(X(\cdot)\big)$ is a bounded supermartingale. Since

$$U^{(c_0)}(\cdot) \leq U^{(c)}(\cdot) \leq U^{(c_0)}(\cdot) + c - c_0$$

(clearly, $U_\theta^{(c_0)}(\cdot) + (c - c_0)$ is $p_\theta-$concave and dominates U), we let $c \downarrow c_0$ and obtain that the process $U^{(c_0)}\big(X(\cdot)\big)$ is a bounded supermartingale.

On the other hand, since $c_0 = U(\mathbf{0})$, the process $X(\cdot \wedge \widetilde{\tau}_\star)$ stops at the origin once it finds itself there; so it never changes the ray it is on, and $L^{\|X\|}(\cdot \wedge \widetilde{\tau}_\star) \equiv 0$. Thus, the one-dimensional generalized Itô rule shows that $U^{(c)}\big(X(\cdot \wedge \widetilde{\tau}_\star)\big)$ is a bounded (local) martingale, following the same idea as above. □

From the Claim 5.10, and for any stopping time $\tau \in \mathcal{J}_X$, we have

$$U^{(c_0)}(x) \geq \mathbb{E}^x\big[U^{(c_0)}\big(X(\tau)\big)\big] \geq \mathbb{E}^x\big[U\big(X(\tau)\big)\big]\,.$$

Furthermore, $U^{(c_0)}(x) = \mathbb{E}^x\big[U^{(c_0)}\big(X(\widetilde{\tau}_\star)\big)\big] = \mathbb{E}^x\big[U\big(X(\widetilde{\tau}_\star)\big)\big]$, where the last equality holds because $X(S) \in \partial B$ and $U^{(c_0)}(\cdot) = U(\cdot)$ on ∂B. These facts come from (5.7), Theorem 4.5, and Proposition 5.7(i). We conclude $U^{(c_0)}(x) = Q(x)$, and the stopping time $\widetilde{\tau}_\star$ $(= \tau_\star)$ is optimal.

The proof of Theorem 5.8 is complete. □

5.3 Solution to the Problem of Optimal Stochastic Control with Discretionary Stopping

Now we go back to the context of Subsection 5.1, and deal with Problem 5.1 of stochastic control with discretionary stopping. We shall provide a characterization of the value function of this problem, as well as an explicit description of a control strategy and of a stopping time, that attain the supremum in (5.4).

We start by introducing, for every $\theta \in [0, 2\pi)$, the maximum of the reward function U on the corresponding ray:

$$U^*_\theta := \max_{0 \leq r \leq 1} U_\theta(r)\,, \tag{5.18}$$

as well as the left-most and right-most locations where this maximum is attained, namely

$$\lambda_\theta := \inf\{r \in [0,1] : U_\theta(r) = U^*_\theta\}\,, \qquad \varrho_\theta := \sup\{r \in [0,1] : U_\theta(r) = U^*_\theta\}\,. \tag{5.19}$$

We assume that there are two pairs $(\boldsymbol{b}_0, \boldsymbol{s}_0)$, $(\boldsymbol{b}_1, \boldsymbol{s}_1)$ of BOREL-measurable functions on $\check{B}$, which

(i) satisfy Condition 3.10 with $\ell(\theta) \equiv 1$, and whose corresponding radial scale functions satisfy (5.7), and

(ii) are such that $(\boldsymbol{b}_i(x), \boldsymbol{s}_i(x)) \in \mathcal{K}(x)$ holds for all $x \in \check{B}$ and $i = 0, 1$, and

$$\frac{\boldsymbol{b}_0(x)}{\boldsymbol{s}_0^2(x)} = \inf\Big\{\frac{\beta}{\sigma^2} : (\beta, \sigma) \in \mathcal{K}(x)\Big\}, \qquad \frac{\boldsymbol{b}_1(x)}{\boldsymbol{s}_1^2(x)} = \sup\Big\{\frac{\beta}{\sigma^2} : (\beta, \sigma) \in \mathcal{K}(x)\Big\}. \tag{5.20}$$

Definition 5.11. Candidate Optimal Control Strategies: For every real constant $c \geq U(\mathbf{0})$, we consider a pair $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)})$ of BOREL-measurable functions on $\check{B}$, which:

(i) Satisfies $(\boldsymbol{b}^{(c)}(x), \boldsymbol{s}^{(c)}(x)) \in \mathcal{K}(x)$ for all $x \in \check{B}$ and Condition 3.10 with $\ell(\theta) \equiv 1$, and the corresponding radial scale functions satisfy (5.7).

(ii) For $\theta \in [0, 2\pi)$ with $U^*_\theta < c$, satisfies $\big(\boldsymbol{b}^{(c)}(r,\theta), \boldsymbol{s}^{(c)}(r,\theta)\big) = \big(\boldsymbol{b}_0(r,\theta), \boldsymbol{s}_0(r,\theta)\big)$ for all $r \in (0,1)$.

(iii) For $\theta \in [0, 2\pi)$ with $U^*_\theta \geq c$, satisfies $\big(\boldsymbol{b}^{(c)}(r,\theta), \boldsymbol{s}^{(c)}(r,\theta)\big) = \big(\boldsymbol{b}_0(r,\theta), \boldsymbol{s}_0(r,\theta)\big)$ for all $r \in (\varrho_\theta, 1)$.

(iv) For $\theta \in [0, 2\pi)$ with $U^*_\theta > c$, satisfies $\big(\boldsymbol{b}^{(c)}(r,\theta), \boldsymbol{s}^{(c)}(r,\theta)\big) = \big(\boldsymbol{b}_1(r,\theta), \boldsymbol{s}_1(r,\theta)\big)$ for all $r \in (0, \lambda_\theta)$.

Remark 5.12. Every real number $c \geq U(\mathbf{0})$ can be seen as a "tentative guess" for the value $V(\mathbf{0})$. If indeed $c = V(\mathbf{0})$, then the optimal control strategy is to take $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)})$ as coefficients, and optimally stop the resulting WALSH diffusion; the value function $Q^{(c)}$ for this optimal stopping problem should then be equal to c at the origin. The way we choose $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)})$ in Definition 5.11 is inspired by [15].

In conjunction with the previous section, we see that if $V(\mathbf{0}) = c$, then the function $U^{(c,p^{(c)})}$ defined as below is the value function of the optimal stopping problem for the WALSH diffusion associated with $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)}, \boldsymbol{\nu})$ and reward function U. This property will help us find $V(\mathbf{0})$.

Definition 5.13. For every real constant $c \geq U(\mathbf{0})$, we define the function $U^{(c,p^{(c)})} : \overline{B} \to \mathbb{R}$ as $U^{(c,p^{(c)})}(r,\theta) \equiv U^{(c,p^{(c)})}_\theta(r)$, where

$$U^{(c,p^{(c)})}_\theta(r) := \inf\Big\{\varphi(r) : \varphi(\cdot) \geq U_\theta(\cdot), \quad \varphi : [0,1] \to \mathbb{R} \text{ is } p^{(c)}_\theta\text{−concave with } \varphi(0) \geq c\Big\}. \tag{5.21}$$

Here $p^{(c)}$ is the radial scale function that corresponds, via (3.10), to the pair of functions $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)})$ in Definition 5.11.

Remark 5.14. In Definition 5.11, we did not specify the values $(\boldsymbol{b}^{(c)}(r,\theta), \boldsymbol{s}^{(c)}(r,\theta))$ in the case $U^*_\theta = c$ and $r \in (0, \varrho_\theta]$, or in the case $U^*_\theta > c$ and $r \in [\lambda_\theta, \varrho_\theta]$.

In these two cases, the values in question need only be chosen suitably, to make the resulting functions $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)})$ satisfy the property (i) of Definition 5.11. For example, $(\boldsymbol{b}_0(r,\theta), \boldsymbol{s}_0(r,\theta))$ and $(\boldsymbol{b}_1(r,\theta), \boldsymbol{s}_1(r,\theta))$ are two allowable choices. We note that this ambiguity does not carry over to the function $U^{(c,p^{(c)})}$.

Proposition 5.15. *For every real constant $c \geq U(\mathbf{0})$, the function $U^{(c,p^{(c)})}$ is uniquely determined, regardless of the ambiguity in the choice of $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)})$ in Definition 5.11.*

Moreover, for any given $\theta \in [0, 2\pi)$, the following hold:

(i) *If $c < U^*_\theta$, we have*

$$D^- U_\theta^{(c,p^{(c)})}(\cdot) \geq 0 \text{ on } (0, \lambda_\theta), \quad D^- U_\theta^{(c,p^{(c)})}(\cdot) \leq 0 \text{ on } (\varrho_\theta, 1), \text{ and } \quad U_\theta^{(c,p^{(c)})}(\cdot) = U^*_\theta \text{ on } [\lambda_\theta, \varrho_\theta].$$

(ii) *If $c = U^*_\theta$, we have*

$$D^- U_\theta^{(c,p^{(c)})}(\cdot) \leq 0 \text{ on } (\varrho_\theta, 1), \text{ and } U_\theta^{(c,p^{(c)})}(\cdot) = U^*_\theta \text{ on } [0, \varrho_\theta].$$

(iii) *If $c > U^*_\theta$, we have*

$$D^- U_\theta^{(c,p^{(c)})}(\cdot) \leq 0 \text{ on } (0, 1).$$

(iv) *With $\big(U^{(c)}, p\big)$ replaced by $\big(U^{(c,p^{(c)})}, p^{(c)}\big)$, the statements of Proposition 5.7 hold here as well.*

Proof. The proof of (i)-(iii) is elementary, using the definition of $U^{(c,p^{(c)})}$; see also the end of Section 3 of [15], where similar properties are considered.

Next, we show the non-ambiguity in the definition of the function $U^{(c,p^{(c)})}$ in (5.21). Let $(\boldsymbol{b}^{(c,1)}, \boldsymbol{s}^{(c,1)})$ and $(\boldsymbol{b}^{(c,2)}, \boldsymbol{s}^{(c,2)})$ be two choices of $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)})$, and $p^{(c,1)}$ and $p^{(c,2)}$ the corresponding radial scale functions. Fix a ray with angle θ. If $U^*_\theta < c$, there is no ambiguity in $(\boldsymbol{b}^{(c)}, \boldsymbol{s}^{(c)})$, and therefore in $U^{(c,p^{(c)})}$, on this ray. If $U^*_\theta = c$, then $U_\theta^{(c,p^{(c)})} = U^*_\theta$ on $[0, \varrho_\theta]$, and it follows that

$$U_\theta^{(c,p^{(c)})}(r) := \inf \Big\{ \varphi(r) : \varphi(\cdot) \geq U_\theta(\cdot), \;\; \varphi : [\varrho_\theta, 1] \to \mathbb{R} \text{ is } p_\theta^{(c)}\text{−concave} \Big\}, \qquad r \in [\varrho_\theta, 1]. \tag{5.22}$$

But when restricted to $[\varrho_\theta, 1]$, we have $\big(\boldsymbol{b}^{(c,1)}(\cdot\,,\theta), \boldsymbol{s}^{(c,1)}(\cdot\,,\theta)\big) = \big(\boldsymbol{b}^{(c,2)}(\cdot\,,\theta), \boldsymbol{s}^{(c,2)}(\cdot\,,\theta)\big)$, and therefore the functions $p_\theta^{(c,1)}(\cdot)$, $p_\theta^{(c,2)}(\cdot)$ are affine transformations of each other. Hence, the two choices $p^{(c)} = p^{(c,1)}$ and $p^{(c)} = p^{(c,2)}$ in (5.22) lead to the same result. The case $U^*_\theta > c$ is dealt with similarly.

Finally, we address (iv). It is easy to see that Proposition 5.7 carries over to the present context essentially unchanged, except for the assertion that the mapping $c \mapsto \int_0^{2\pi} D^+ U_\theta^{(c,p^{(c)})}(0)\, \boldsymbol{\nu}(\mathrm{d}\theta)$ is continuous and strictly decreasing. For this assertion it is enough to show that the mapping $c \mapsto D^+ U_\theta^{(c,p^{(c)})}(0)$ is continuous and strictly decreasing, given any $\theta \in [0, 2\pi)$. We now observe that we have the freedom to choose

$$\big(\boldsymbol{b}^{(U^*_\theta)}(\cdot,\theta), \boldsymbol{s}^{(U^*_\theta)}(\cdot,\theta)\big) \; = \; \big(\boldsymbol{b}_0(\cdot,\theta), \boldsymbol{s}_0(\cdot,\theta)\big),$$

so that $(\boldsymbol{b}^{(c)}(\cdot,\theta), \boldsymbol{s}^{(c)}(\cdot,\theta))$ is the same for all $c \in [U^*_\theta, \infty)$. Then the proof of Proposition 5.7 (iv) yields that the mapping $c \mapsto D^+ U_\theta^{(c,p^{(c)})}(0)$ is continuous and strictly decreasing on $[U^*_\theta, \infty)$. The argument is similar for $c \in [U(\mathbf{0}), U^*_\theta]$. ☐

We can state now and prove the following fundamental result, regarding the optimal control problem with discretionary stopping for WALSH semimartingales of the present section. We regard the fact, that such a problem can be shown to admit a very explicit solution, as testament to the power of the stochastic calculus developed in the present paper.

Theorem 5.16. *With the previous assumptions and notation, the value function V of the control problem with discretionary stopping in (5.4), is given by*

$$V(x) \,=\, U^{(c_*,p^{(c_*)})}(x)\,, \qquad c_* := \,\inf \left\{ c \geq U(\mathbf{0}) : \int_0^{2\pi} D^+ U_\theta^{(c,p^{(c)})}(0)\, \boldsymbol{\nu}(\mathrm{d}\theta) \leq 0 \right\}, \tag{5.23}$$

and therefore satisfies $V(\mathbf{0}) = c_*$.

The supremum in (5.4) is attained by the Walsh *diffusion $X^*(\cdot)$ associated with the triple $\big(\boldsymbol{b}^{(c_*)}, \boldsymbol{s}^{(c_*)}, \boldsymbol{\nu}\big)$, and the corresponding stopping time*

$$\tau_* := \,\inf \big\{ t \geq 0 : \, U\big(X^*(t)\big) = V\big(X^*(t)\big) \big\}. \tag{5.24}$$

Remark 5.17. *On Interpretation:* In conjunction with Definition 5.11 this result states that, before entering the stopping region $\{x \in \overline{B} : \, U(x) = V(x)\}$, it is optimal to control the state process $X(\cdot)$ thus:

(i) Along any ray of angle θ with $U_\theta^* > V(\mathbf{0})$: maximize the “signal-to-noise” ratio β/σ^2 on the interval $(0, \lambda_\theta)$; minimize the “signal-to-noise” ratio β/σ^2 on the interval $(\rho_\theta, 1)$; and follow on the interval $[\lambda_\theta, \varrho_\theta]$ any strategy that will bring the process $X(\cdot)$ to one of its endpoints.

(ii) Along any ray of angle θ with $U_\theta^* = V(\mathbf{0})$: minimize the “signal-to-noise” ratio β/σ^2 on the interval $(\varrho_\theta, 1)$, and follow on the interval $(0, \rho_\theta]$ any strategy that will bring the process $X(\cdot)$ to one of its endpoints.

(iii) Along any ray of angle θ with $U_\theta^* < V(\mathbf{0})$: minimize the “signal-to-noise” ratio β/σ^2.

Since the function V is obtained via (5.23), the above strategy can indeed be implemented.

Proof. We first show that $U^{(c_*,\,p^{(c_*)})}(x) \geq V(x)$. Let us fix a starting point $x \in B$, pick up an arbitrary process $X(\cdot) \in \mathcal{A}(x)$, a stopping time $\tau \in \mathcal{J}_X$, and recall the dynamics of (5.2). We claim that we have

$$U^{(c_*,\,p^{(c_*)})}(x) \,\geq\, \mathbb{E}\Big[U^{(c_*,\,p^{(c_*)})}\big(X(\tau)\big)\Big]\,. \tag{5.25}$$

This implies $U^{(c_*,p^{(c_*)})}(x) \geq \mathbb{E}\big[U\big(X(\tau)\big)\big]$ for all $X(\cdot) \in \mathcal{A}(x),\ \tau \in \mathcal{J}_X$, thus also $U^{(c_*,p^{(c_*)})}(x) \geq V(x)$.

• Now we establish the claim (5.25). Assume first that $c_* > U(\mathbf{0})$. Proposition 5.15 (iv) gives then $\int_0^{2\pi} D^+ U_\theta^{(c_*,p^{(c_*)})}(0)\, \boldsymbol{\nu}(\mathrm{d}\theta) = 0$ and $U^{(c_*,p^{(c_*)})} \in \mathfrak{C}_B$. In the same manner as in the derivation of (5.16), (5.17), and recalling the stopping times S, S_n given there, we obtain here

$$\begin{aligned}
U^{(c_*,p^{(c_*)})}\big(X(\cdot \wedge S_n)\big) &= U^{(c_*,p^{(c_*)})}(x) + \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t)\neq \mathbf{0}\}}\, D^- U_{\Theta(t)}^{(c_*,p^{(c_*)})}\big(\|X(t)\|\big) \left[\beta(t)\, \mathrm{d}t + \sigma(t)\, \mathrm{d}W(t)\right] \\
&\quad + \sum_{\theta \in [0,2\pi)} \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t)\neq \mathbf{0},\ \Theta(t)=\theta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r)\, D^2 U_\theta^{(c_*,p^{(c_*)})}(\mathrm{d}r) \\
&\leq U^{(c_*,p^{(c_*)})}(x) + \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t)\neq \mathbf{0}\}} D^- U_{\Theta(t)}^{(c_*,p^{(c_*)})}\big(\|X(t)\|\big) \left[\frac{\boldsymbol{b}^{(c_*)}\big(X(t)\big)}{(\boldsymbol{s}^{(c_*)})^2\big(X(t)\big)} \cdot \sigma^2(t)\, \mathrm{d}t + \sigma(t)\, \mathrm{d}W(t)\right] \\
&\quad + \sum_{\theta \in [0,2\pi)} \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t)\neq \mathbf{0},\ \Theta(t)=\theta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r)\, D^2 U_\theta^{(c_*,p^{(c_*)})}(\mathrm{d}r) \\
&= U^{(c_*,p^{(c_*)})}(x) + \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t)\neq \mathbf{0}\}} D^- U_{\Theta(t)}^{(c_*,p^{(c_*)})}\big(\|X(t)\|\big)\, \sigma(t)\, \mathrm{d}W(t) \\
&\quad + \sum_{\theta \in [0,2\pi)} \int_0^{\cdot \wedge S_n} \mathbf{1}_{\{X(t)\neq \mathbf{0},\ \Theta(t)=\theta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r)\, (p_\theta^{(c_*)})'(r)\, \mathrm{d}\big(D^- \widetilde{U}_\theta^{(c_*,p^{(c_*)})}(p_\theta(r))\big)\,,
\end{aligned} \tag{5.26}$$

where $\widetilde{U}_\theta^{(c_*, p^{(c_*)})} : [0, p_\theta(1-)] \to \mathbb{R}$ is concave, and such that $U_\theta^{(c_*, p^{(c_*)})}(\cdot) = \widetilde{U}_\theta^{(c_*, p^{(c_*)})}\big(p_\theta(\cdot)\big)$.
We have used Definition 5.11 and Proposition 5.15 (i)-(iii) for the above inequality; namely, we observe

$$D^- U_{\Theta(t)}^{(c_*, p^{(c_*)})}\big(\|X(t)\|\big) \ > (<) \ 0 \quad \Longrightarrow \quad \frac{\boldsymbol{b}^{(c_*)}\big(X(t)\big)}{\big(\boldsymbol{s}^{(c_*)}\big)^2\big(X(t)\big)} \ \geq (\leq) \ \frac{\beta(t)}{\sigma^2(t)} .$$

The claim for the case $c_* > U(\mathbf{0})$ now follows readily from (5.26), by localization.

Next, we consider the case $c_* = U(\mathbf{0})$. Then we have $\int_0^{2\pi} D^+ U_\theta^{(c, p^{(c)})}(0)\, \boldsymbol{\nu}(\mathrm{d}\theta) < 0$ and $U^{(c, p^{(c)})} \in \mathfrak{C}_B$, for any $c > c_*$. Thus, similarly as above, we see that

$$U^{(c, p^{(c)})}(x) \ \geq \ \mathbb{E}\big[U^{(c, p^{(c)})}\big(X(\tau)\big)\big] .$$

On the strength of the following paragraph, we may let $c \downarrow c_*$ and obtain the claim in this case.

Fix $\theta \in [0, 2\pi)$. By making $(\boldsymbol{b}^{(c)}(\cdot\,, \theta), \boldsymbol{s}^{(c)}(\cdot\,, \theta))$ the same for all $c \geq U_\theta^*$ (cf. the proof of Proposition 5.15 (iv)), we note that there exists an $\varepsilon(\theta) > 0$ such that $p_\theta^{(c)}(\cdot)$ is the same for $c \in [c_*, c_* + \varepsilon(\theta)]$. Thus $U^{(c, p^{(c)})}(\cdot\,, \theta) \leq U^{(c_*, p^{(c_*)})}(\cdot\,, \theta) + c - c_*$ for $c \in [c_*, c_* + \varepsilon(\theta)]$.

• On the other hand, $U^{(c_*, p^{(c_*)})}(x) \leq V(x)$ follows from the fact that, by Theorem 5.8, $U^{(c_*, p^{(c_*)})}$ is the value function of the optimal stopping problem for the same reward function U and the WALSH diffusion $X^*(\cdot)$ associated with the triple $(\boldsymbol{b}^{(c_*)}, \boldsymbol{s}^{(c_*)}, \boldsymbol{\nu})$.

We conclude that $U^{(c_*, p^{(c_*)})}(x) = V(x)$; the other claims of the theorem follow then directly. ☐

5.4 The Underlying Dynamic Programming Equations

Combining the features of control and stopping (e.g. Theorems 3.6 and 4.5 in [21]), we may write informally the following HAMILTON-JACOBI-BELLMAN-type variational inequalities for the value function $(r, \theta) \mapsto V(r, \theta) = V_\theta(r)$, namely

$$\min \Big\{ - \sup_{(\beta, \sigma) \in \mathcal{K}(x)} \big\{ \beta\, DV_\theta(r) \ + \ \frac{1}{2}\, \sigma^2\, D^2 V_\theta(r) \big\}\,,\ V(x) - U(x) \Big\} \ = \ 0\,, \quad x = (r, \theta) \in B\,, \tag{5.27}$$

and

$$\min \Big\{ - \int_0^{2\pi} D^+ V_\theta(0)\, \boldsymbol{\nu}(\mathrm{d}\theta)\,,\ \ V(\mathbf{0}) - U(\mathbf{0}) \Big\} \ = \ 0\,. \tag{5.28}$$

The importance – and advantage – of the purely probabilistic approach we have developed, is that it obviates the need to give rigorous meaning to the above fully nonlinear variational inequality; it constructs, rather, the value function and the optimal control and stopping strategies of the problem *from first principles* and using educated guesses, in conjunction with the refined stochastic calculus developed in this paper.

6 Appendix: Proofs of Selected Results

PROOF OF PROPOSITION 3.5: Let $E_t := \big\{ \int_0^{t \wedge S} \mathbf{1}_{\{X(u) \neq \mathbf{0}\}}\, \boldsymbol{s}^2(X(u)) \mathrm{d}u \ = \infty \big\}$. Following the idea of the solutions to Problem 3.4.11 and Problem 5.5.3 in [14], we have $\underline{\lim}_{n \to \infty} \|X(t \wedge S_n)\| \ = \ 0$ and $\overline{\lim}_{n \to \infty} \|X(t \wedge S_n)\| = \infty$, a.e. on E_t. Thus $\mathbb{P}(E_t) = 0$ by the continuity of $X(\cdot)$ in the tree-topology.

Therefore, $\int_0^{t \wedge S} \mathbf{1}_{\{X(t) \neq \mathbf{0}\}}\, \boldsymbol{s}^2(X(u)) \mathrm{d}u < \infty$ holds a.e., and we obtain the existence in $\mathbb{R}^2$ of the limit $\lim_{n \to \infty} X(t \wedge S_n)$ in the tree-topology, in the same spirit as in the second-to-last paragraph in the proof of Proposition 3.4. Thus $X(t \wedge S)$ is valued in $\mathbb{R}^2$, a.e., for every $t \geq 0$, and consequently $S = \infty$ a.e. ☐

PROOF OF THEOREM 3.7: Omitting from the notation the underlying probability space, we begin with a standard one-dimensional Brownian motion $\{\widetilde{B}(s), \widetilde{\mathcal{G}}(s); 0 \leq s < \infty\}$ and an independent two-dimensional random variable ξ with distribution $\boldsymbol{\mu}$. Let $\{Z(\cdot), \mathcal{G}(\cdot)\}$ be a WALSH Brownian motion starting at $Z(0) = \xi$ and driven by the Brownian motion $B(\cdot) = \|\xi\| + \widetilde{B}(\cdot)$, with angular measure $\boldsymbol{\nu}$. This WALSH Brownian motion can be constructed as in the proof of Theorem 2.1 in [12] (even though in that proof the process starts at a nonrandom point, the same method applies to a random initial condition). Let

$$T(s) := \int_0^{s+} \frac{\mathbf{1}_{\{Z(u) \neq \mathbf{0}\}} \, \mathrm{d}u}{\boldsymbol{s}^2\big(Z(u)\big)}, \quad 0 \leq s < \infty, \qquad A(t) := \inf\big\{s \geq 0 : T(s) > t\big\}, \ 0 \leq t < \infty.$$

Lemma 6.1. *We have* $T(\infty) = \infty$*, a.s.*

Proof of Lemma 6.1: Consider the stopping times $\{\tau_k^\varepsilon\}_{k \in \mathbb{N}_{-1}}$ as in (2.6), with X replaced by Z. Since $Z(\cdot)$ is time-homogeneous strongly-Markovian (as a WALSH Brownian motion) and $Z(\tau_{2m}^\varepsilon) \equiv \mathbf{0}$, $\forall m \in \mathbb{N}_0$, we deduce that the random variables

$$\widehat{T}_m := \int_{\tau_{2m}^\varepsilon}^{\tau_{2m+2}^\varepsilon} \frac{\mathbf{1}_{\{Z(u) \neq \mathbf{0}\}} \, \mathrm{d}u}{\boldsymbol{s}^2(Z(u))}, \qquad m \in \mathbb{N}_0$$

are I.I.D and strictly positive. Therefore, we have $T(\infty) \geq \sum_{m \in \mathbb{N}_0} \widehat{T}_m = \infty$, a.e. □

We also note that $T(\cdot)$ is strictly increasing when it is finite, because $Z(\cdot)$ spends zero amount of time at the origin $\mathbf{0}$. Now it is easy to see that the analogue of relationships (5.10)-(5.14) at the beginning of Section 5.5.A in [14], as well as the discussions between them, all hold here as well. Define

$$R := \inf\big\{s \geq 0 : \ Z(s) \in \mathcal{I}(\boldsymbol{s})\big\}. \tag{6.1}$$

Lemma 6.2. *We have* $R = A(\infty)$*, a.s.*

Proof of Lemma 6.2: The proof of $R \leq A(\infty)$ follows as in the proof of Lemma 5.5.2 in [14], with the help of Condition 3.6, Lemma 2.10, and the tree-metric.

As for the reverse inequality $A(\infty) \leq R$, it suffices to prove it on the event $\{R \leq n\}$ for every $n \in \mathbb{N}$. We define the standard Brownian motion $B_n(\cdot) := B\big((R \wedge n) + \cdot\big) - B(R \wedge n)$, and the stopping time $\tau := \{s \geq 0 : B_n(s) \leq -\eta\}$. Then on the event $\{R \leq n\}$, we have for any $0 < s < \tau$ the comparison

$$\int_0^{R+s} \frac{\mathbf{1}_{\{Z(u) \neq \mathbf{0}\}} \, \mathrm{d}u}{\boldsymbol{s}^2\big(Z(u)\big)} \geq \int_R^{R+s} \frac{\mathbf{1}_{\{Z(u) \neq \mathbf{0}\}} \, \mathrm{d}u}{\boldsymbol{s}^2\big(Z(u)\big)} = \int_0^s \frac{\mathrm{d}u}{\boldsymbol{s}^2\big(\|Z(R)\| + B_n(u), \arg\big(Z(R)\big)\big)}.$$

The last equality comes here from the fact $Z(\cdot) \neq \mathbf{0}$ holds on the interval $[R, R + \tau)$, which is because $Z(R) \in \mathcal{I}(\boldsymbol{s}) \subseteq \{(r, \theta) : r \geq \eta, \ 0 \leq \theta < 2\pi\}$. It follows from Lemma 3.6.26 in [14] that the last integral above is infinite, thus $T(R) = \infty$ holds on $\{R \leq n\}$, and therefore $A(\infty) \leq R$ holds on $\{R \leq n\}$. □

• Now we adapt the proof of Theorem 5.5.4 in [14]; i.e., we shall show that, under the Condition 3.6, a WALSH Diffusion with state-space I associated with the triple $(\mathbf{0}, \boldsymbol{s}, \boldsymbol{\nu})$ exists, if and only if $\mathcal{I}(\boldsymbol{s}) \subseteq \mathcal{Z}(\boldsymbol{s})$.
(i) Let us first assume $\mathcal{I}(\boldsymbol{s}) \subseteq \mathcal{Z}(\boldsymbol{s})$ and define

$$X(t) := Z(A(t)), \qquad U(t) := B(A(t)), \qquad \mathcal{F}(t) := \mathcal{G}(A(t)), \qquad 0 \leq t < \infty. \tag{6.2}$$

It follows that $X(T(u)) = Z(u)$ for $u < A(\infty)$. Thus, for every $t \in [0, \infty)$ we have

$$\int_0^t \mathbf{1}_{\{X(v) = \mathbf{0}\}} \, \mathrm{d}v = \int_0^{A(t)} \mathbf{1}_{\{X(T(u)) = \mathbf{0}\}} \, \mathrm{d}T(u) = \int_0^{A(t)} \mathbf{1}_{\{Z(u) = \mathbf{0}\}} \, \mathrm{d}T(u) = 0, \tag{6.3}$$

verifying the first part of (3.3). Moreover, with (6.3) and all the previous preparations, we can proceed as in the proof of Theorem 5.5.4 in [14], and obtain that the process $U(\cdot) - \|\xi\|$ is a scalar local martingale with $\langle U\rangle(\cdot) = A(\cdot)$, as well as the representation

$$A(t) = \int_0^t \mathbf{1}_{\{X(v)\neq \mathbf{0}\}} \, \boldsymbol{s}^2(X(v))\,\mathrm{d}v\,, \qquad 0 \le t < \infty\,. \tag{6.4}$$

Then there exists a Brownian motion $W(\cdot)$ on a possibly extended probability space, with the property $U(t) = \|\xi\| + \int_0^t \mathbf{1}_{\{X(v)\neq\mathbf{0}\}}\, \boldsymbol{s}(X(v))\mathrm{d}W(v)\,,\ \ 0 \le t < \infty$.

Let us note that $\|Z(\cdot)\|$ is the SKOROKHOD reflection of $B(\cdot)$; thus the same relationship is true for $\|X(\cdot)\|$ and $U(\cdot)$ by (6.2), and so (3.1) gives

$$\|X(\cdot)\| = \|\xi\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}}\, \boldsymbol{s}\big(X(t)\big)\mathrm{d}W(t) + L^{\|X\|}(\cdot)\,. \tag{6.5}$$

Finally, the latter part of (3.3) and the continuity in the tree-topology for $X(\cdot)$ are both inherited from $Z(\cdot)$, as the proof of Proposition 3.4 illustrates. We have thus verified that the just constructed $X(\cdot)$ is a WALSH diffusion as described in the Theorem.

(ii) Conversely, let us assume the existence of the WALSH diffusion $X(\cdot)$ described in Theorem 3.7, with any given initial condition. Consider such a WALSH diffusion $X(\cdot)$ with $X(0) = x \in \mathcal{Z}(\boldsymbol{s})^c$ and the underlying Brownian motion $W(\cdot)$. We introduce the scalar local martingale

$$U(\cdot) := \|X(\cdot)\| - L^{\|X\|}(\cdot) = \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}}\, \boldsymbol{s}(X(t))\mathrm{d}W(t)\,. \tag{6.6}$$

Then $\|X(\cdot)\|$ is the SKOROKHOD reflection of $U(\cdot)$, and therefore $X(\cdot)$ is a WALSH semimartingale driven by $U(\cdot)$. By Proposition 3.4, there exists a WALSH Brownian motion $Z(\cdot)$ on a possibly extended probability space, such that $X(\cdot) = Z(\langle U\rangle(\cdot))$. We can follow the proof of Theorem 5.5.4 in [14] with $T(s) := \inf\{t \ge 0 : \langle U\rangle(t) > s\}$, first to derive that

$$\int_0^{s\wedge\langle U\rangle(\infty)} \frac{\mathbf{1}_{\{Z(u)\neq\mathbf{0}\}}\,\mathrm{d}u}{\boldsymbol{s}^2(Z(u))} = \int_0^{T(s)} \frac{\mathbf{1}_{\{X(v)\neq\mathbf{0}\}}\,\mathrm{d}\langle U\rangle(v)}{\boldsymbol{s}^2(X(v))} = \int_0^{T(s)} \mathbf{1}_{\{X(v)\neq\mathbf{0},\ \boldsymbol{s}(X(v))\neq 0\}}\mathrm{d}v \le T(s)$$

holds for all $0 \le s < \infty$, then to argue $\mathbb{P}\big(T(s) < \infty,\ \langle U\rangle(\infty) > 0\big) > 0$ for sufficiently small $s > 0$, and finally to show that $x \in \mathcal{I}(\boldsymbol{s})$ cannot hold. It follows that $\mathcal{I}(\boldsymbol{s}) \subseteq \mathcal{Z}(\boldsymbol{s})$.

• Next, we assume the validity of Condition 3.6 and $\mathcal{I}(\boldsymbol{s}) \subseteq \mathcal{Z}(\boldsymbol{s})$, and show that uniqueness in distribution is then equivalent to the condition $\mathcal{I}(\boldsymbol{s}) \supseteq \mathcal{Z}(\boldsymbol{s})$.

(i) First, we suppose that the inclusion $\mathcal{I}(\boldsymbol{s}) \supseteq \mathcal{Z}(\boldsymbol{s})$ does not hold. By picking a starting point $x \in \mathcal{Z}(\boldsymbol{s})\backslash\mathcal{I}(\boldsymbol{s})$, we see that uniqueness in distribution is violated for the WALSH diffusion described in Theorem 3.7 and starting at x, in the spirit of Remark 5.5.6 in [14].

(ii) Conversely, let us assume in addition that $\mathcal{I}(\boldsymbol{s}) \supseteq \mathcal{Z}(\boldsymbol{s})$ holds. Let $X(\cdot)$ be a WALSH diffusion described in Theorem 3.7 and with an arbitrarily given initial distribution $\boldsymbol{\mu}$. With $U(\cdot)$ as in (6.6), we can adapt the proof of Theorem 5.5.7 in [14] in a manner similar to what we did before, and obtain the existence of a WALSH Brownian motion $Z(\cdot)$ such that $X(\cdot) = Z(\langle U\rangle(\cdot))$ and

$$\langle U\rangle(t) \,=\, \inf\left\{ s \ge 0 : \int_0^{s+} \frac{\mathbf{1}_{\{Z(u)\neq\mathbf{0}\}}\mathrm{d}u}{\boldsymbol{s}^2(Z(u))} > t \right\}, \qquad 0 \le t < \infty\,. \tag{6.7}$$

It develops that the process $X(\cdot)$ can be expressed as a measurable functional of the WALSH Brownian motion $Z(\cdot)$, with initial distribution $\boldsymbol{\mu}$ and angular measure $\boldsymbol{\nu}$. Since this $Z(\cdot)$ has a uniquely determined probability distribution, thanks to Proposition 7.2 in [12] (again, this can be generalized from a nonrandom starting point to a random initial condition), we deduce the uniqueness of $X(\cdot)$ in distribution. $\square$

ON THE PROOF OF THEOREM 4.9: We need some preparation before proving Theorem 4.9. By analogy with Section 5.5.C in [14], we define a sequence $\{u_n\}_{n=0}^{\infty}$ of functions on I via $u_0 \equiv 1$ and

$$u_n(r,\theta) := \int_0^r p'_\theta(y) \int_0^y u_{n-1}(z,\theta)\boldsymbol{m}_\theta(\mathrm{d}z)\,, \qquad (r,\theta) \in I\,, \quad n \in \mathbb{N}, \tag{6.8}$$

recursively. Note that $u_1 \equiv v$. We have the following analogue of Lemma 5.5.26 in [14].

Lemma 6.3. *Under Condition 3.10, the series*

$$u(r,\theta) := \sum_{n=0}^{\infty} u_n(r,\theta)\,, \qquad (r,\theta) \in I \tag{6.9}$$

converges on I *and defines a function in the class* $\mathfrak{D}_I$. *Furthermore, for every* $\theta \in [0,2\pi)$, *the mapping* $r \mapsto u_\theta(r) := u(r,\theta)$ *is strictly increasing on* $[0,\ell(\theta))$, *and satisfies* $u_\theta(0) = 1$, $u'_\theta(0+) = 0$, *as well as*

$$\boldsymbol{b}(r,\theta)u'_\theta(r) + \frac{1}{2}\boldsymbol{s}^2(r,\theta)u''_\theta(r) \;=\; u_\theta(r)\,, \qquad \textit{a.e. } r \in (0,\ell(\theta))\,. \tag{6.10}$$

Moreover, we have $1 + v(x) \le u(x) \le e^{v(x)}$, $\forall\, x \in I$.

Proof. Apart from (iii) of Definition 3.9 for the claim that $u \in \mathfrak{D}_I$, Lemma 6.3 can be proved in the same way as in the proof of Lemma 5.5.26 in [14]. And (iii) of Definition 3.9 for u can be seen through Condition 3.10, (6.10), the fact $u_\theta(r) \le e^{v_\theta(r)}$, as well as the fact $u'_\theta(r) \le v'_\theta(r) \cdot e^{v_\theta(r)}$ derived from the proof of Lemma 5.5.26 in [14]. □

Proof of Theorem 4.9: Thanks to Lemma 6.3, we can apply Theorem 2.13 to u and obtain that the process $\{e^{-t\wedge S_n}u(X(t\wedge S_n));\ 0 \le t < \infty\}$ is a local martingale for every $n \in \mathbb{N}$. But this process is also nonnegative, thus a supermartingale. Then we may let $n \to \infty$ to obtain that $\{e^{-t\wedge S}u(X(t\wedge S));\ 0 \le t < \infty\}$ is a nonnegative supermartingale, thus

$$\lim_{t\uparrow S} e^{-t}u(X(t)) \quad \text{exists and is finite,} \quad \mathbb{P}^{\mathbf{0}} - \text{a.e.} \tag{6.11}$$

Proof of **(i)**. By (3.5), $\lim_{t\uparrow S} X(t)$ exists in $\{(r,\theta) :\ r = \ell(\theta),\ 0 \le \theta < 2\pi\}$, $\mathbb{P}^{\mathbf{0}}-$a.e. on $\{S < \infty\}$. Since $\boldsymbol{\nu}(\{\theta :\ v_\theta(\ell(\theta)-) < \infty\}) = 0$, Proposition 4.1 implies that $\lim_{t\uparrow S} v(X(t)) = \infty$, $\mathbb{P}^{\mathbf{0}}-$a.e. on $\{S < \infty\}$. Thus $\lim_{t\uparrow S} u(X(t)) = \infty$, $\mathbb{P}^{\mathbf{0}}-$a.e. on $\{S < \infty\}$, by Lemma 6.3. It follows that $\lim_{t\uparrow S} e^{-t}u(X(t)) = \infty$ holds $\mathbb{P}^{\mathbf{0}}-$a.e. on $\{S < \infty\}$. Comparing this with (6.11), we deduce $\mathbb{P}^{\mathbf{0}}\big(S < \infty\big) = 0$.

Proof of **(ii)**. With

$$A^p := \{\theta :\ p_\theta(\ell(\theta)-) < \infty\} \qquad \text{and} \qquad A^v := \{\theta :\ v_\theta(\ell(\theta)-) < \infty\}\,,$$

we have $A^v \subseteq A^p$ by Proposition 4.3(iii) and $\boldsymbol{\nu}(A^v) > 0$ by assumption, thus $\boldsymbol{\nu}(A^p) > 0$. By Theorem 4.5, the limit $\lim_{t\uparrow S} X(t)$ exists $\mathbb{P}^{\mathbf{0}}-$a.e. in $\{(r,\theta) :\ r = \ell(\theta),\ 0 \le \theta < 2\pi\}$, and $\mathbb{P}^{\mathbf{0}}\big(\Theta(S) \in A^p\big) = 1$. We also have the assumption $\boldsymbol{\nu}(A^p \setminus A^v) = 0$, thus $\mathbb{P}^{\mathbf{0}}\big(\Theta(S) \in A^p \setminus A^v\big) = 0$ and therefore $\mathbb{P}^{\mathbf{0}}\big(\Theta(S) \in A^v\big) = 1$. For every $n \in \mathbb{N}$, let us define

$$\ell^v_n(\theta) := \sup\{r :\ 0 \le r < \ell(\theta),\ v_\theta(r) \le n\}\,, \quad I^v_n := \{(r,\theta) :\ 0 \le r < \ell^v_n(\theta),\ 0 \le \theta < 2\pi\}\,, \tag{6.12}$$

$$S^v_n := \inf\{t \ge 0 :\ \|X(t)\| \ge \ell^v_n(\Theta(t))\} \;=\; \inf\{t \ge 0 :\ X(t) \notin I^v_n\}\,. \tag{6.13}$$

By Proposition 4.4, we have $\mathbb{E}^{\mathbf{0}}[S^v_n] < \infty$, thus $\mathbb{P}^{\mathbf{0}}\big(S^v_n < \infty\big) = 1$, $\forall\, n \in \mathbb{N}$.

Therefore, there is an event $\Omega^\star \in \mathcal{F}$ with $\mathbb{P}^{\mathbf{0}}(\Omega^\star) = 1$, such that for every $\omega \in \Omega^\star$, we have that: $\lim_{t\uparrow S(\omega)} X(t,\omega)$ exists in $\{(r,\theta) :\ r = \ell(\theta),\ 0 \le \theta < 2\pi\}$; that $\Theta(S(\omega),\omega) \in A^v$; and that $S^v_n(\omega) < \infty$ for every $n \in \mathbb{N}$. We fix now an $\omega \in \Omega^\star$. Since $\Theta(S(\omega),\omega) \in A^v$, the limit $\lim_{t\uparrow S(\omega)} v(X(t,\omega))$ exists and is finite. Thus we can choose $n(\omega) \in \mathbb{N}$, such that $n(\omega) > \sup_{t\in[0,S(\omega))} v(X(t,\omega))$.

Claim 6.4. *We have* $S^v_{n(\omega)}(\omega) = S(\omega)$ *, thus* $S(\omega) < \infty$ *.*

Proof. Since $S^v_{n(\omega)}(\omega) < \infty$, we have $X(S^v_{n(\omega)}(\omega),\, \omega) \in \{(r,\theta) : r = \ell^v_{n(\omega)}(\theta),\ 0 \le \theta < 2\pi\}$. With $A^v_n := \{\theta :\ v_\theta(\ell(\theta)-) \le n\}$ for every $n \in \mathbb{N}$, we claim that $\Theta(S^v_{n(\omega)}(\omega),\, \omega) \in A^v_{n(\omega)}$.

Indeed, whenever $\theta \notin A^v_{n(\omega)}$, we have $v_\theta(\ell(\theta)-) > n(\omega)$ and therefore $v_\theta(\ell^v_{n(\omega)}(\theta)) = n(\omega)$. But $v(X(S^v_{n(\omega)}(\omega),\, \omega)) \le \sup_{t\in[0,S(\omega))} v(X(t,\omega)) < n(\omega)$, so we must have $\Theta(S^v_{n(\omega)}(\omega),\, \omega) \in A^v_{n(\omega)}$.

We also observe that, whenever $\theta \in A^v_{n(\omega)}$, we have $v_\theta(\ell(\theta)-) \le n(\omega)$ and therefore $\ell^v_{n(\omega)}(\theta) = \ell(\theta)$. Thus the fact $\Theta(S^v_{n(\omega)}(\omega),\, \omega) \in A^v_{n(\omega)}$ implies that $X(S^v_{n(\omega)}(\omega),\, \omega) \in \{(r,\theta) : r = \ell(\theta),\ 0 \le \theta < 2\pi\}$. We have then $S^v_{n(\omega)}(\omega) = S(\omega)$, and $S(\omega) < \infty$ follows. □

Since Claim 6.4 holds for every $\omega \in \Omega^\star$, the proof of **(ii)** is complete.

Proof of **(iii)**. Since $\boldsymbol{\nu}(\{\theta :\ v_\theta(\ell(\theta)-) < \infty\}) > 0$, we can choose an integer $N \in \mathbb{N}$, such that $A^v_N = \{\theta :\ v_\theta(\ell(\theta)-) \le N\}$ satisfies $\boldsymbol{\nu}(A^v_N) > 0$. Recalling (6.12) and (6.13), we have by Proposition 4.3(iii) that $p_\theta(\ell^v_N(\theta)) < \infty$ for all $\theta \in [0, 2\pi)$. Then an application of Theorem 4.5 yields $\mathbb{P}^{\mathbf{0}}\big(\Theta(S^v_N) \in A^v_N\big) > 0$.

We have also $S^v_N = S$, $\mathbb{P}^{\mathbf{0}}$-a.e. on $\{\Theta(S^v_N) \in A^v_N\}$, in light of the last paragraph of the proof of Claim 6.4. Thus $\mathbb{P}^{\mathbf{0}}\big(S^v_N = S\big) > 0$. But $\mathbb{P}^{\mathbf{0}}\big(S^v_N < \infty\big) = 1$, so $\mathbb{P}^{\mathbf{0}}\big(S < \infty\big) > 0$ follows.

It remains only to show that $\mathbb{P}^{\mathbf{0}}\big(S < \infty\big) < 1$ holds under the assumptions of **(iii)**. We have

$$\boldsymbol{\nu}(A^p \setminus A^v) = \boldsymbol{\nu}\big(\big\{\theta :\ v_\theta(\ell(\theta)-) = \infty,\, p_\theta(\ell(\theta)-) < \infty\big\}\big) > 0$$

by assumption. Another application of Theorem 4.5 yields that $\lim_{t\uparrow S} X(t)$ exists $\mathbb{P}^{\mathbf{0}}-$a.e. in the set $\{(r,\theta) :\ r = \ell(\theta),\ 0 \le \theta < 2\pi\}$, and that $\mathbb{P}^{\mathbf{0}}\big(\Theta(S) \in A^p \setminus A^v\big) > 0$. But since $\lim_{t\uparrow S} v(X(t)) = \infty$ and therefore $\lim_{t\uparrow S} u(X(t)) = \infty$ on $\{\Theta(S) \in A^p \setminus A^v\}$, we may recall (6.11) to obtain $S = \infty$, $\mathbb{P}^{\mathbf{0}}-$a.e. on $\{\Theta(S) \in A^p \setminus A^v\}$. It follows that $\mathbb{P}^{\mathbf{0}}\big(S = \infty\big) > 0$, thus $\mathbb{P}^{\mathbf{0}}\big(S < \infty\big) < 1$. □

PROOF OF THEOREM 5.3: *Step 1.* In this first step we extend Proposition 2.9 and Lemma 2.11 to functions in the class $\mathfrak{C}$. Except for Lemma 2.11(ii), it is straightforward to state and prove the extension. For the extension of Lemma 2.11 (ii), we shall show that whenever $g \in \mathfrak{C}$, the process

$$\sum_{\theta\in[0,2\pi)} \int_0^T \mathbf{1}_{\{X(t)\neq\mathbf{0},\ \Theta(t)=\theta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r)\, D^2 g_\theta(\mathrm{d}r)\,, \qquad 0 \le T < \infty \tag{6.14}$$

is well-defined, adapted, continuous and of finite variation on compact intervals. Following the idea and notation in the proof of Lemma 2.10 and using (iii) of Definition 5.2, we derive

$$\sum_{\theta\in[0,2\pi)} \int_0^T \mathbf{1}_{\{X(t)\neq\mathbf{0},\ \Theta(t)=\theta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r)\, \big|D^2 g_\theta(\mathrm{d}r)\big|$$

$$\le \sum_{\theta\in[0,2\pi)} \int_0^T \mathbf{1}_{\{0<\|X(t)\|\le\eta,\ \Theta(t)=\theta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r)\, \mu(\mathrm{d}r) + \sum_{\{\ell:\ \tau^\eta_{2\ell+1}<T\}} \int_0^\infty L^{\|X\|}(T, r)\, \big|D^2 g_{\Theta(\tau^\eta_{2\ell+1})}(\mathrm{d}r)\big|\,.$$

The second term in the above expression represents a continuous process of finite variation on compact intervals; indeed, the process $\int_0^\infty L^{\|X\|}(\cdot\,, r)\, \big|D^2 g_\theta(\mathrm{d}r)\big|$ has these properties for every fixed $\theta \in [0, 2\pi)$, and the set $\{\ell :\ \tau^\eta_{2\ell+1} < T\}$ is almost surely finite. On the other hand, the first term can be written as

$$\int_0^T \mathbf{1}_{\{0<\|X(t)\|\le\eta\}} \int_0^\infty L^{\|X\|}(\mathrm{d}t, r)\, \mu(\mathrm{d}r) \;=\; \int_0^\eta L^{\|X\|}(T, r)\, \mu(\mathrm{d}r)$$

via interchanging first the summation and the integration, then the two integrals; this is justified by the finiteness of the last expression above. It is now easy to see that the process given by (6.14) is well-defined, continuous and of finite variation on compact intervals.

For adaptedness, it is standard to show, by the BOREL-measurability of g and the joint measurability of $(t,r,\omega)\mapsto L^{\|X\|}(t,r,\omega)$, that for any $T\in[0,\infty)$ the mapping

$$(t,\theta,\omega)\ \longmapsto\ \int_0^\infty L^{\|X\|}(t,r,\omega)\,D^2 g_\theta(\mathrm{d}r)$$

is $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{B}\big([0,2\pi)\big)\otimes\mathcal{F}(T)-$measurable when restricted to $[0,\infty)\times[0,T]\times\Omega$. Let $(s_{k,T},t_{k,T})$, $k\in\mathbb{N}$ be an enumeration of all excursion intervals of the path $\|X(t)\|$, $0<t<T$ away from 0, such that $s_{k,T},t_{k,T}$, $k\in\mathbb{N}$ are all $\mathcal{F}(T)-$measurable. Let $\Theta(t)=\theta_{k,T}$ for all $t\in(s_{k,T},t_{k,T})$, and thus $\theta_{k,T}$ is also $\mathcal{F}(T)-$measurable. Since (6.14) may be rewritten as

$$\sum_{k=1}^{\infty}\int_0^\infty\Big(L^{\|X\|}(t_{k,T},r,\omega)-L^{\|X\|}(s_{k,T},r,\omega)\Big)\,D^2 g_{\theta_{k,T}}(\mathrm{d}r)\,,\qquad 0\le T<\infty\,,$$

it is thus adapted to the filtration $\mathbb{F}$. Step 1 is now complete.

Step 2. With Proposition 2.9 and Lemma 2.11 having been extended, we can follow exactly the same arguments as in the proof of Theorem 2.13 to prove (5.5) (we note that Theorem 3.7.1 (v) in [14] should be used here for the generalized ITÔ's rule). Finally, we note that any function f as in Lemma 2.10 is the second derivative (in the sense of Definition 2.8) of some function in $\mathfrak{D}$, hence also in $\mathfrak{C}$. Thus, both Theorem 2.13 and the just obtained (5.5) apply; comparing the results, we obtain (5.6). □

PROOF OF PROPOSITION 5.7: (i). *Step 1.* We shall show in this step that for every $\theta\in[0,2\pi)$, $U_\theta^{(c)}(\cdot)$ is continuous on $[0,1]$ with $U_\theta^{(c)}(0)=c$ and $U_\theta^{(c)}(1)=U_\theta(1)$. It is easy to show that, $U_\theta^{(c)}(\cdot)$ itself is also $p_\theta-$concave and therefore continuous on $(0,1)$ with finite limits at the two endpoints, such that $\lim_{r\downarrow 0}U_\theta^{(c)}(r)\ge U_\theta^{(c)}(0)\ge c$. Thus to finish this step, it suffices to show $\lim_{r\downarrow 0}U_\theta^{(c)}(r)\le c$ (the situation at 1 can be treated in the same way, thanks to condition (5.7)).

We need only construct, for every $c'>c$, a continuous and $p_\theta-$concave function φ on $[0,1]$ with $\varphi(\cdot)\ge U_\theta(\cdot)$ and $\varphi(0)=c'$. Let $M:=\sup_{x\in\overline{B}}|U(x)|<\infty$. If $c'\ge M$, we take $\varphi\equiv c'$. If $c'<M$, by the continuity of $U_\theta(\cdot)$, we choose $r'>0$ such that $U_\theta(\cdot)\le c'$ on $[0,r']$, and take

$$\varphi(r)=c'+\Big(M-c'\Big)\,\frac{p_\theta(r)}{p_\theta(r')}\,,\quad r\in[0,r']\,;\qquad\quad \varphi(r)=M\,,\quad r\in[r',1]\,.$$

Step 2. By Step 1, the only remaining issue in proving (i), is the continuity in the tree-topology at the origin. By $p_\theta-$concavity we have

$$U_\theta^{(c)}(r)\ \ge\ U_\theta^{(c)}(1)\,\frac{p_\theta(r)}{p_\theta(1-)}+c\Big(1-\frac{p_\theta(r)}{p_\theta(1-)}\Big)\ \ge\ c-\big(c+M\big)\,\frac{p_\theta(r)}{p_\theta(1-)}\,,\quad r\in(0,1). \tag{6.15}$$

Since p is continuous in the tree-topology and $p_\theta(1-)$ is bounded away from zero, we see that

$$\lim_{r\downarrow 0}\ \inf_{\tilde r\le r,\,\theta\in[0,2\pi)}U_\theta^{(c)}(\tilde r)\ge c\,.$$

On the other hand, given $\varepsilon>0$, since U is continuous in the tree-topology, we can choose $\delta>0$, such that $U(r,\theta)\le U(\mathbf{0})+\varepsilon$ for all $r\le 2\delta$ and $\theta\in[0,2\pi)$. Fixing $r\le\delta$ and $\theta\in[0,2\pi)$, we distinguish two cases:

Case 1. $U_\theta^{(c)}(r)=U(r,\theta)$.

Then $U_\theta^{(c)}(r)\le U(\mathbf{0})+\varepsilon\le c+\varepsilon$.

Case 2. The point r belongs to some connected component (r_1, r_2) of the set $\{\rho \in (0,1) : U_\theta^{(c)}(\rho) > U(\rho,\theta)\}$. By (ii) of this proposition (whose proof will not use the continuity of $U^{(c)}$ at the origin under the tree-topology), $U_\theta^{(c)}$ is a linear function of p_θ on $[r_1, r_2]$.

If $r_2 \le 2\delta$, then $U_\theta^{(c)}(r_i) = U_\theta(r_i) \le c + \varepsilon$ for $i = 1, 2$, and it follows that $U_\theta^{(c)}(r) \le c + \varepsilon$. If on the other hand $r_2 > 2\delta$, then the slope of the just mentioned linear function does not exceed $\frac{\max(M,c)+M}{p_\theta(2\delta)-p_\theta(\delta)}$, because $r_1 < r \le \delta$. Therefore,

$$U_\theta^{(c)}(r) \le U_\theta^{(c)}(r_1) + \frac{\max(M,c)+M}{p_\theta(2\delta)-p_\theta(\delta)}\big(p_\theta(r) - p_\theta(r_1)\big) \le c + \varepsilon + \frac{\max(M,c)+M}{p_\theta(2\delta)-p_\theta(\delta)} p_\theta(r).$$

By Condition 3.10, the mapping $\theta \mapsto p_\theta(2\delta) - p_\theta(\delta)$ is bounded away from zero when $2\delta \le \eta$. Thus we obtain $\lim_{r\downarrow 0} \sup_{\tilde{r}\le r,\, \theta\in[0,2\pi)} U_\theta^{(c)}(\tilde{r}) \le c + \varepsilon$ from the above two cases. It is now clear that $U^{(c)}$ is continuous at the origin in the tree-topology.

(ii). Without loss of generality, we may assume $p_\theta(r) \equiv r$. By way of contradiction, we assume that there exist some $\theta \in [0, 2\pi)$ and $(r_1, r_2) \subset [0,1]$, such that $U_\theta^{(c)}(r) > U_\theta(r)$ holds for $r \in (r_1, r_2)$, yet $r \mapsto U_\theta^{(c)}(r)$ is *not* linear on $[r_1, r_2]$. We shall then construct a concave function φ on $[0,1]$ that dominates U_θ and satisfies $\varphi(0) = c$ – yet does not dominate $U_\theta^{(c)}$, thus contradicting (5.10).

Since $U_\theta^{(c)}(\cdot)$ is concave and not linear on $[r_1, r_2]$, we have $D^- U_\theta^{(c)}(r_2) < D^+ U_\theta^{(c)}(r_1)$. Choose $[r_3, r_4] \subset (r_1, r_2)$ with $D^- U_\theta^{(c)}(r_4) < D^+ U_\theta^{(c)}(r_3)$, and we have

$$\min_{r\in[r_3,r_4]} (U_\theta^{(c)}(r) - U_\theta(r)) > 0\,.$$

Thus by dividing $[r_3, r_4]$ into small enough subintervals, we can find $[r_5, r_6] \subset [r_3, r_4]$, such that

$$D^- U_\theta^{(c)}(r_6) < D^+ U_\theta^{(c)}(r_5) \qquad \text{and} \qquad \max_{r\in[r_5,r_6]} U_\theta(r) < \min_{r\in[r_5,r_6]} U_\theta^{(c)}(r)\,. \tag{6.16}$$

Let $\varphi : [0,1] \to \mathbb{R}$ be a linear function on $[r_5, r_6]$ which equals $U_\theta^{(c)}$ on $[0,1] \setminus (r_5, r_6)$. Then φ is concave and satisfies $\varphi(0) = U_\theta^{(c)}(0) = c$; also, the two inequalities of (6.16) imply $\varphi(r) < U_\theta^{(c)}(r)$ for $r \in (r_5, r_6)$, and $\varphi(\cdot) \ge U_\theta(\cdot)$, respectively. Thus φ is our desired function that leads to the contradiction.

(iii). Property (i) in Definition 5.2 is obvious for $U^{(c)}$. For the BOREL-measurability, we may write (in the spirit of the proposition in Section 3 of [15])

$$U_\theta^{(c)}(r) \;=\; \inf\big\{a_n(\theta) p_\theta(r) + b_n(\theta)\,,\;\; n \in \mathbb{N}\big\}, \tag{6.17}$$

where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are two sequences of measurable functions on $[0, 2\pi)$, such that for every $\theta \in [0, 2\pi)$, the set $\{(a_n(\theta), b_n(\theta)),\; n \in \mathbb{N}\}$ is the collection of all rational pairs (a, b) with $b \ge c$, and for which $a p_\theta(\cdot) + b$ dominates $U_\theta(\cdot)$. This is due to the continuity in r and the measurability of both $(r,\theta) \mapsto p_\theta(r)$ and $(r,\theta) \mapsto U_\theta(r)$. The representation (6.17) yields the BOREL-measurability of $U^{(c)}$.

Now let us assume $c > U(\mathbf{0})$. Since both functions $U^{(c)}$ and U are continuous in the tree-topology, we can find an $\eta > 0$, such that $U_\theta^{(c)}(\cdot) > U_\theta(\cdot)$ on $[0, \eta)$, for all $\theta \in [0, 2\pi)$. Hence, we may write $U_\theta^{(c)}(r) = a_\theta\, p_\theta(r) + c$ for $r \in [0, \eta]$, and thus

$$a_\theta = \frac{U_\theta^{(c)}(\eta) - c}{p_\theta(\eta)} \qquad \text{but} \qquad -\frac{c+M}{p_\theta(\eta)} \;\le\; \frac{U_\theta^{(c)}(\eta) - c}{p_\theta(\eta)} \;\le\; \max\Big(0, \frac{M-c}{p_\theta(\eta)}\Big)\,.$$

As the function $\theta \mapsto p_\theta(\eta)$ is bounded away from zero, we see that $\theta \mapsto a_\theta$ is bounded. Thus property (ii) in Definition 5.2 holds for $U^{(c)}$. Property (iii) also follows, using in addition that $p \in \mathfrak{C}_B$.

(iv). The inequality (6.15) shows that the function $\theta \mapsto D^+ U_\theta^{(c)}(0)$ is bounded from below, so the function Φ is well-defined by (5.11) and takes values in $\mathbb{R} \bigcup \{\infty\}$. In fact, from the just proved property (iii), we see that Φ takes the value ∞ only possibly at $U(\mathbf{0})$.

For the other two claimed properties for Φ, it suffices to show that the mapping $c \mapsto D^+ U_\theta^{(c)}(0)$ is continuous and strictly decreasing for every $\theta \in [0, 2\pi)$. Fix $\theta \in [0, 2\pi)$ and consider $c_2 > c_1 \ge U(\mathbf{0})$. With

$$r_{2,\theta} := \inf\left\{r \in [0,1] : U_\theta^{(c_2)}(r) = U_\theta(r)\right\} > 0\,,$$

the function $U_\theta^{(c_2)}(\cdot)$ is a linear transformation of $p_\theta(\cdot)$ on $[0, r_{2,\theta}]$, and $U_\theta^{(c_2)}(r_{2,\theta}) = U_\theta(r_{2,\theta})$. Hence

$$D^+ U_\theta^{(c_2)}(0) = \frac{U_\theta(r_{2,\theta}) - c_2}{p_\theta(r_{2,\theta})} \qquad \text{and} \qquad D^+ U_\theta^{(c_1)}(0) \ge \frac{U_\theta^{(c_1)}(r_{2,\theta}) - c_1}{p_\theta(r_{2,\theta})} > \frac{U_\theta(r_{2,\theta}) - c_2}{p_\theta(r_{2,\theta})}\,,$$

thanks to $p_\theta-$concavity; we have also used the fact $p'_\theta(0+) \equiv 1$. We have thus obtained the strict decrease of the mapping $c \mapsto D^+ U_\theta^{(c)}(0)$. Therefore, we may let $c_2 \downarrow c_1$ then $r \downarrow 0$ in the observation

$$D^+ U_\theta^{(c_2)}(0) \ge \frac{U_\theta^{(c_2)}(r) - c_2}{p_\theta(r)} \ge \frac{U_\theta^{(c_1)}(r) - c_2}{p_\theta(r)}\,,$$

and obtain the right-continuity of $c \mapsto D^+ U_\theta^{(c)}(0)$.

To show the left-continuity, we assume now $c_2 > c_1 > U(\mathbf{0})$ and set $r_{1,\theta} := \inf\{r \in [0,1] : U_\theta^{(c_1)}(r) = U_\theta(r)\} > 0$. It follows that $U_\theta^{(c)}(\cdot)$ is a linear transformation of $p_\theta(\cdot)$ on $[0, r_{1,\theta}]$ whenever $c \ge c_1$. Thus for $r \in (0, r_{1,\theta}]$, we have

$$D^+ U_\theta^{(c_2)}(0) = \frac{U_\theta^{(c_2)}(r) - c_2}{p_\theta(r)}\,, \qquad \text{and} \qquad D^+ U_\theta^{(c)}(0) = \frac{U_\theta^{(c)}(r) - c}{p_\theta(r)} \le \frac{U_\theta^{(c_2)}(r) - c}{p_\theta(r)}\,, \quad c \in [c_1, c_2]\,.$$

Letting $c \uparrow c_2$, we obtain the left-continuity of $c \mapsto D^+ U_\theta^{(c)}(0)$. □

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