Turnover in the INTECH Diversity Index℠

The INTECH Diversity Index℠ is a functionally generated portfolio (see Fernholz (1997)) of the stocks in the S&P 500® Index. Since the Diversity Index℠ has dynamically changing weights, any actual portfolio following it will have to be rebalanced from time to time in order to maintain the actual portfolio weights close enough to the theoretical index weights. In this report we shall estimate the turnover generated by rebalancing such a portfolio.

The generating function for the Diversity Index℠ is

$$\mathbf{D}_p(x) = \Big(\sum_{i=1}^{n} x_i^p\Big)^{1/p},$$

with $p = .76$. The corresponding portfolio weights are

$$\pi_i(t) = \frac{\mu_i^p(t)}{\big(\mathbf{D}_p(\mu(t))\big)^p}, \tag{1}$$

where the $\mu_i(t)$ are the cap weights of the S&P 500® . The drift process for the Diversity Index℠ is

$$\Theta(t) = (1-p)\gamma_\pi^*(t).$$

We shall consider the case in which the portfolio is traded when the actual portfolio weights differ from the desired weights in (1) by a fixed multiple δ of the desired weights. Suppose that a period of time dt passes, along with a change in the i-th desired weight to $\pi_i(t) + d\pi_i(t)$. Over this period, the i-th actual portfolio weight will become

$$\begin{aligned}\pi_i(t)\frac{X_i(t)+dX_i(t)}{X_i(t)}\frac{Z_\pi(t)}{Z_\pi(t)+dZ_\pi(t)} &= \pi_i(t)\bigg(1+\frac{dX_i(t)}{X_i(t)}\bigg)\bigg(1-\frac{dZ_\pi(t)}{Z_\pi(t)}+O(dt)\bigg)\\ &= \pi_i(t)\bigg(1+\frac{dX_i(t)}{X_i(t)}-\frac{dZ_\pi(t)}{Z_\pi(t)}+O(dt)\bigg).\end{aligned}$$

When

$$\bigg|\pi_i(t)\bigg(1+\frac{dX_i(t)}{X_i(t)}-\frac{dZ_\pi(t)}{Z_\pi(t)}+O(dt)\bigg)-\pi_i(t)-d\pi_i(t)\bigg| = \delta\pi_i(t), \tag{2}$$

a trade is made. We shall estimate the expected time between such trades, and use this to estimate the expected turnover.

Now, (2) is equivalent to

$$\left(\frac{dX_i(t)}{X_i(t)} - \frac{dZ_\pi(t)}{Z_\pi(t)} - \frac{d\pi_i(t)}{\pi_i(t)} + O(dt)\right)^2 = \delta^2.$$

By Itô's lemma,

$$\begin{aligned}\left(\frac{dX_i(t)}{X_i(t)} - \frac{dZ_\pi(t)}{Z_\pi(t)} - \frac{d\pi_i(t)}{\pi_i(t)} + O(dt)\right)^2 \\ &= \big(d\log X_i(t) - d\log Z_\pi(t) - d\log \pi_i(t) + O(dt)\big)^2 \\ &= \big(d\log \mu_i(t) - d\log\big(Z_\pi(t)/Z(t)\big) - d\log \pi_i(t) + O(dt)\big)^2 \\ &= d\langle \log\mu_i - \log(Z_\pi/Z) - \log\pi_i\rangle_t.\end{aligned}$$

Now,

$$d\log\pi_i(t) = p\,d\log\mu_i(t) - p\,d\log\mathbf{D}_p(\mu(t)),$$

so,

$$\begin{aligned}d\log\mu_i(t) - d\log\pi_i(t) &= (1-p)\,d\log\mu_i(t) - d\log\big(Z_\pi(t)/Z(t)\big) + p\,d\log\mathbf{D}_p(\mu(t)) \\ &= (1-p)\,d\log\mu_i(t) - (1-p)\,d\log\big(Z_\pi(t)/Z(t)\big) + p\Theta(t)\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}d\langle\log\mu_i - \log(Z_\pi/Z) - \log\pi_i\rangle_t &= d\langle(1-p)\log\mu_i - (1-p)\log(Z_\pi/Z)\rangle_t \\ &= (1-p)^2(\tau_{ii}(t) - 2\tau_{i\pi}(t) + \tau_{\pi\pi}(t))\,dt.\end{aligned}$$

Martingale arguments imply that the expected time for (2) to occur is

$$\delta^2/\big((1-p)^2(\tau_{ii}(t) - 2\tau_{i\pi}(t) + \tau_{\pi\pi}(t))\big). \tag{3}$$

If we assume that a unit of time is one year, and all parameters are measured accordingly, then the theory of recurrent events implies that the expected number of such trades per year is the reciprocal of (3). Since the proportion of the portfolio value traded in each such trade is $\pi_i(t)\delta$, the annual trading generated by the i-th stock will be

$$\pi_i(t)(1-p)^2(\tau_{ii}(t) - 2\tau_{i\pi}(t) + \tau_{\pi\pi}(t))/\delta.$$

It follows that the total annual trading will be

$$(1-p)^2\sum_{i=1}^n \pi_i(t)(\tau_{ii}(t) - 2\tau_{i\pi}(t) + \tau_{\pi\pi}(t))/\delta = 2(1-p)^2\gamma_\pi^*(t)/\delta. \tag{4}$$

Since $\Theta(t) = (1-p)\gamma_\pi^*(t)$ and $p = .76$, (4) becomes

$$\begin{aligned}\text{annual trading} &= 2(1-p)\Theta(t)/\delta \\ &= .48\Theta(t)/\delta.\end{aligned}$$

In Fernholz, Garvy, and Hannon (1998), Θ is estimated at 61 basis points a year (this is without corrections for dividends or leakage). Hence,

$$\text{annual trading} \simeq .0029/\delta.$$

With $\delta = .1$, this implies that annual trading will be about 2.9% a year, which is 1.45% a year annual turnover (which pairs buy and sell trades). If trading were to be done at the bid-ask for each stock, this would mean that for a spread of 1/8 on a $40 stock,

$$\begin{aligned}\delta &\simeq \frac{1/16}{40} \\ &\simeq .0016,\end{aligned}$$

and this would result in turnover of about 93% a year.

References

Fernholz, R. (1997). Portfolio generating functions. Technical report, INTECH, Princeton, NJ.

Fernholz, R., R. Garvy, and J. Hannon (1998). Diversity weighted indexing. *Journal of Portfolio Management,* to appear.

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