SHORT-TERM RELATIVE ARBITRAGE IN VOLATILITY-STABILIZED MARKETS

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Abstract

We answer in the affirmative the following open question posed in Fernholz & Karatzas (2005): do there exist relative arbitrage opportunities over arbitrarily short time horizons in the context of certain volatility-stabilized market models?

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1 Introduction

Several recent results in stochastic portfolio theory have been concerned with the existence of relative arbitrage in equity markets. Roughly speaking, relative arbitrage over a given time horizon occurs if there exists a pair of all-long portfolios of equal initial value such that the first portfolio is guaranteed not to underperform the second, and such that the probability of outperformance is nonzero. We shall also consider the special case of strong relative arbitrage, wherein the first portfolio outperforms the second with probability one.

In Fernholz, Karatzas & Kardaras (2005), it is shown that strong relative arbitrage exists over arbitrarily short time horizons in the context of equity market models which resemble actual equity markets. The key property of these so-called weakly diverse markets, so far as relative arbitrage is concerned, is that the relative volatility (with respect to the market) of the stock of largest capitalization admits an a.s. positive lower bound.

On the other hand, two of the above authors have developed a market model with extreme volatility at the low end of the capitalization scale, namely the *volatility-stabilized model* of Fernholz & Karatzas (2005). In the aforementioned article, it is shown that the extreme volatilities enjoyed by the small-cap stocks in this model lead to strong relative arbitrage on time horizons greater than a fixed constant which depends on the number of stocks in the market. The article poses the following question: does a relative arbitrage exist in this market model on arbitrarily short time horizons?

We answer this question in the affirmative. Section 2 establishes the details of the market model, while Section 3 provides a formal definition of weak and strong relative arbitrage opportunities. In Section 4, we present an argument due to R. Fernholz that weak arbitrage opportunities exist over arbitrarily short time horizons, while Section 5 contains our main result: strong arbitrage opportunities exist in the volatility-stabilized market model over arbitrarily short time horizons. Finally, having resolved one open question, we retaliate with another in Section 6.

2 Preliminaries

We shall work in the context of the volatility-stabilized model given by

$$d\left(\log X_i(t)\right) = \frac{\alpha}{2\mu_i}\,dt + \frac{1}{\sqrt{\mu_i(t)}}\,dW_i(t), \quad i = 1, \ldots, n, \tag{2.1}$$

which was first described in Fernholz & Karatzas (2005). In the above model, the quantity $X_i(t)$ denotes the value of the i^{th} stock at time $t \in [0, \infty)$; the *market weights* $\{\mu_i(\cdot)\}_{i=1}^n$ are given by

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \qquad \text{for } t \in [0, \infty), \quad i = 1, \ldots, n;$$

the parameter α is constant and nonnegative; and $W_1(\cdot), \ldots, W_n(\cdot)$ are independent standard Brownian motions. The above processes are defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and are adapted to a given filtration which satisfies the "usual conditions" of right-continuity and augmentation by P-negligible sets.

A *portfolio* is a progressively measurable process $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))'$ on $[0, \infty) \times \Omega$ with values satisfying

$$\pi_1(t) \geq 0, \quad \ldots, \quad \pi_n(t) \geq 0 \quad \text{and} \quad \sum_{i=1}^n \pi_i(t) = 1, \qquad t \in [0, \infty).$$

The quantity $\pi_i(t)$ represents the proportion of wealth invested in the i^{th} stock at time t. The nonnegativity of the portfolio weights $\{\pi_i(\cdot)\}$ indicates that short-selling of stocks is not permitted. The *value process* $Z^\pi(\cdot)$ corresponding to this portfolio is given by

$$\frac{dZ^\pi(t)}{Z^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \cdot \frac{dX_i(t)}{X_i(t)},$$

where $Z^\pi(0) > 0$ is the initial fortune. In the special case where $\pi_i(t) = \mu_i(t)$ for all $i = 1, \ldots, n$ and $t \in [0, \infty)$, the resulting portfolio "mirrors the market", in the sense that the ratio of its value to the total market capitalization $X_1(t) + \cdots + X_n(t)$ is a.s. constant over time. Consequently, this portfolio is referred to as the *market portfolio*. Note that the market portolio process $(\mu_1(\cdot), \ldots, \mu_n(\cdot))$ always lies in the simplex Δ^n defined by

$$\Delta^n := \{x = (x_1, \ldots, x_n) \in R^n : x_1 > 0, \ldots, x_n > 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i = 1\}. \tag{2.2}$$

Finally, suppose that S is a positive C^2 function defined on a neighborhood U of Δ^n, such that the mapping $x \mapsto x_i D_i \log S(x)$ is bounded on U for all $i = 1, \ldots, n$. (Here D_i denotes differentiation with respect to the ith coordinate.) Furthemore, suppose that for all $x \in \Delta^n$, the Hessian $D^2S(x) := \{D^2_{ij}S(x)\}_{1 \le i,j \le n}$ has at most one positive eigenvalue, and if such an eigenvalue exists, the corresponding eigenvector is orthogonal to Δ^n. Then the assignment

$$\pi_i(t) := \left[D_i \log S(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log S(\mu(t)) \right] \cdot \mu_i(t),$$

for $i = 1, \ldots, n$ and $t \in [0, \infty)$, defines a portfolio $\pi(\cdot)$ which is said to be *generated* by S. Theorem 3.1.5 of Fernholz (2002) gives the relative return decomposition

$$\log\left(\frac{Z^\pi(t)}{Z^\mu(t)}\right) = \log \mathbf{S}(\mu(t)) - \log \mathbf{S}(\mu(0)) + \int_0^t \frac{-1}{2\mathbf{S}(\mu(s))} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu(s))\mu_i(s)\mu_j(s)\tau_{ij}(s)\, ds \tag{2.3}$$

for all $t \in [0, \infty)$, almost surely. The conditions on the Hessian $D^2S(x)$ ensure that the integrand on the right-hand side of (2.3) is always nonnegative.

3 Relative arbitrage opportunities

Given a fixed time horizon $[t_0, T]$, we shall say that a *weak relative arbitrage opportunity* over $[t_0, T]$ is a pair of portfolios $(\pi(\cdot), \rho(\cdot))$ such that there exists a constant $q = q_{\pi,\rho,t_0,T} > 0$

satisfying

$$P\left(\frac{Z^\pi(t)}{Z^\rho(t)} \geq q, \text{ for all } t_0 \leq t \leq T\right) = 1, \tag{3.1}$$

$$P[Z^\pi(T) \geq Z^\rho(T)] = 1 \qquad \text{and} \qquad P[Z^\pi(T) > Z^\rho(T)] > 0 \tag{3.2}$$

whenever the value processes $Z^\pi(\cdot)$ and $Z^\rho(\cdot)$ have the same fortune at time t_0; that is, $Z^\pi(t_0) = Z^\rho(t_0) = z > 0$. If the conditions (3.2) are replaced by the stronger condition

$$P[Z^\pi(T) > Z^\rho(T)] = 1, \tag{3.3}$$

then we say that the portfolio pair $(\pi(\cdot), \rho(\cdot))$ constitutes a *strong relative arbitrage opportunity* over $[t_0, T]$.

In Fernholz & Karatzas (2005), it is shown that a strong relative arbitrage opportunity exists in the market model of (2.1) above over the time horizon $[t_0, T]$, for any T strictly greater than $t_0 + T_*$, where

$$T_* := \frac{2S(\mu(t_0))}{n-1}; \tag{3.4}$$

here $S(\cdot)$ is the *entropy function* on Δ^n defined by

$$S(x) := -\sum_{i=1}^{n} x_i \log(x_i), \qquad x = (x_1, \ldots, x_n) \in \Delta^n. \tag{3.5}$$

A pair of portfolios which provides the arbitrage opportunity is given by $(\pi(\cdot), \mu(\cdot))$, where μ is the market portfolio and $\pi(\cdot)$ is the "modified entropy-weighted portfolio" which is generated by $C+S$, for some sufficiently large constant C depending on T. In the case where the market is equally-weighted at time $t = t_0$, the quantity T_* is precisely $2\log(n)/(n-1)$. Note that $T_* \to 0$ as the number of stocks n tends to ∞. This observation naturally leads to the following question:

Does there exist a weak relative arbitrage opportunity in the model of (2.1) over arbitrarily short time horizons, regardless of the value of n?

This is posed as an open question at the end of Section 4 of Fernholz & Karatzas (2005). In Section 5 below, we shall show that, in fact, a **strong** relative arbitrage opportunity exists in the model of (2.1) over the time horizon $[0, T]$, for arbitrary $T > 0$. Although this answers the above question in the affirmative, we shall first look at a simpler argument which establishes the existence of strictly weak relative arbitrage over $[0, T]$.

4 Weak relative arbitrage over arbitrary time horizons

In this section, we prove the following proposition:

Proposition 1 *For any $T > 0$, a weak relative arbitrage opportunity exists in the market model (2.1) over the time horizon* $[0, T]$.

Proof: We adapt the argument found in Fernholz (2005). Consider the set

$$\mathcal{A} = \left\{ x \in \Delta^n : \frac{2S(x)}{n-1} < \frac{T}{4} \right\},$$

where $S(\cdot)$ is the entropy function of (3.5) above. We define a portfolio $\tilde{\pi}(\cdot)$ by setting

$$\tilde{\pi}(t) = \begin{cases} \mu_i(t) & \text{if} \quad \leq T/2 \quad \text{or} \quad \mu(T/2) \notin \mathcal{A} \\ \pi_i(t) & \text{otherwise.} \end{cases}$$

Here $\pi(\cdot)$ is the "modified entropy-weighted portfolio" generated by $C + S$, as described in Section 3 above. The constant C is chosen large enough to ensure that $(\pi(\cdot), \mu(\cdot))$ provides a strong relative arbitrage opportunity on the time horizon $[T/2, T]$, under the assumption that $\mu(T/2) \in \mathcal{A}$. That is, if $\mu(T/2) \in \mathcal{A}$ and $Z^\pi(T/2) = Z^\mu(T/2)$, then

$$P(Z^\pi(T) > Z^\mu(T)) = 1. \tag{4.1}$$

This is consistent with the definition of T_* given in (3.4) above, since $T_* < T/4$ whenever $\mu(T/2) \in \mathcal{A}$.

The portfolio $\tilde{\pi}(\cdot)$ defined above replicates the market portfolio $\mu(\cdot)$ up until time $T/2$. At that time, if the vector of market weights lies in the set $\mathcal{A}$, the portfolio $\tilde{\pi}(\cdot)$ switches to the portfolio $\pi(\cdot)$; otherwise, $\tilde{\pi}(\cdot)$ remains identical to the market portfolio.

Since the pair $(\pi(\cdot), \mu(\cdot))$ satisfies the condition (3.1) over $[0, T]$, it is obvious that $(\tilde{\pi}(\cdot), \mu(\cdot))$ also satisfies this condition. We need to show that this pair also satisfies (3.2) under the assumption that $Z_\mu(0) = Z_{\tilde{\pi}}(0)$. If $\mu(T/2) \notin \mathcal{A}$, then $\tilde{\pi}(t) = \mu(t)$ for all t in $[0, T]$, so we clearly have $P[Z^{\tilde{\pi}}(T) \geq Z^\mu(T)] = 1$. Otherwise, since $Z^{\tilde{\pi}}(T/2) = Z^\mu(T/2)$ and $\tilde{\pi}(t) = \pi(t)$ for all t in $[T/2, T]$, equation (4.1) above shows that

$$P(Z^{\tilde{\pi}}(T) > Z^\mu(T)) = 1.$$

All that remains is to show that $P(\mu(T/2) \in \mathcal{A}) > 0$ (since otherwise the inequality in (3.2) will fail). In fact, $P(\mu(T/2) \in A) > 0$ for any $A \subset \Delta^n$ of positive Lebesgue measure (and certainly our set $\mathcal{A}$ satisfies this). This assertion follows easily from the representation of $X_i(t)$ in terms of time-changed Bessel processes, as given in equation (6.6) of Fernholz & Karatzas (2005). □

5 Strong relative arbitrage over arbitrary time horizons

The portfolio $\tilde{\pi}$ of the previous section is equivalent to the market portfolio μ, except in the event that the market weights lie in a special subset of the simplex at the specified time $T/2$. For short time horizons, the probability of this event occurring is quite small. Although there is some room for improvement in the above proof, the fact remains that any similar portfolio will replicate the market over the entire time horizon with a high probability.

On the other hand, for any given time horizon, it is possible to construct a portfolio which is guaranteed to beat the market portfolio over that time horizon:

Proposition 2 *For any $T > 0$, a strong relative arbitrage opportunity exists in the market model (2.1) over the time horizon $[0, T]$.*

Proof: In the model (2.1), the variance relative to the market of the ith stock is easily seen to be

$$\tau_{ii}(t) = \frac{1}{\mu_i(t)} - 1, \qquad \text{a.s.} \tag{5.1}$$

for $i = 1, \ldots, n$. Suppose that the portfolio π is generated by $\mathbf{S}$, where

$$\mathbf{S}(x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i); \tag{5.2}$$

here $f : [0,1] \to \mathbf{R}$ is a given nonnegative, bounded, increasing and concave function which is C^2 on $(0,1)$, such that the function $y \mapsto yf'(y)$ is bounded on $(0,1)$. We shall also require that

$$-f''(y)(y - y^2) \quad \text{is decreasing in } y \text{ on } (0, 1/n); \qquad \text{and} \tag{5.3}$$

$$\int_0^{1/n} \frac{f'(y)}{-f''(y)(y - y^2)}\, dy < \infty. \tag{5.4}$$

Also suppose that the initial values $Z^{\pi}(0)$ and $Z^{\mu}(0)$ of the portfolio π and the market portfolio μ, respectively, are equal. It then follows from (2.3) and (5.1) that

$$\log\left(\frac{Z^{\pi}(t)}{Z^{\mu}(t)}\right) = \log \mathbf{S}(\mu(t)) - \log \mathbf{S}(\mu(0)) + \int_0^t \frac{1}{2\mathbf{S}(\mu(s))} \sum_{i=1}^{n} -f''(\mu_i(s))\left(\mu_i(s) - \mu_i^2(s)\right) ds, \tag{5.5}$$

almost surely. Now, for $x \in \Delta^n$, put $x_{(n)} := \min\{x_1, \ldots, x_n\}$ and note that $0 < x_{(n)} \leq 1/n$. In Section 7 below, we prove the estimates

$$\mathbf{S}(x) = \sum_{i=1}^{n} f(x_i) \leq f(x_{(n)}) + (n-1) f\left(\frac{1 - x_{(n)}}{n-1}\right) \leq n f(1/n) \tag{5.6}$$

and

$$\mathbf{S}(x) \geq (n-1)f(x_{(n)}) + f(1-(n-1)x_{(n)}) \geq (n-1)f(0) + f(1). \tag{5.7}$$

Setting $y_\cdot := \mu_{(n)}(\cdot)$ for convenience, we see that the above estimates and (5.5) lead to

$$\begin{aligned}
\log\left(\frac{Z^\pi(t)}{Z^\mu(t)}\right) &\geq \log \mathbf{S}(\mu(t)) - \log \mathbf{S}(\mu(0)) + \int_0^t \frac{1}{2\mathbf{S}(\mu(s))}\left(-f''(y_s)\right)(y_s - y_s^2)\, ds \\
&\geq \left[\log((n-1)f(y_t) + f(1-(n-1)y_t))\right] - \log(nf(1/n)) \\
&\quad + \int_0^t \frac{-f''(y_s)(y_s - y_s^2)}{2\left(f(y_s) + (n-1)f\left(\frac{1-y_s}{n-1}\right)\right)}\, ds \qquad \text{a.s.} \\
&=: S_1(y_t) - \log(nf(1/n)) + \int_0^t \Theta_1(y_s)\, ds.
\end{aligned} \tag{5.8}$$

In light of (5.7) and the nonnegativity of $f(\cdot)$ and $\Theta_1(\cdot)$, we see that the condition (3.1) is satisfied for the pair $(\pi(\cdot), \mu(\cdot))$ with $q = \log(f(1)) - \log(nf(1/n))$. We now claim that

$$\Theta_1(\cdot) \text{ is decreasing on } (0, 1/n). \tag{5.9}$$

Indeed, by our assumption (5.3), the numerator $-f''(r)(r - r^2)$ of $\Theta_1(r)$ is decreasing in r. As for the denominator, we note that

$$\frac{1-r}{n-1} > r \qquad \text{for } r \in (0, 1/n);$$

now, since f is concave, f' is decreasing, so

$$f'(r) - f'\left(\frac{1-r}{n-1}\right) > 0 \qquad \text{for } r \in (0, 1/n).$$

The left-hand side of this inequality is one-half of the derivative of the denominator of $\Theta_1(r)$ (with respect to r). This shows that this denominator is increasing in r, hence $\Theta_1(\cdot)$ is indeed decreasing on $(0, 1/n)$.

Now, fix $t_0 \geq 0$, and define a function $T_1(\cdot)$ on $[0, 1/n]$ by

$$\begin{aligned}
T_1(Y) &:= t_0 + \int_{1/n}^{Y} -\frac{S_1'(r)}{\Theta_1(r)}\, dr \\
&= t_0 + \int_Y^{1/n} \frac{(n-1)f'(r) - (n-1)f'(1-(n-1)r)}{(n-1)f(r) + f(1-(n-1)r)} \left(\frac{-f''(r)(r-r^2)}{2\left(f(r) + (n-1)f\left(\frac{1-r}{n-1}\right)\right)}\right)^{-1} dr.
\end{aligned} \tag{5.10}$$

To see that $T_1(0)$ is well defined, note that

$$T_1(0) \le t_0 + \frac{(n-1)2nf(1/n)}{(n-1)f(0)+f(1)} \int_0^{1/n} \frac{f'(r)}{-f''(r)(r-r^2)}\,dr \tag{5.11}$$

$$\le t_0 + 2n(n-1) \int_0^{1/n} \frac{f'(r)}{-f''(r)(r-r^2)}\,dr < \infty; \tag{5.12}$$

here we have used the estimates (5.6) and (5.7), as well as the fact that $f'(1-(n-1)r) > 0$ for all r in $(0, 1/n)$, to obtain (5.11), while the first inequality in (5.12) follows from the simple observations $f(1/n) \le f(1)$ and $f(0) \ge 0$. The finiteness of the expression in (5.12) is a consequence of our assumption (5.4).

The function $T_1(\cdot)$ satisfies the differential equation $T_1'(Y) = -S_1'(Y)/\Theta_1(Y)$. Since $S_1'(Y) > 0$ and $\Theta_1(Y) > 0$ for all $Y \in (0, 1/n)$, we see that $T_1(\cdot)$ is decreasing. It therefore possesses an inverse $Y(\cdot)$, defined on the interval $[t_0, T_1(0)]$. Since $Y'(t) = 1/T_1'(Y(t))$, we have $S_1'(Y(t))Y'(t) + \Theta_1(Y(t)) = 0$. In light of the initial condition $Y(t_0) = T_1^{-1}(t_0) = 1/n$, we see that $Y(\cdot)$ satisfies the integral equation

$$S_1(Y(t)) + \int_{t_0}^{t} \Theta_1(Y(s))\,ds \equiv S_1(1/n) = \log(nf(1/n)), \qquad t \in [t_0, T_1(0)]. \tag{5.13}$$

We now set

$$f(y) := \Gamma(c+1, -\log y),$$

where c is a positive real number to be determined, and $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function defined by

$$\Gamma(c, z) = \int_z^{\infty} e^{-r} r^{c-1}\,dr$$

for $c \in \mathbf{R}^+$, $z \in \mathbf{R}^+ \cup \{0\}$. The resulting generating function $\mathbf{S}$, as given by (5.2) above, is a generalization of the "modified entropy function" $1 + S(\cdot)$; here S is the entropy function of (3.5). In fact, an integration by parts shows that $\mathbf{S} \equiv 1 + S$ when $c = 1$. In general, it is easy to check that

$$f'(y) = (-\log y)^c, \qquad f''(y) = -\frac{c(-\log y)^{c-1}}{y} \qquad \text{on } (0,1),$$

and that f satisfies all the assumptions leading up to, and including, the conditions (5.3) and (5.4) above. With this choice of f, (5.12) becomes

$$T_1(0) \le t_0 + 2n(n-1) \int_0^{1/n} \frac{(-\log r)^c}{\frac{c(-\log r)^{c-1}}{r}(r-r^2)}\,dr = t_0 + \frac{2n(n-1)}{c} \int_0^{1/n} \frac{-\log r}{1-r}\,dr = t_0 + \frac{A_n}{c}. \tag{5.14}$$

Here the finite constant A_n is independent of c.

To establish that strong relative arbitrage exists in the volatility-stabilized market of (2.1) on an arbitary time horizon $[0, T]$ for some $T > 0$, first set $c = 2A_n/T$, so that

$$\frac{A_n}{c} = T/2. \tag{5.15}$$

Set $t_0 = T/2$, let $Y(\cdot)$ be as in (5.10) above, and put

$$T_0 = \inf\{t \geq T/2 : y_t > Y(t)\}.$$

(Recall that $y_\cdot := \mu_{(n)}(\cdot)$.) Clearly T_0 is a stopping time, and we also claim that $T_0 \leq T$ a.s.; indeed, since $y_{T_1(0)} > 0$ a.s. (for the function $T_1(\cdot)$ defined in (5.10) above), we must have $T_0 \leq T_1(0)$. On the other hand, (5.14) and (5.15) show that $T_1(0) \leq T$, so $T_0 \leq T$ a.s., as claimed.

Define a portfolio $\tilde{\pi}(\cdot)$ by setting

$$\tilde{\pi}(t) = \begin{cases} \pi(t), & t < T_0 \\ \mu(t), & t \geq T_0. \end{cases}$$

Since the condition (3.1) is satisfied for the pair $(\pi(\cdot), \mu(\cdot))$ and $q = \log(f(1)) - \log(nf(1/n))$, it is clearly also satisfied for the pair $(\tilde{\pi}(\cdot), \mu(\cdot))$ with the same value of q. It remains to establish the condition (3.3) for the pair $(\tilde{\pi}(\cdot), \mu(\cdot))$.

We now return to the estimate (5.8). Using the facts that $y_t \leq 1/n$ on $[0, T/2]$, $y_t \leq Y(t)$ on $[T/2, T_0]$, and $\Theta_1(\cdot)$ is decreasing, as well as (5.13), we have

$$\begin{aligned}
\log\left(\frac{Z^{\tilde{\pi}}(T)}{Z^{\mu}(T)}\right) &= \log\left(\frac{Z^{\pi}(T_0)}{Z^{\mu}(T_0)}\right) \\
&\geq S_1(y_{T_0}) - \log(nf(1/n)) + \int_0^{T/2} \Theta_1(y_s)\, ds + \int_{T/2}^{T_0} \Theta_1(y_s)\, ds \\
&\geq S_1(Y(T_0)) - \log(nf(1/n)) + \int_0^{T/2} \Theta_1(1/n)\, ds + \int_{t_0}^{T_0} \Theta_1(Y(s))\, ds \\
&= \int_0^{T/2} \Theta_1(1/n)\, ds = (T/2)\Theta_1(1/n), \qquad \text{a.s.}
\end{aligned}$$

This establishes the desired relative strong arbitrage, since the quantity $(T/2)\Theta_1(1/n)$ is a positive constant (depending on T, c and n). □

It is interesting to note that the portfolio $\tilde{\pi}$ switches from the functionally-generated portfolio π to the market portfolio μ, while the corresponding portfolio in the proof of Proposition 1 does the opposite. Also note that the precise form of the growth rate term $\alpha/2\mu_i(t)\, dt$ of (2.1) does not appear in the above proof. In other words, the relative arbitrage is driven purely by volatility considerations. In principle, the growth rate term could be replaced by another suitable growth rate term $\gamma_i(t)\,dt$, although the resulting market may lack the long-term stability of the market of (2.1).

Finally, we briefly examine the constants A_n and $(T/2)\Theta_1(1/n)$ which appear in the above proof. For fixed large n, we have

$$A_n := 2n(n-1)\int_0^{1/n} \frac{-\log r}{1-r}\,dr \approx 2(n-1)\log n.$$

Since $T/2 = A_n/c$, we have

$$\begin{aligned}(T/2)\Theta_1(1/n) &= \frac{A_n}{c}\frac{c(\log n)^{c-1}(1-1/n)}{2n\Gamma(c+1,\log n)} \approx \frac{(n-1)^2}{n^2}\frac{(\log n)^c}{\Gamma(c+1,\log n)}\\ &\approx \frac{(\log n)^{2(n-1)\log n/T}}{\Gamma(1+2(n-1)\log n/T,\log n)}.\end{aligned}$$

Setting $C_0 = \log n$, $C_1 = 2(n-1)\log n$ and $U = 1/T$, we see that the above quantity is $C_0^{C_1 U}/\Gamma(1+C_1U, C_0)$. The denominator grows at the order of $[U]!$, so the above quantity tends to 0 very quickly as $T \to 0^+$.

6 Another open question

Propositions 3.1 and 3.8 of Fernholz & Karatzas (2005) states that strong relative arbitrage opportunities exist over long enough time horizons in any market satisfying the condition

$$\Gamma(t) \le \int_0^t \gamma_*^{\mu,p}(s)\,ds < \infty, \qquad \text{a.s.} \tag{6.1}$$

for some $p > 0$ and continuous, strictly increasing function $\Gamma : [0,\infty) \to [0,\infty)$ with $\Gamma(0) = 0$ and $\Gamma(\infty) = \infty$. In the above equation, $\gamma_*^{\mu,p}(\cdot)$ is the *generalized excess growth rate* of the market, which can be expressed as a weighted-average relative volatility of the market:

$$\gamma_*^{\mu,p}(t) = \frac{1}{2}\sum_{i=1}^n (\mu_i(t))^p \tau_{ii}(t).$$

The existence of long-term strong relative arbitrage opportunities in the model of (2.1) follows as a corollary, since $\gamma_*^{\mu,1}(t)$ is identically equal to $(n-1)/2$ in this model. Provided that the market portfolio is not confined to a subset of Δ^n whose complement in Δ^n has positive Lebesgue measure, it turns out that the argument in Section 4 can be adapted to show that short-term weak relative arbitrage opportuntities in models stasifying (6.1).

On the other hand, the proof of Proposition 2 relies on the precise structure of the model of (2.1). It is straightforward to adapt the proof to similar models where the variance term $dW_i(t)/\sqrt{\mu_i(t)}$ is replaced by $dW_i(t)/(\mu_i(t))^p$ for some other power $p > 1/2$, but we have not been able to generalize to the broader class of models satisfying condition (6.1). Consequently, we conclude as follows:

Open question: do strong relative arbitrage opportunities exist over arbitrarily short time horizons in any market model satisfying the condition (6.1)?

7 Appendix: some properties of concave functions

We now prove equations (5.6) and (5.7) from Section 5 by using the following pair of lemmas relating to concave functions:

Lemma 7.1 *If f is concave on $[A, B]$ and $a_1, \ldots, a_m \in [A, B]$, then*

$$\sum_{i=1}^{m} f(a_i) \leq m f\left(\frac{1}{m}\sum_{i=1}^{m} a_i\right).$$

Lemma 7.2 *Suppose that f is concave on $[A, B]$ and $a_1, \ldots, a_m \in [A, B]$ are chosen such that $a - (m-1)A \leq B$, where $a := \sum_{i=1}^{m} a_i$. Then*

$$\sum_{i=1}^{m} f(a_i) \geq (m-1) f(A) + f(a - (m-1)A). \tag{7.2}$$

To prove these lemmas, we first recall that a concave function on $[A, B]$ satisfies the inequality

$$\sum_{i=1}^{m} w_i f(a_i) \leq f\left(\sum_{i=1}^{m} w_i a_i\right) \tag{7.3}$$

whenever $a_1, \ldots, a_m \in [A, B]$ and $w := (w_1, \ldots, w_m) \in \Delta^m$. (The simplex Δ^m is defined in (2.2) above.) Note that (7.3) follows easily (by induction) from the more commonly-quoted property

$$(1-\lambda) f(a_1) + \lambda f(a_2) \leq f((1-\lambda)a_1 + \lambda a_2), \tag{7.4}$$

valid for any $0 \leq \lambda \leq 1$ and a_1, a_2 in the domain of the concave function f. In any case, Lemma 7.1 follows from (7.3) by setting $w_i = 1/m$ for all $i = 1, \ldots, m$. As for Lemma 7.2, we set $\lambda_i = (a_i - A)/(a - mA)$. It is straightforward to check that $0 \leq \lambda_i \leq 1$ and that

$$(1-\lambda_i)A + \lambda_i(a - (m-1)A) = a_i.$$

It follows from (7.4), with λ replaced by λ_i, that

$$f(a_i) \geq (1-\lambda_i) f(A) + \lambda_i f(a - (m-1)A)$$

for any $i = 1, \ldots, m$. Adding all m inequalities establishes (7.2), since $\sum_{i=1}^m \lambda_i = 1$. □

In the case where $x := (x_1, \ldots, x_n)$ lies in Δ^n, and $x_{(n)} := \min\{x_1, \ldots, x_n\}$, we may as well suppose that $x_n = x_{(n)}$; then by Lemma 7.1 with $A = 0$, $B = 1$, $m = n-1$ and $a_i = x_i$ for each $i = 1, \ldots, m$, we have

$$\sum_{i=1}^{n} f(x_i) = f(x_{(n)}) + \sum_{i=1}^{n-1} f(x_i) \leq f(x_{(n)}) + (n-1) f\left(\frac{1 - x_{(n)}}{n-1}\right).$$

Reapplying Lemma 7.1, this time with $m = n$, $a_1 = \cdots = a_{n-1} = (1 - x_{(n)})/(n-1)$ and $a_n = x_{(n)}$, we get

$$f(x_{(n)}) + (n-1) f\left(\frac{1 - x_{(n)}}{n-1}\right) \leq n f(1/n).$$

Taken together, the previous two inequalities give (5.6). As for (5.7), an application of Lemma 7.2 with $m = n$, $A = x_{(n)}$, $B = 1$ and $a_i = x_i$ for $i = 1, \ldots, n$ gives

$$\sum_{i=1}^{n} f(x_i) \geq (n-1) f(x_{(n)}) + f(1 - (n-1) x_{(n)}).$$

Reapplying the lemma with $A = 0$, $a_1 = \cdots = a_{n-1} = x_{(n)}$ and $a_n = 1 - (n-1)x_{(n)}$ gives

$$(n-1) f(x_{(n)}) + f(1 - (n-1) x_{(n)}) \geq (n-1) f(0) + f(1).$$

The previous two inequalities give (5.7). □

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References

[1] Fernholz, E.R. (2002). *Stochastic Portfolio Theory.* Springer-Verlag, New York.

[2] Fernholz, R. (2005) Short-term arbitrage. Unpublished research note.

[3] Fernholz, R., Karatzas, I. & Kardaras, C. (2005) Diversity and arbitrage in equity markets. *Finance & Stochastics* **9**, 1-27.

[4] Fernholz, R. & Karatzas, I. (2005) Relative arbitrage in volatility-stabilized markets. *Annals of Finance* **1**, 149-177.