Teacher Edition

Eureka Math
Grade 8
Module 7

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Mathematics Curriculum

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Introduction to Irrational Numbers Using Geometry

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[1]Each lesson is ONE day, and ONE day is considered a 45-minute period.

Topics A through D (assessment 2 days, return 1 day, remediation or further applications 3 days)

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Grade 8 • Module 7

Introduction to Irrational Numbers Using Geometry

OVERVIEW

The module begins with work related to the Pythagorean theorem and right triangles. Before the lessons of this module are presented to students, it is important that the lessons in Modules 2 and 3 related to the Pythagorean theorem are taught (i.e., Module 2 Lessons 15 and 16 and Module 3 Lessons 13 and 14). In Modules 2 and 3, students used the Pythagorean theorem to determine the unknown side length of a right triangle. In cases where the side length was an integer, students computed the length. When the side length was not an integer, students left the answer in the form of $x^2 = c$, where c was not a perfect square number. Those solutions are revisited and are the motivation for learning about square roots and irrational numbers in general.

In Topic A, students learn the notation related to roots. The definition for irrational numbers relies on students' understanding of rational numbers; that is, students know that rational numbers are points on a number line and that every quotient of integers (with a nonzero divisor) is a rational number. Then, irrational numbers are numbers that can be placed in their approximate positions on a number line and not expressed as a quotient of integers. Though the term *irrational* is not introduced until Topic B, students learn that irrational numbers exist and are different from rational numbers. Students learn to find positive square roots and cube roots of expressions and know that there is only one such number. Topic A includes some extension work on simplifying perfect square factors of radicals in preparation for Algebra I.

In Topic B, students learn that to get the decimal expansion of a number, they must develop a deeper understanding of the long division algorithm learned in Grades 6 and 7. Students show that the decimal expansion for rational numbers repeats eventually, in some cases with zeros, and they can convert the decimal form of a number into a fraction. Students learn a procedure to get the approximate decimal expansion of numbers like $\sqrt{2}$ and $\sqrt{5}$ and compare the size of these irrational numbers using their rational approximations. At this point, students learn that the definition of an irrational number is a number that is not equal to a rational number. In the past, irrational numbers may have been described as numbers that are infinite decimals that cannot be expressed as a fraction, like the number π. This may have led to confusion about irrational numbers because until now, students did not know how to write repeating decimals as fractions; additionally, students frequently approximated π using $\frac{22}{7}$, which led to more confusion about the definition of irrational numbers. Defining irrational numbers as those that are not equal to rational numbers provides an important guidepost for students' knowledge of numbers. Students learn that an irrational number is something quite different from other numbers they have studied before. They are infinite decimals that can only be expressed by a decimal approximation. Now that students know that

irrational numbers can be approximated, they extend their knowledge of the number line gained in Grade 6 to include being able to position irrational numbers on a line diagram in their approximate locations.

Topic C revisits the Pythagorean theorem and its applications in a context that now includes the use of square roots and irrational numbers. Students learn another proof of the Pythagorean theorem involving areas of squares off of each side of a right triangle. Another proof of the converse of the Pythagorean theorem is presented, which requires an understanding of congruent triangles. With the concept of square roots firmly in place, students apply the Pythagorean theorem to solve real-world and mathematical problems to determine an unknown side length of a right triangle and the distance between two points on the coordinate plane.

In Topic D, students learn that radical expressions naturally arise in geometry, such as the height of an isosceles triangle or the lateral length of a cone. The Pythagorean theorem is applied to three-dimensional figures in Topic D as students learn some geometric applications of radicals and roots. In order for students to determine the volume of a cone or sphere, they must first apply the Pythagorean theorem to determine the height of the cone or the radius of the sphere. Students learn that truncated cones are solids obtained by removing the top portion above a plane parallel to the base. Students know that to find the volume of a truncated cone they must access and apply their knowledge of similar figures learned in Module 3. Their work with truncated cones is an exploration of solids that is not formally assessed. In general, students solve real-world and mathematical problems in three dimensions in Topic D. For example, now that students can compute with cube roots and understand the concept of rate of change, students compute the average rate of change in the height of the water level when water is poured into a conical container at a constant rate. Students also use what they learned about the volume of cylinders, cones, and spheres to compare volumes of composite solids.

It is recommended that students have access to a calculator to complete the End-of-Module Assessment but that they complete the Mid-Module Assessment without one.

The discussion of infinite decimals and the conversion of fractions to decimals in this module is taken from the following source:

H. Wu, *Mathematics of the Secondary School Curriculum, Volume III* (to appear in 2016).

Focus Standards

Know that there are numbers that are not rational, and approximate them by rational numbers.

- ■ Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

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- Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get a better approximation.*

Work with radicals and integer exponents.[2]

- Use square root and cube root symbols to represent solutions to the equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

Understand and apply the Pythagorean Theorem.

- Explain a proof of the Pythagorean Theorem and its converse.

- Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

- Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

- Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.[3]

Foundational Standards

Compute fluently with multi-digit numbers and find common factors and multiples.

- Fluently divide multi-digit numbers using the standard algorithm.

Apply and extend previous understandings of numbers to the system of rational numbers.

- Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

  □ Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$ and that 0 is its own opposite.

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[2]The balance of this cluster is taught in Module 1.
[3]Solutions that introduce irrational numbers are allowed in this module.

       □   Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

       □   Find and position integers and other rational numbers on a horizontal and vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

     ■   Apply and extend previous understandings of multiplication and division of fractions to multiply and divide rational numbers.

       □   Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

       □   Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-\frac{p}{q} = \frac{-p}{q} = \frac{p}{-q}$. Interpret quotients of rational numbers by describing real-world contexts.

       □   Apply properties of operations as strategies to multiply and divide rational numbers.

       □   Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Draw, construct, and describe geometrical figures and describe the relationships between them.

     ■   Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

     ■   Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

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Focus Standards for Mathematical Practice

- **Attend to precision**. Students begin attending to precision by recognizing and identifying numbers as rational or irrational. Students know the definition of an irrational number and can represent the number in different ways (e.g., as a root, as a non-repeating decimal block, or as a symbol such as π). Students will attend to precision when clarifying the difference between an exact value of an irrational number compared to the decimal approximation of the irrational number. Students use appropriate symbols and definitions when they work through proofs of the Pythagorean theorem and its converse. Students know and apply formulas related to volume of cones and truncated cones.

- **Look for and make use of structure**. Students learn that a radicand can be rewritten as a product and that sometimes one or more of the factors of the product can be simplified to a rational number. Students look for structure in repeating decimals, recognize repeating blocks, and know that every fraction is equal to a repeating decimal. Additionally, students learn to see composite solids as made up of simpler solids. Students interpret numerical expressions as representations of volumes of complex figures.

- **Look for and express regularity in repeated reasoning**. While using the long division algorithm to convert fractions to decimals, students recognize that when a sequence of remainders repeats, the decimal form of the number will contain a repeat block. Students recognize that when the decimal expansion of a number does not repeat or terminate, the number is irrational and can be represented with a method of rational approximation using a sequence of rational numbers to get closer and closer to the given number.

Terminology

New or Recently Introduced Terms

- **Cube Root** (A *cube root of the number b* is a number whose cube is b. It is denoted by $\sqrt[3]{b}$.)
- **Decimal Expansion (description)** (A whole number (e.g., 3) and an infinite sequence of single-digit numbers (e.g., 1, 4, 1, 5, 9, 2, …) is called a *decimal expansion* and is written as a finite decimal together with ellipses to indicate the infinite sequence (i.e., 3.141592…).)
- **Decimal Expansion of a Negative Number** (A *decimal expansion of a negative number* is a decimal expansion of the absolute value of the number together with a negative sign in front of the expansion.)

- **Decimal Expansion of a Positive Real Number (description)** (A *decimal expansion of a positive real number x* is a decimal expansion with the property that for any and all whole numbers n, if a is the nth finite decimal of that decimal expansion, then $|x - a| \leq \frac{1}{10^n}$.

  Using the decimal expansion 3.141…, the 1ˢᵗ finite decimal 3.1 satisfies $|\pi - 3.1| \leq \frac{1}{10}$, the 2ⁿᵈ finite decimal 3.14 satisfies $|\pi - 3.14| \leq \frac{1}{100}$, the 3ʳᵈ finite decimal 3.141 satisfies $|\pi - 3.141| \leq \frac{1}{1000}$, and so on. Hence, $\pi = 3.1415…$ is a true statement assuming that difference of π and the nth finite decimal of the decimal expansion continues to be less than or equal to $\frac{1}{10^n}$.)

- **Decimal System** (The *decimal system* is a positional numeral system for representing real numbers by their decimal expansions.

  The decimal system extends the whole number place value system and the place value systems to decimal representations with an infinite number of digits.)

- **Irrational Number** (An *irrational number* is a real number that cannot be expressed as $\frac{p}{q}$ for integers p and q with $q \neq 0$.

  An irrational number has a decimal expansion that is neither terminating nor repeating.)

- **The nth Decimal Digit of a Decimal Expansion** (The nth single-digit number in the infinite sequence is called the nth *decimal digit* of the decimal expansion. The whole number is called the *whole number part* of the decimal expansion. For example, the whole number part of 3.141592… is 3, and the 4ᵗʰ decimal digit is 5.)

- **The nth Finite Decimal of a Decimal Expansion** (The nth *finite decimal of a decimal expansion* is the number represented by the finite decimal obtained by discarding all the digits in the decimal expansion after the nth decimal digit. For example, the 2ⁿᵈ finite decimal of 3.141592… is 3.14.)

- **Perfect Square** (A *perfect square* is a number that is the square of an integer.)

- **Rational Approximation (description)** (A *rational approximation* of a real number r is a rational number a with absolute error less than some specified number. Rational approximations are usually found by taking the nth finite decimal a of a decimal expansion of r, which approximates r with absolute error less than or equal to $\frac{1}{10^n}$.)

- **Real Number (description)** (A *real number* is a number that can be represented by a point on the number line.

  Any point on the number line corresponds to a real number. (Recall that a number line is a line equipped with a coordinate system.))

- **A Square Root of a Number** (A *square root of the number b* is a number whose square is b.

  In symbols, a square root of b is a number a such that $a^2 = b$. Negative numbers do not have any square roots, zero has exactly one square root, and positive numbers have two square roots.)

- **The Square Root of a Number** (Every positive real number a has a unique positive square root called *the square root of the number b* or *principle square root* of b; it is denoted $\sqrt{b}$. The square root of zero is zero.)

- **Truncated Cone (description)** (Given a cone, a *truncated cone* is a solid obtained by taking all points of the cone that lie between two planes that are both parallel to its base together with the points of the cone that lie in both planes.)

Familiar Terms and Symbols[4]

- Decimal Expansion
- Finite Decimals
- Number Line
- Rate of Change
- Rational Number
- Volume

Suggested Tools and Representations

- 3-D models (truncated cone, pyramid)
- Scientific Calculator

Assessment Summary

Assessment Type	Administered	Format
Mid-Module Assessment Task	After Topic B	Constructed response with rubric
End-of-Module Assessment Task	After Topic D	Constructed response with rubric

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[4]These are terms and symbols students have seen previously.

Topic A

Square and Cube Roots

<table>
<tr>
<td>Focus Standards:</td>
<td>■</td>
<td>Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.</td>
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<tr>
<td></td>
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<td>Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue to get better approximations.</td>
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<tr>
<td></td>
<td>■</td>
<td>Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.</td>
</tr>
<tr>
<td colspan="3">Instructional Days: 5</td>
</tr>
<tr>
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<td>The Pythagorean Theorem (P)[1]</td>
</tr>
<tr>
<td colspan="2">Lesson 2:</td>
<td>Square Roots (S)</td>
</tr>
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<td colspan="2">Lesson 3:</td>
<td>Existence and Uniqueness of Square Roots and Cube Roots (S)</td>
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<td colspan="2">Lesson 4:</td>
<td>Simplifying Square Roots (Optional) (P)</td>
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<tr>
<td colspan="2">Lesson 5:</td>
<td>Solving Equations with Radicals (P)</td>
</tr>
</table>

The use of the Pythagorean theorem to determine side lengths of right triangles motivates the need for students to learn about square roots and irrational numbers in general. While students have previously applied the Pythagorean theorem using perfect squares, students begin by estimating the length of an unknown side of a right triangle in Lesson 1 by determining which two perfect squares a squared number is between. This leads them to know between which two positive integers the length must be. In Lesson 2, students are introduced to the notation and meaning of square roots. The term and formal definition for

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

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irrational numbers is not given until Topic B, but students know that many of this type of number exist between the positive integers on the number line. That fact allows students to place square roots on a number line in their approximate position using perfect square numbers as reference points. In Lesson 3, students are given proof that the square or cube root of a number exists and is unique. Students then solve simple equations that require them to find the square root or cube root of a number. These will be in the form $x^2 = p$ or $x^3 = p$, where p is a positive rational number. In the optional Lesson 4, students learn that a square root of a number can be expressed as a product of its factors and use that fact to simplify the perfect square factors. For example, students know that they can rewrite $\sqrt{18}$ as $\sqrt{3^2 \times 2} = \sqrt{3^2} \times \sqrt{2} = 3 \times \sqrt{2} = 3\sqrt{2}$. The work in this lesson prepares students for what they may need to know in Algebra I to simplify radicals related to the quadratic formula. Some solutions in subsequent lessons are in simplified form, but these may be disregarded if Lesson 4 is not used. In Lesson 5, students solve multi-step equations that require students to use the properties of equality to transform an equation until it is in the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number.

📝 Lesson 1: The Pythagorean Theorem

Student Outcomes

- Students know that they can estimate the length of a side of a right triangle as a number between two integers and identify the integer to which the length is closest.

Lesson Notes

Before beginning this lesson, it is imperative that students are familiar with the lessons in Modules 2 and 3 that relate to the Pythagorean theorem. This lesson assumes knowledge of the theorem and its basic applications. Students should not use calculators during this lesson.

In this lesson, students are exposed to expressions that involve irrational numbers, but they will not learn the definition of an irrational number until Topic B. It is important for students to understand that these irrational numbers can be approximated, but it is not yet necessary that they know the definition.

> *Scaffolding:*
> In teaching about right triangles and guiding students in learning to identify the hypotenuse, it may be necessary to provide additional support in addressing the differences among the terms *long, longer,* and *longest,* as comparative words like these (with the same root) may not yet be familiar to English language learners.

Classwork

Opening (5 minutes)

Show students the three triangles below. Give students the direction to determine as much as they can about the triangles. If necessary, give the direction to apply the Pythagorean theorem, in particular. Then, have a discussion with students about their recollection of the theorem. Basic points should include the theorem, the converse of the theorem, and the fact that when the theorem leads them to an answer of the form $c^2 = x^2$, then $c = x$ (perfect squares).
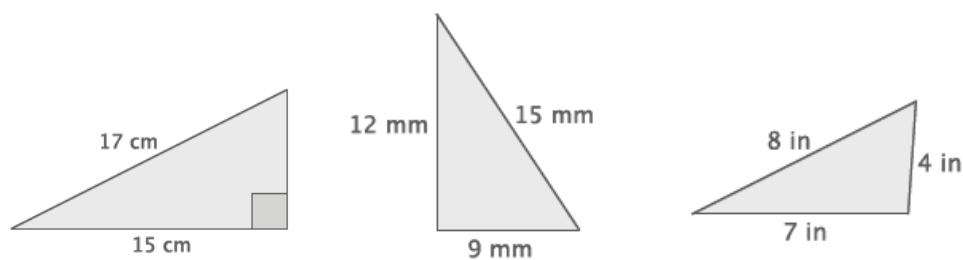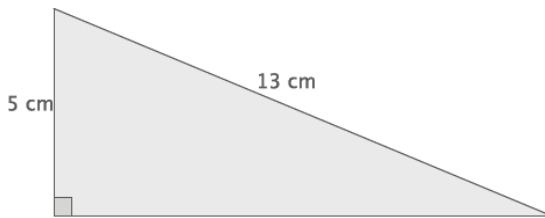

In the first triangle, students are required to use the Pythagorean theorem to determine the unknown side length. Let the unknown side have a length of x cm. Then, x is 8 because, by the Pythagorean theorem, $17^2 - 15^2 = x^2$, and $64 = x^2$. Since 64 is a perfect square, then students should identify the length of the unknown side as 8 cm. In the second triangle, students are required to use the converse of the Pythagorean theorem to determine that it is a right triangle. In the third triangle, students are required to use the converse of the Pythagorean theorem to determine that it is not a right triangle.

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Example 1 (3 minutes)

- Recall the Pythagorean theorem and its converse for right triangles.

  □ *The Pythagorean theorem states that a right triangle with leg lengths a and b and hypotenuse c will satisfy $a^2 + b^2 = c^2$. The converse of the theorem states that if a triangle with side lengths a, b, and c satisfies the equation $a^2 + b^2 = c^2$, then the triangle is a right triangle.*

Example 1

Write an equation that allows you to determine the length of the unknown side of the right triangle.

13 cm

5 cm

- Write an equation that allows you to determine the length of the unknown side of the right triangle.

Note: Students may use a different symbol to represent the unknown side length.

  □ *Let b cm represent the unknown side length. Then, $5^2 + b^2 = 13^2$.*

Scaffolding:

Consider using cutouts of the triangles in this lesson to further illustrate the difference between triangles with whole number hypotenuses and those without. Then, call on students to measure the lengths directly for Examples 1–3. Cutouts drawn to scale are provided at the end of the lesson.

Verify that students wrote the correct equation; then, allow them to solve it. Ask them how they knew the correct answer was 12. They should respond that $13^2 - 5^2 = 144$, and since 144 is a perfect square, they knew that the unknown side length must be 12 cm.

Example 2 (5 minutes)

Example 2

Write an equation that allows you to determine the length of the unknown side of the right triangle.

9 cm

4 cm
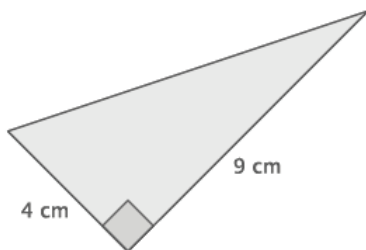
- Write an equation that allows you to determine the length of the unknown side of the right triangle.

  □ *Let c cm represent the length of the hypotenuse. Then, $4^2 + 9^2 = c^2$.*

- There is something different about this triangle. What is the length of the missing side?   If you cannot find the length of the missing side exactly, then find a good approximation.

Provide students time to find an approximation for the length of the unknown side. Select students to share their answers and explain their reasoning. Use the points below to guide their thinking as needed.

- How is this problem different from the last one?
    - *The answer is $c^2 = 97$. Since 97 is not a perfect square, the exact length cannot be represented as an integer.*
- Since 97 is not a perfect square, we cannot determine the exact length of the hypotenuse as an integer; however, we can make an estimate. Think about all of the perfect squares we have seen and calculated in past discussions. The number 97 is between which two perfect squares?
    - *The number 97 is between 81 and 100.*
- If c^2 were 81, what would be the length of the hypotenuse?
    - *The length would be 9 cm.*
- If c^2 were 100, what would be the length of the hypotenuse?
    - *The length would be 10 cm.*
- At this point, we know that the length of the hypotenuse is somewhere between 9 cm and 10 cm. Think about the length to which it is closest. The actual length of the hypotenuse is determined by the equation $c^2 = 97$. To which perfect square number, 100 or 81, is 97 closer?
    - *The number 97 is closer to the perfect square 100 than to the perfect square 81.*
- Now that we know that the length of the hypotenuse of this right triangle is between 9 cm and 10 cm, but closer to 10 cm, let's try to get an even better estimate of the length. Choose a number between 9 and 10 but closer to 10. Square that number. Do this a few times to see how close you can get to the number 97.

Provide students time to check a few numbers between 9 and 10. Students should see that the length is somewhere between 9.8 cm and 9.9 cm because $9.8^2 = 96.04$ and $9.9^2 = 98.01$. Some students may even check 9.85; $9.85^2 = 97.0225$. This activity shows students that an estimation of the length being between 9 cm and 10 cm is indeed accurate, and it helps students develop an intuitive sense of how to estimate square roots.

Example 3 (4 minutes)

Example 3

Write an equation to determine the length of the unknown side of the right triangle.

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- Write an equation to determine the length of the unknown side of the right triangle.
  - *Let c* cm *represent the length of the hypotenuse. Then,* $3^2 + 8^2 = c^2$.

Verify that students wrote the correct equation, and then allow them to solve it. Instruct them to estimate the length, if necessary. Then, let them continue to work. When most students have finished, ask the questions below.

- Could you determine an answer for the length of the hypotenuse as an integer?
  - *No. Since* $c^2 = 73$, *the length of the hypotenuse is not a perfect square.*

Optionally, you can ask, "Can anyone find the exact length of the hypotenuse as a rational number?" It is important that students recognize that no one can determine the exact length of the hypotenuse as a rational number at this point.

- Since 73 is not a perfect square, we cannot determine the exact length of the hypotenuse as a whole number. Let's estimate the length. Between which two whole numbers is the length of the hypotenuse? Explain.
  - *Since* 73 *is between the two perfect squares* 64 *and* 81, *we know the length of the hypotenuse must be between* 8 cm *and* 9 cm.
- Is the length closer to 8 cm or 9 cm? Explain.
  - *The length is closer to* 9 cm *because* 73 *is closer to* 81 *than it is to* 64.
- The length of the hypotenuse is between 8 cm and 9 cm but closer to 9 cm.

Example 4 (8 minutes)

Example 4

In the figure below, we have an equilateral triangle with a height of 10 Inches. What do we know about an equilateral triangle?
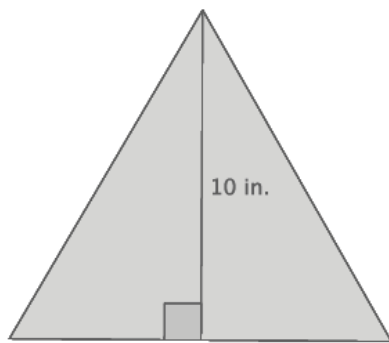

- In the figure below, we have an equilateral triangle with a height of 10 inches. What do we know about an equilateral triangle?
  - *Equilateral triangles have sides that are all of the same length and angles that are all of the same degree, namely* 60°.

▪ Let's say the length of the sides is x inches. Determine the approximate length of the sides of the triangle.
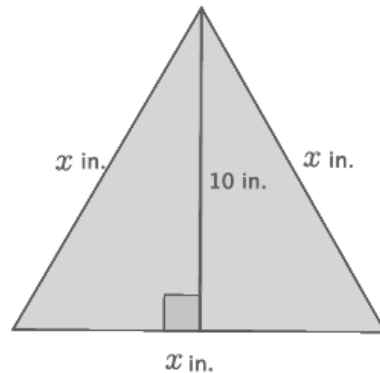
▪ What we actually have here are two congruent right triangles.

Trace one of the right triangles on a transparency, and reflect it across the line representing the height of the triangle to convince students that an equilateral triangle is composed of two congruent right triangles.
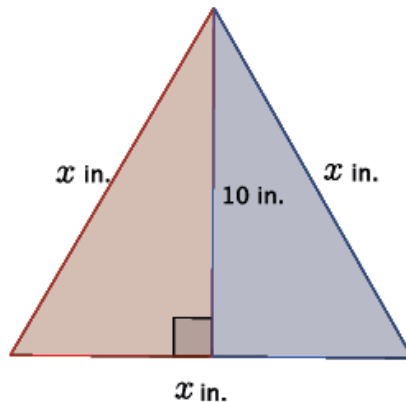
▪ With this knowledge, we need to determine the length of the base of one of the right triangles. If we know that the length of the base of the equilateral triangle is x inches, then what is the length of the base of one of the right triangles? Explain.

Lesson 1:   The Pythagorean Theorem

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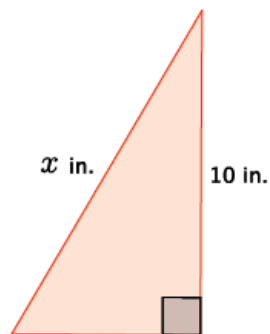
- □ *The length of the base of one of the right triangles must be $\frac{1}{2}x$ inches because the equilateral triangle has a base of length x inches. Since the equilateral triangle is composed of two congruent right triangles, we know that the base of each of the right triangles is of the same length (reflections preserve lengths of segments). Therefore, each right triangle has a base length of $\frac{1}{2}x$ inches*

- ▪ Now that we know the length of the base of the right triangle, write an equation for this triangle using the Pythagorean theorem.
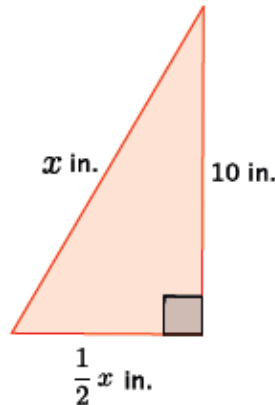
- □ $\left(\frac{1}{2}x\right)^2 + 10^2 = x^2$

Verify that students wrote the correct equation, and then ask students to explain the meaning of each term of the equation. Allow students time to solve the equation in pairs or small groups. Instruct them to make an estimate of the length, if necessary. Then, let them continue to work. When most students have finished, continue with the discussion below.

- ▪ Explain your thinking about this problem. What did you do with the equation $\left(\frac{1}{2}x\right)^2 + 10^2 = x^2$?

If students are stuck, ask them questions that help them work through the computations below. For example, you can ask them what they recall about the laws of exponents to simplify the term $\left(\frac{1}{2}x\right)^2$ or how to use the properties of equality to get the answer in the form of x^2 equal to a constant.

- □ *We had to solve for x:*

$$\left(\frac{1}{2}x\right)^2 + 10^2 = x^2$$
$$\frac{1}{4}x^2 + 100 = x^2$$
$$\frac{1}{4}x^2 - \frac{1}{4}x^2 + 100 = x^2 - \frac{1}{4}x^2$$
$$100 = \frac{3}{4}x^2$$
$$\frac{400}{3} = x^2$$
$$133.3 \approx x^2$$

- Now that we know that $x^2 \approx 133.3$, find a whole number estimate for the length x inches. Explain your thinking.

  □ *The value of x is approximately* 12*, which means that the equilateral triangle has side lengths of approximately* 12 *inches The number* 133.3 *is between the perfect squares* 121 *and* 144*. Since* 133.3 *is closer to* 144 *than* 121*, we know that the value of x is between* 11 *and* 12 *but closer to* 12*.*

Exercises 1–3 (7 minutes)

Students complete Exercises 1–3 independently.

Exercises

1. Use the Pythagorean theorem to find a whole number estimate of the length of the unknown side of the right triangle. Explain why your estimate makes sense.

6 cm

11 cm
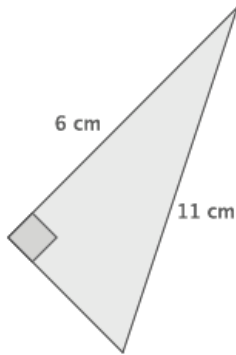
Let x cm be the length of the unknown side.

$$6^2 + x^2 = 11^2$$
$$36 + x^2 = 121$$
$$x^2 = 85$$

The length of the unknown side of the triangle is approximately 9 *cm. The number* 85 *is between the perfect squares* 81 *and* 100*. Since* 85 *is closer to* 81 *than* 100*, then the length of the unknown side of the triangle is closer to* 9 *cm than it is to* 10 *cm.*

2. Use the Pythagorean theorem to find a whole number estimate of the length of the unknown side of the right triangle. Explain why your estimate makes sense.

6 in

10 in

Let c in. be the length of the hypotenuse.

$$6^2 + 10^2 = c^2$$
$$36 + 100 = c^2$$
$$136 = c^2$$

The length of the hypotenuse is approximately 12 *in. The number* 136 *is between the perfect squares* 121 *and* 144*. Since* 136 *is closer to* 144 *than* 121*, the length of the unknown side of the triangle is closer to* 12 *in. than it is to* 11 *in.*

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3. Use the Pythagorean theorem to find a whole number estimate of the length of the unknown side of the right triangle. Explain why your estimate makes sense.

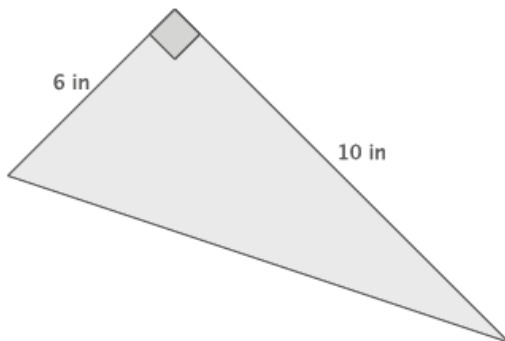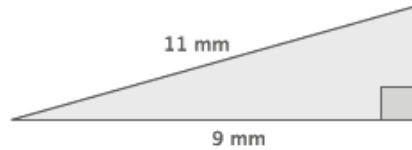*Let x mm be the length of the unknown side.*

$$9^2 + x^2 = 11^2$$
$$81 + x^2 = 121$$
$$x^2 = 40$$

The length of the hypotenuse is approximately 6 mm. The number 40 is between the perfect squares 36 and 49. Since 40 is closer to 36 than 49, then the length of the unknown side of the triangle is closer to 6 mm than it is to 7 mm.

(Triangle diagram: labeled 11 mm on the upper side and 9 mm on the bottom side, with a right angle mark.)

Discussion (3 minutes)

- Our estimates for the lengths in the problems in this lesson are acceptable, but we can do better. Instead of saying that a length is between two particular whole numbers and closer to one compared to the other, we will soon learn how to make more precise estimates.

- Obviously, since the lengths are been between two integers (e.g., between 8 and 9), we need to look at the numbers between the integers: the rational numbers (fractions). That means we need to learn more in general about rational numbers and all numbers between the integers on the number line.

- The examination of those numbers is the focus of the next several lessons.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know what a perfect square is.

- We know that when the square of the length of an unknown side of a triangle is not equal to a perfect square, we can estimate the side length by determining which two perfect squares the square of the length is between.

- We know that we need to look more closely at the rational numbers in order to make better estimates of the lengths of unknown sides of a right triangle.

Lesson Summary

Perfect square numbers are those that are a product of an integer factor multiplied by itself. For example, the number 25 is a perfect square number because it is the product of 5 multiplied by 5.

When the square of the length of an unknown side of a right triangle is not equal to a perfect square, you can estimate the length as a whole number by determining which two perfect squares the square of the length is between.

Example:

Let c in. represent the length of the hypotenuse. Then,

$$3^2 + 7^2 = c^2$$
$$9 + 49 = c^2$$
$$58 = c^2.$$

The number 58 is not a perfect square, but it is between the perfect squares 49 and 64. Therefore, the length of the hypotenuse is between 7 in. and 8 in. but closer to 8 in. because 58 is closer to the perfect square 64 than it is to the perfect square 49.

Exit Ticket (5 minutes)

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Name _____   Date _____

Lesson 1: The Pythagorean Theorem

Exit Ticket

1. Determine the length of the unknown side of the right triangle. If you cannot determine the length exactly, then determine which two integers the length is between and the integer to which it is closest.
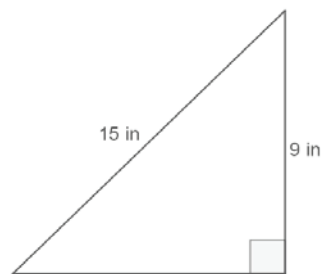
2. Determine the length of the unknown side of the right triangle. If you cannot determine the length exactly, then determine which two integers the length is between and the integer to which it is closest.
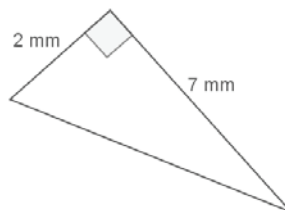
Exit Ticket Sample Solutions

1. Determine the length of the unknown side of the right triangle. If you cannot determine the length exactly, then determine which two integers the length is between and the integer to which it is closest.

Let x in. be the length of the unknown side.

$$9^2 + x^2 = 15^2$$
$$81 + x^2 = 225$$
$$x^2 = 144$$
$$x = 12$$

The length of the unknown side is 12 in. The Pythagorean theorem led me to the fact that the square of the value of the unknown length is 144. We know 144 is a perfect square, and 144 is equal to 12^2; therefore, $x = 12$, and the unknown length of the triangle is 12 in.
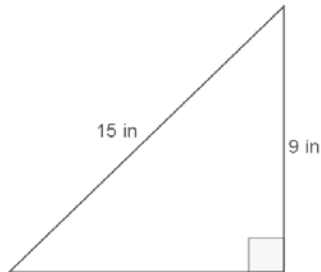
2. Determine the length of the unknown side of the right triangle. If you cannot determine the length exactly, then determine which two integers the length is between and the integer to which it is closest.

Let x mm be the length of the unknown side.

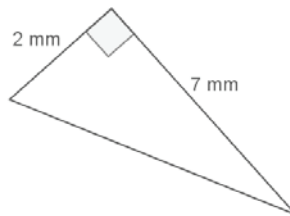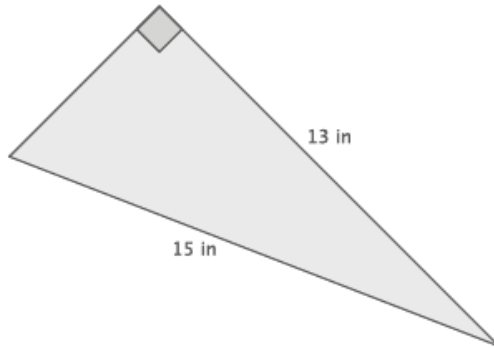$$2^2 + 7^2 = x^2$$
$$4 + 49 = x^2$$
$$53 = x^2$$

The number 53 is between the perfect squares 49 and 64. Since 53 is closer to 49 than 64, the length of the unknown side of the triangle is closer to 7 mm than 8 mm.

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Problem Set Sample Solutions

1. Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.

Let x in. be the length of the unknown side.

$$13^2 + x^2 = 15^2$$
$$169 + x^2 = 225$$
$$x^2 = 56$$

The number 56 is between the perfect squares 49 and 64. Since 56 is closer to 49 than it is to 64, the length of the unknown side of the triangle is closer to 7 in. than it is to 8 in.

2. Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.

Let x cm be the length of the unknown side.

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 = 25$$
$$x = 5$$

The length of the unknown side is 5 cm. The Pythagorean theorem led me to the fact that the square of the value of the unknown length is 25. Since 25 is a perfect square, 25 is equal to 5^2; therefore, $x = 5$, and the unknown length of the triangle is 5 cm.

3. Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.

Let c in. be the length of the hypotenuse.

$$4^2 + 12^2 = c^2$$
$$16 + 144 = c^2$$
$$160 = c^2$$

The number 160 is between the perfect squares 144 and 169. Since 160 is closer to 169 than it is to 144, the length of the hypotenuse of the triangle is closer to 13 in. than it is to 12 in.

4. Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.

Let x cm be the length of the unknown side.

$$x^2 + 11^2 = 13^2$$
$$x^2 + 121 = 169$$
$$x^2 = 48$$

The number 48 is between the perfect squares 36 and 49. Since 48 is closer to 49 than it is to 36, the length of the unknown side of the triangle is closer to 7 cm than it is to 6 cm.

5. Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.
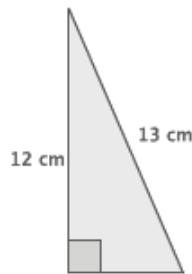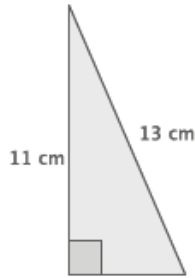
Let c in. be the length of the hypotenuse.

$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$100 = c^2$$
$$10 = c$$

The length of the hypotenuse is 10 in. The Pythagorean theorem led me to the fact that the square of the value of the unknown length is 100. We know 100 is a perfect square, and 100 is equal to 10^2; therefore, $c = 10$, and the length of the hypotenuse of the triangle is 10 in.

6. Determine the length of the unknown side of the right triangle. Explain how you know your answer is correct.

Let c cm be the length of the hypotenuse.

$$7^2 + 4^2 = c^2$$
$$49 + 16 = c^2$$
$$65 = c^2$$

The number 65 is between the perfect squares 64 and 81. Since 65 is closer to 64 than it is to 81, the length of the hypotenuse of the triangle is closer to 8 cm than it is to 9 cm.

7. Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.
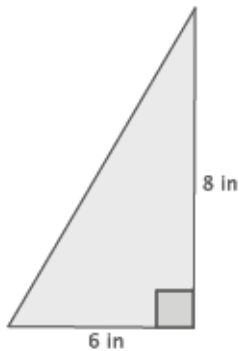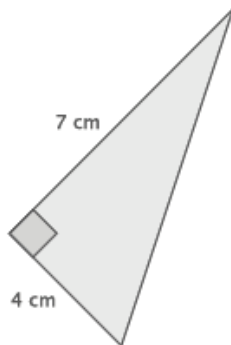

Let x mm be the length of the unknown side.

$$3^2 + x^2 = 12^2$$
$$9 + x^2 = 144$$
$$x^2 = 135$$

The number 135 is between the perfect squares 121 and 144. Since 135 is closer to 144 than it is to 121, the length of the unknown side of the triangle is closer to 12 mm than it is to 11 mm.

8. The triangle below is an isosceles triangle. Use what you know about the Pythagorean theorem to determine the approximate length of the base of the isosceles triangle.
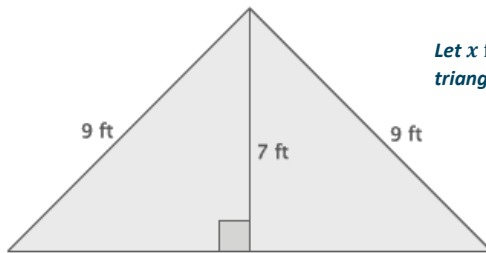
Let x ft. represent the length of the base of one of the right triangles of the isosceles triangle.

$$x^2 + 7^2 = 9^2$$
$$x^2 + 49 = 81$$
$$x^2 = 32$$

Since 32 is between the perfect squares 25 and 36 but closer to 36, the approximate length of the base of the right triangle is 6 ft. Since there are two right triangles, the length of the base of the isosceles triangle is approximately 12 ft.

9. Give an estimate for the area of the triangle shown below. Explain why it is a good estimate.

Let x cm represent the length of the base of the right triangle.

$$x^2 + 3^2 = 7^2$$
$$x^2 + 9 = 49$$
$$x^2 = 40$$

Since 40 is between the perfect squares 36 and 49 but closer to 36, the approximate length of the base is 6 cm.

$$A = \frac{1}{2}(6)(3) = 9$$

So, the approximate area of the triangle is 9 cm^2. This is a good estimate because of the approximation of the length of the base. Further, since the hypotenuse is the longest side of the right triangle, approximating the length of the base as 6 cm makes mathematical sense because it has to be shorter than the hypotenuse.

Example 1

Example 2

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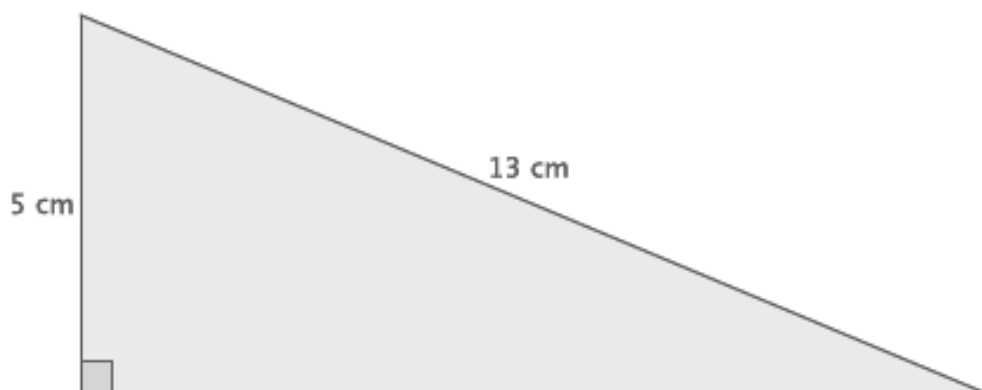
Example 3
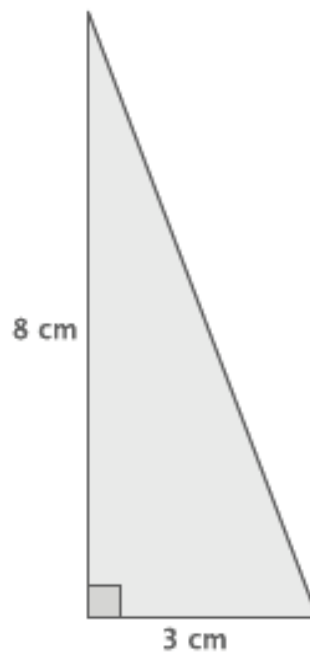
Lesson 2: Square Roots

Student Outcomes

- Students are introduced to the notation for square roots.
- Students approximate the location of square roots of whole numbers on the number line.

Classwork

Discussion (10 minutes)

As an option, the discussion can be framed as a challenge. Distribute compasses, and ask students, "How can we determine an estimate for the length of the diagonal of the unit square?"

- Consider a unit square, a square with side lengths equal to 1. How can we determine the length of the diagonal, s, of the unit square?
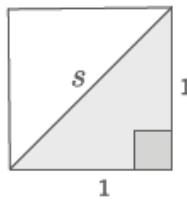
  - *We can use the Pythagorean theorem to determine the length of the diagonal.*

$$1^2 + 1^2 = s^2$$
$$2 = s^2$$

- What number, s, times itself is equal to 2?
  - *We don't know exactly, but we know the number has to be between 1 and 2.*

- We can show that the number must be between 1 and 2 if we place the unit square on a number line as shown. Then, consider a circle with center O and radius equal to the length of the hypotenuse, segment OA, of the triangle.
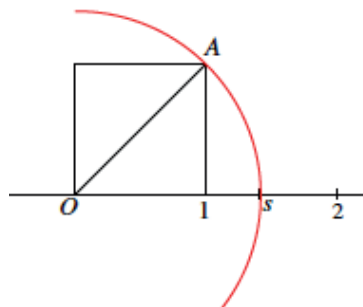
> *Scaffolding:*
>
> Depending on students' experience, it may be useful to review or teach the concept of square numbers and perfect squares.

- We can see that length OA is somewhere between 1 and 2 but precisely at point s. But what is that number s?

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- From our work with exponents, specifically squared numbers, we know that 2 is not a perfect square. Thus, the length of the diagonal must be between the two integers 1 and 2, and that is confirmed on the number line. To determine the number s, we should look at that part of the number line more closely. To do so, we need to discuss what kinds of numbers lie between the integers on a number line. What do we already know about those numbers?

Lead a discussion about the types of numbers found between the integers on a number line. Students should identify that rational numbers, such as fractions and decimals, lie between the integers. Have students give concrete examples of numbers found between the integers 1 and 2. Consider asking students to write a rational number, x, so that $1 < x < 2$, on a sticky note and then to place it on a number line drawn on a poster or white board. At the end of this part of the discussion, make clear that all of the numbers students identified are rational and in the familiar forms of fractions, mixed numbers, and decimals. Then, continue with the discussion below about square roots.

> *Scaffolding:*
>
> Students may benefit from an oral recitation of square roots of perfect squares here and throughout the module. Consider some repeated *quick practice*, calling out, for example: "What is the square root of 81?" and "What is the square root of 100?" Ask for choral or individual responses.

- There are other numbers on the number line between the integers. Some of the square roots of whole numbers are equal to whole numbers, but most lie between the integers on the number line. A positive number whose square is equal to a positive number b is denoted by the symbol $\sqrt{b}$. The symbol $\sqrt{b}$ automatically denotes a positive number (e.g., $\sqrt{4}$ is always 2, not -2). The number $\sqrt{b}$ is called *a positive square root of b*. We will soon learn that it is *the* positive square root (i.e., there is only one).

- What is $\sqrt{25}$, that is, the positive square root of 25? Explain.
  - *The positive square root of 25 is 5 because $5^2 = 25$.*
- What is $\sqrt{9}$, that is, the positive square root of 9? Explain.
  - *The positive square root of 9 is 3 because $3^2 = 9$.*

> *Scaffolding:*
>
> If students are struggling with the concept of a square root, it may help to refer to visuals that relate numbers and their squares. Showing this visual
>
>
>
> and asking questions (e.g., "What is the square root of 9?") builds students' understanding of square roots through their understanding of squares.
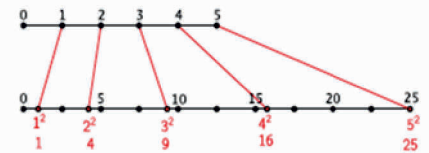
Exercises 1–4 (5 minutes)

Students complete Exercises 1–4 independently.

Exercises 1–4

1. Determine the positive square root of 81, if it exists. Explain.

   The square root of 81 is 9 because $9^2 = 81$.

2. Determine the positive square root of 225, if it exists. Explain.

   The square root of 225 is 15 because $15^2 = 225$.

3. Determine the positive square root of -36, if it exists. Explain.

   The number -36 does not have a square root because there is no number squared that can produce a negative number.

4. Determine the positive square root of 49, if it exists. Explain.

   The square root of 49 is 7 because $7^2 = 49$.

Discussion (15 minutes)

- Now, back to our unit square. We said that the length of the diagonal is s, and $s^2 = 2$. Now that we know about square roots, we can say that the length of s is $\sqrt{2}$ and that the number $\sqrt{2}$ is between integers 1 and 2. Let's look at the number line more generally to see if we can estimate the value of $\sqrt{2}$.

- Take a number line from 0 to 4:
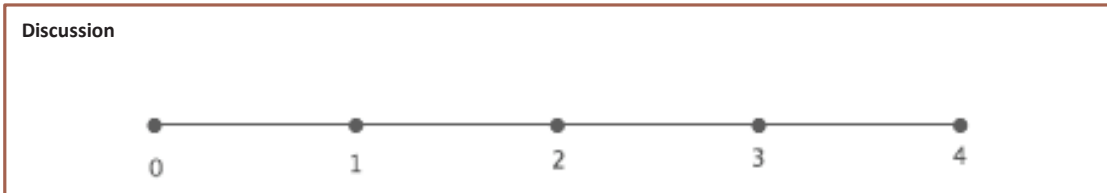
- Place the numbers $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, and $\sqrt{16}$ on the number line, and explain how you knew where to place them.

Solutions are shown below in red.

- Place the numbers $\sqrt{2}$ and $\sqrt{3}$ on the number line. Be prepared to explain your reasoning.

Solutions are shown below in red. Students should reason that the numbers $\sqrt{2}$ and $\sqrt{3}$ belong on the number line between $\sqrt{1}$ and $\sqrt{4}$. They might be more specific and suggest that the numbers $\sqrt{2}$ and $\sqrt{3}$ sit equally spaced in the interval between 1 and 2. This idea suggests that $1\frac{1}{3}$ might be a good approximation for $\sqrt{2}$ and $1\frac{2}{3}$ for $\sqrt{3}$. Of course, this suggested spacing is just speculation for now.
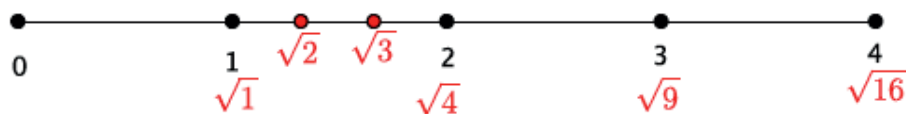
- Place the numbers $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{8}$ on the number line. Be prepared to explain your reasoning.

Solutions are shown below in red. The discussion about placement should be similar to the previous one.

Lesson 2:   Square Roots

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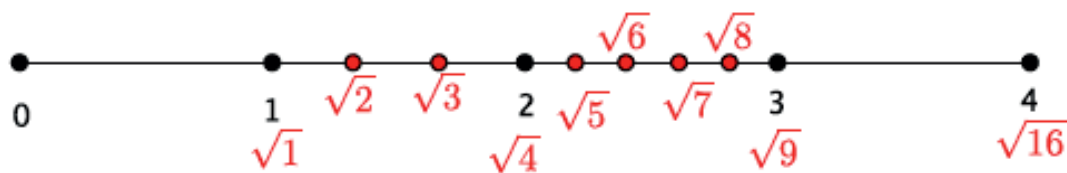
▪ Place the numbers $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, and $\sqrt{15}$ on the number line. Be prepared to explain your reasoning.

Solutions are shown below in red. The discussion about placement should be similar to the previous one.
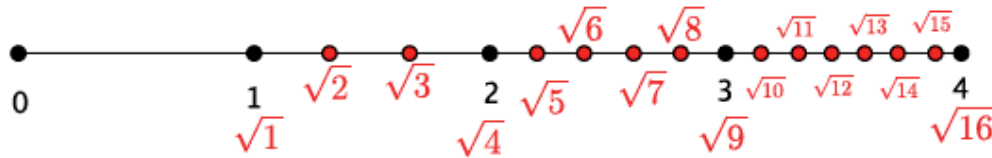
▪ Our work on the number line shows that there are many more square roots of whole numbers that are not perfect squares than those that are perfect squares. On the number line above, we have four perfect square numbers and twelve that are not! After we do some more work with roots, in general, we will cover exactly how to describe these numbers and how to approximate their values with greater precision. For now, we will estimate their locations on the number line using what we know about perfect squares.

Exercises 5–9 (5 minutes)

Students complete Exercises 5–9 independently. Calculators may be used for approximations.

Exercises 5–9

Determine the positive square root of the number given. If the number is not a perfect square, determine which whole number the square root would be closest to, and then use *guess and check* to give an approximate answer to one or two decimal places.

5. $\sqrt{49}$

    7

6. $\sqrt{62}$

    The square root of 62 is close to 8. The square root of 62 is approximately 7.9 because $7.9^2 = 62.41$.

7. $\sqrt{122}$

    The square root of 122 is close to 11. Students may guess a number between 11 and 11.1 because $11.05^2 = 122.1025$.

8. $\sqrt{400}$

    20

9. Which of the numbers in Exercises 5–8 are not perfect squares? Explain.

    The numbers 62 and 122 are not perfect squares because there is no integer x to satisfy $x^2 = 62$ or $x^2 = 122$.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that there are numbers on the number line between the integers. The ones we looked at in this lesson are square roots of whole numbers that are not perfect squares.

- We know that when a positive number x is squared and the result is b, then $\sqrt{b}$ is equal to x.

- We know how to approximate the square root of a whole number and its location on a number line by figuring out which two perfect squares it is between.

Lesson Summary

A positive number whose square is equal to a positive number b is denoted by the symbol $\sqrt{b}$. The symbol $\sqrt{b}$ automatically denotes a positive number. For example, $\sqrt{4}$ is always 2, not -2. The number $\sqrt{b}$ is called *a positive square root of b*.

The square root of a perfect square of a whole number is that whole number. However, there are many whole numbers that are not perfect squares.

Exit Ticket (5 minutes)

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Name _____   Date _____

Lesson 2: Square Roots

Exit Ticket

1. Write the positive square root of a number x in symbolic notation.

2. Determine the positive square root of 196. Explain.

3. The positive square root of 50 is not an integer. Which whole number does the value of $\sqrt{50}$ lie closest to? Explain.

4. Place the following numbers on the number line in approximately the correct positions: $\sqrt{16}$, $\sqrt{9}$, $\sqrt{11}$, and 3.5.

Exit Ticket Sample Solutions

1. Write the square root of a number x in symbolic notation.

   $\sqrt{x}$

2. Determine the positive square root of 196. Explain.

   $\sqrt{196} = 14$ *because* $14^2 = 196$.

3. The positive square root of 50 is not an integer. Which whole number does the value $\sqrt{50}$ lie closest to? Explain.

   $\sqrt{50}$ *is between 7 and 8 but closer to 7. The reason is that* $7^2 = 49$, *and* $8^2 = 64$. *The number 50 is between 49 and 64 but closer to 49. Therefore, the square root of 50 is close to 7.*

4. Place the following numbers on the number line in approximately the correct positions: $\sqrt{16}, \sqrt{9}, \sqrt{11}$, and 3.5.

   Solutions are shown in red below.
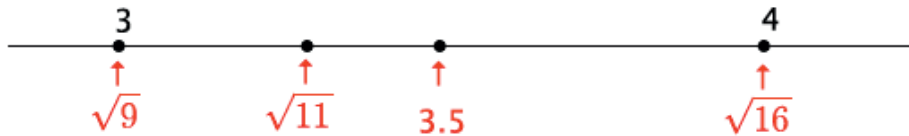
Problem Set Sample Solutions

Determine the positive square root of the number given. If the number is not a perfect square, determine the integer to which the square root would be closest.

1. $\sqrt{169}$

   13

2. $\sqrt{256}$

   16

3. $\sqrt{81}$

   9

4. $\sqrt{147}$

   The number 147 is not a perfect square. It is between the perfect squares 144 and 169 but closer to 144. Therefore, the square root of 147 is close to 12.

5. $\sqrt{8}$

   The number 8 is not a perfect square. It is between the perfect squares 4 and 9 but closer to 9. Therefore, the square root of 8 is close to 3.

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6. Which of the numbers in Problems 1–5 are not perfect squares? Explain.

    The numbers 147 and 8 are not perfect squares because there is no integer x so that $x^2 = 147$ or $x^2 = 8$.

7. Place the following list of numbers in their approximate locations on a number line.

    $$\sqrt{32},\ \sqrt{12},\ \sqrt{27},\ \sqrt{18},\ \sqrt{23},\ \text{and}\ \sqrt{50}$$
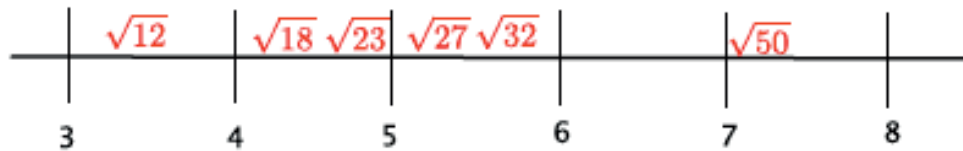
    Answers are noted in red.

8. Between which two integers will $\sqrt{45}$ be located? Explain how you know.

    The number 45 is not a perfect square. It is between the perfect squares 36 and 49 but closer to 49. Therefore, the square root of 45 is between the integers 6 and 7 because $\sqrt{36} = 6$ and $\sqrt{49} = 7$ and $\sqrt{36} < \sqrt{45} < \sqrt{49}$.

Lesson 3: Existence and Uniqueness of Square Roots and Cube Roots

Student Outcomes

- Students know that the positive square root and the cube root exist for all positive numbers and both a square root of a number and a cube root of a number are unique.
- Students solve simple equations that require them to find the square root or cube root of a number.

Lesson Notes

This lesson has two options for showing the existence and uniqueness of positive square roots and cube roots. Each option has an Opening Exercise and a Discussion that follows. The first option has students explore facts about numbers on a number line, leading to an understanding of the trichotomy law, followed by a discussion of how the law applies to squares of numbers, which should give students a better understanding of what square roots and cube roots are and how they are unique. The second option explores numbers and their squares via a Find the Rule exercise, followed by a discussion that explores how square roots and cube roots are unique. The first option includes a discussion of the basic inequality property, a property referred to in subsequent lessons. The basic inequality property states that if x, y, w, and z are positive numbers so that $x < y$ and $w < z$, then $xw < yz$. Further, if $x = w$ and $y = z$, when $x < y$, then $x^2 < y^2$. Once the first or second option is completed, the lesson continues with a discussion of how to solve equations using square roots.

Throughout this and subsequent lessons, we ask students to find only the positive values of x that satisfy an equation containing squared values of x. The reason is that students will not learn how to solve these problems completely until Algebra I. Consider the following equation and solution below:

$$4x^2 + 12x + 9 = 49$$
$$(2x + 3)^2 = 49$$
$$\sqrt{(2x + 3)^2} = \sqrt{49}$$
$$2x + 3 = \pm 7$$

$$2x + 3 = 7 \quad \text{or} \quad 2x + 3 = -7$$
$$x = 2 \qquad\qquad x = -5$$

At this point, students have not learned how to factor quadratics and will solve all equations using the square root symbol, which means students are only responsible for finding the positive solution(s) to an equation.

Classwork

Opening (4 minutes): Option 1

Ask students the following to prepare for the discussion that follows.

- Considering only the positive integers, if $x^2 = 4$, what must x be to make the statement a true number sentence? Could x be any other number?
  - *x is the number 2 and cannot be any other number.*

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- Can you find a pair of positive numbers c and d so that $c < d$ and $c^2 < d^2$?
  - *Yes! Answers will vary.*
- Can you find a pair of positive numbers c and d so that $c < d$ and $c^2 = d^2$?
  - *No*
- Can you find a pair of positive numbers c and d so that $c < d$ and $c^2 > d^2$?
  - *No*

Discussion (9 minutes): Option 1

(An alternative discussion is provided below.) Once this discussion is complete, continue with the discussion that begins on page 40.

- We will soon be solving equations that include roots. For this reason, we want to be sure that the answer we get when we simplify is correct. Specifically, we want to be sure that we can get an answer, that it exists, that the answer we get is correct, and that it is unique to the given situation.

- To this end, existence requires us to show that given a positive number b and a positive integer n, there is one and only one positive number c, so that when $c^n = b$, we say "c is the positive n^{th} root of b." When $n = 2$, we say "c is the positive square root of b" and when $n = 3$, we say "c is the positive cube root of b." Uniqueness requires us to show that given two positive numbers c and d and $n = 2$: If $c^2 = d^2$, then $c = d$. This statement implies uniqueness because both c and d are the positive square root of b, that is, $c^2 = b$ and $d^2 = b$; since $c^2 = d^2$, then $c = d$. Similarly, when $n = 3$, if $c^3 = d^3$, then $c = d$. The reasoning is the same since both c and d are the positive cube root of b, that is, $c^3 = b$ and $d^3 = b$; since $c^3 = d^3$, then $c = d$. Showing uniqueness also shows existence, so we focus on proving the uniqueness of square and cube roots.

> *Scaffolding:*
>
> The number line can help students make sense of the trichotomy law. Give students two numbers, and ask them which of the three possibilities for those numbers is true. In each case, only one of three is true for any pair of numbers. For example, given the numbers $c = 2$ and $d = 3$, which of the following is true: $c = d$, $c < d$, or $c > d$?
>
>

- To show that $c = d$, we use the trichotomy law. The trichotomy law states that given two numbers c and d, one and only one of the following three possibilities is true.

  (i) $c = d$

  (ii) $c < d$

  (iii) $c > d$

  We show $c = d$ by showing that $c < d$ and $c > d$ cannot be true.

- If x, y, w, and z are positive numbers so that $x < y$ and $w < z$, is it true that $xw < yz$? Explain.
  - *Yes, it is true that $xw < yz$. Since all of the numbers are positive and both x and w are less than y and z, respectively, then their product must also be less. For example, since $3 < 4$ and $5 < 6$, then $3 \times 5 < 4 \times 6$.*

- This basic inequality property can also be explained in terms of areas of a rectangle. The picture on the right clearly shows that when $x < y$ and $w < z$, then $xw < yz$.

- We use this fact to show that $c < d$ and $c > d$ cannot be true. We begin with $c^n < d^n$ when $n = 2$. By the basic inequality, $c^2 < d^2$. Now, we look at the case where $n = 3$. We use the fact that $c^2 < d^2$ to show $c^3 < d^3$. What can we do to show $c^3 < d^3$?

  □ *We can multiply c^2 by c and d^2 by d. The basic inequality guarantees that since $c < d$ and $c^2 < d^2$, that $c^2 \times c < d^2 \times d$, which is the same as $c^3 < d^3$.*

- Using $c^3 < d^3$, how can we show $c^4 < d^4$?

  □ *We can multiply c^3 by c and d^3 by d. The basic inequality guarantees that since $c < d$ and $c^3 < d^3$, $c^3 \times c < d^3 \times d$, which is the same as $c^4 < d^4$.*

- We can use the same reasoning for any positive integer n. We can use similar reasoning to show that if $c > d$, then $c^n > d^n$ for any positive integer n.

- Recall that we are trying to show that if $c^n = d^n$, then $c = d$ for $n = 2$ or $n = 3$. If we assume that $c < d$, then we know that $c^n < d^n$, which contradicts our hypothesis of $c^n = d^n$. By the same reasoning, if $c > d$, then $c^n > d^n$, which is also a contradiction of the hypothesis. By the trichotomy law, the only possibility left is that $c = d$. Therefore, we have shown that the square root or cube root of a number is unique and also exists.

Opening (6 minutes): Option 2

Begin by having students *find the rule* given numbers in two columns. The goal is for students to see the relationship between the square of a number and its square root and the cube of a number and its cube root. Students have to figure out the rule, find the missing values in the columns, and then explain their reasoning. Provide time for students to do this independently. If necessary, allow students to work in pairs.

- The numbers in each column are related. Your goal is to determine how they are related, determine which numbers belong in the blank parts of the columns, and write an explanation for how you know the numbers belong there.

Opening

Find the Rule Part 1

1	1
2	4
3	9
9	81
11	121
15	225
7	49
10	100
12	144
13	169
m	m^2
$\sqrt{n}$	n

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Note: Students will not know how to write the cube root of a number using the proper notation, but this activity will be a good way to launch into the discussion below.

<div style="border:1px solid">

Find the Rule Part 2

1	1
2	8
3	27
5	125
6	216
11	1,331
4	64
10	1,000
7	343
14	2,744
p	p^3
$\sqrt[3]{q}$	q

</div>

Discussion (7 minutes): Option 2

Once the Find the Rule exercise is finished, use the discussion points below; then, continue with the Discussion that follows on page 39.

- For Find the Rule Part 1, how were you able to determine which number belonged in the blank?
  - *To find the numbers that belonged in the blanks in the right column, I had to square the number in the left column. To find the numbers that belonged in the left column, I had to take the square root of the number in the right column.*
- When given the number m in the left column, how did we know the number that belonged in the right column?
  - *Given m on the left, the number that belonged on the right was m^2.*
- When given the number n in the right column, how did we know the number that belonged in the left column?
  - *Given n on the right, the number that belonged on the left was $\sqrt{n}$.*
- For Find the Rule Part 2, how were you able to determine which number belonged in the blank?
  - *To find the number that belonged in the blank in the right column, I had to multiply the number in the left column by itself three times. To find the number that belonged in the left column, I had to figure out which number multiplied by itself three times equaled the number that was in the right column.*
- When given the number p in the left column, how did we note the number that belonged in the right column?
  - *Given p on the left, the number that belonged on the right was p^3.*
- When given the number q in the right column, the notation we use to denote the number that belongs in the left column is similar to the notation we used to denote the square root. Given the number q in the right column, we write $\sqrt[3]{q}$ in the left column. The 3 in the notation shows that we must find the number that multiplied by itself 3 times is equal to q.

- Were you able to write more than one number in any of the blanks?
  - *No, there was only one number that worked.*
- Were there any blanks that could not be filled?
  - *No, in each case there was a number that worked.*
- For Find the Rule Part 1, you were working with squared numbers and square roots. For Find the Rule Part 2, you were working with cubed numbers and cube roots. Just like we have perfect squares, there are also perfect cubes. For example, 27 is a perfect cube because it is the product of 3^3. For Find the Rule Part 2, you cubed the number on the left to fill the blank on the right and took the cube root of the number on the right to fill the blank on the left.
- We could extend the Find the Rule exercise to include an infinite number of rows, and in each case, we would be able to fill the blanks. Therefore, we can say that positive square roots and cube roots exist. Because only one number worked in each of the blanks, we can say that the positive roots are unique.
- We must learn about square roots and cube roots to solve equations. The properties of equality allow us to add, subtract, multiply, and divide the same number to both sides of an equal sign. We want to extend the properties of equality to include taking the square root and taking the cube root of both sides of an equation.
- Consider the equality $25 = 25$. What happens when we take the square root of both sides of the equal sign? Do we get a true number sentence?
  - *When we take the square root of both sides of the equal sign, we get $5 = 5$. Yes, we get a true number sentence.*
- Consider the equality $27 = 27$. What happens when we take the cube root of both sides of the equal sign? Do we get a true number sentence?
  - *When we take the cube root of both sides of the equal sign, we get $3 = 3$. Yes, we get a true number sentence.*
- At this point, we only know that the properties of equality can extend to those numbers that are perfect squares and perfect cubes, but it is enough to allow us to begin solving equations using square and cube roots.

Discussion (5 minutes)

The properties of equality have been proven for rational numbers, which are central in school mathematics. As we begin to solve equations that require roots, we are confronted with the fact that we may be working with irrational numbers, which have not yet been defined for students. Therefore, we make the assumption that all of the properties of equality for rational numbers are also true for irrational numbers (i.e., the real numbers, as far as computations are concerned). This is sometimes called the fundamental assumption of school mathematics (FASM). In the discussion below, we reference n^{th} roots. You may choose to discuss square and cube roots only.

- In the past, we have determined the length of the missing side of a right triangle, x, when $x^2 = 25$. What is that value, and how did you get the answer?
  - *The value of x is 5 because x^2 means $x \cdot x$. Since $5 \times 5 = 25$, x must be 5.*
- If we did not know that we were trying to find the length of the side of a triangle, then the answer could also be -5 because $-5 \times -5 = 25$. However, because we were trying to determine the length of the side of a triangle, the answer must be positive because a length of -5 does not make sense.
- Now that we know that positive square roots exist and are unique, we can begin solving equations that require roots.

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- When we solve equations that contain roots, we do what we do for all properties of equality (i.e., we apply the operation to both sides of the equal sign). In terms of solving a radical equation, if we assume x is positive, then

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$x = \sqrt{25}$$
$$x = 5.$$

- Explain the first step in solving this equation.

  □ *The first step is to take the square root of both sides of the equation.*

- It is by definition that when we use the symbol $\sqrt{}$, it automatically denotes a positive number; therefore, the solution to this equation is 5. In Algebra I, you learn how to solve equations of this form without using the square root symbol, which means the possible values for x can be both 5 and -5 because $5^2 = 25$ and $(-5)^2 = 25$, but for now, we will only look for the positive solution(s) to our equations.

- The symbol $\sqrt[n]{}$ is called a *radical*. An equation that contains that symbol is referred to as a *radical equation*. So far, we have only worked with square roots (i.e., $n = 2$). Technically, we would denote a square root as $\sqrt[2]{}$, but it is understood that the symbol $\sqrt{}$ alone represents a square root.

- When $n = 3$, then the symbol $\sqrt[3]{}$ is used to denote the cube root of a number. Since $x^3 = x \cdot x \cdot x$, the cube root of x^3 is x (i.e., $\sqrt[3]{x^3} = x$).

- For what value of x is the equation $x^3 = 8$ true?

  □ $x^3 = 8$
  $$\sqrt[3]{x^3} = \sqrt[3]{8}$$
  $$x = \sqrt[3]{8}$$
  $$x = 2$$

- The nth root of a number is denoted by $\sqrt[n]{}$. In the context of our learning, we limit our work with radicals to square and cube roots.

Exercises 1–6 (8 minutes)

Students complete Exercises 1–6 independently. Allow them to use a calculator to check their answers. Also consider showing students how to use the calculator to find the square root of a number.

Exercises

Find the positive value of x that makes each equation true. Check your solution.

1. $x^2 = 169$

   a. Explain the first step in solving this equation.

   The first step is to take the square root of both sides of the equation.

---

    b.   **Solve the equation, and check your answer.**

$$x^2 = 169$$ *Check:*
$$\sqrt{x^2} = \sqrt{169}$$
$$13^2 = 169$$
$$x = \sqrt{169}$$
$$169 = 169$$
$$x = 13$$

2.   **A square-shaped park has an area of 324 yd^2. What are the dimensions of the park? Write and solve an equation.**

$$x^2 = 324$$ *Check:*
$$\sqrt{x^2} = \sqrt{324}$$
$$18^2 = 324$$
$$x = \sqrt{324}$$
$$324 = 324$$
$$x = 18$$

The square park is 18 yd. in length and 18 yd. in width.

3. $625 = x^2$

$$625 = x^2$$ *Check:*
$$\sqrt{625} = \sqrt{x^2}$$
$$625 = 25^2$$
$$\sqrt{625} = x$$
$$625 = 625$$
$$25 = x$$

4.   **A cube has a volume of 27 in^3. What is the measure of one of its sides? Write and solve an equation.**

$$27 = x^3$$ *Check:*
$$\sqrt[3]{27} = \sqrt[3]{x^3}$$
$$27 = 3^3$$
$$\sqrt[3]{27} = x$$
$$27 = 27$$
$$3 = x$$

The cube has side lengths of 3 in.

5.   **What positive value of x makes the following equation true: $x^2 = 64$? Explain.**

$$x^2 = 64$$ *Check:*
$$\sqrt{x^2} = \sqrt{64}$$
$$8^2 = 64$$
$$x = \sqrt{64}$$
$$64 = 64$$
$$x = 8$$

To solve the equation, I need to find the positive value of x so that when it is squared, it is equal to 64. Therefore, I can take the square root of both sides of the equation. The square root of x^2, $\sqrt{x^2}$, is x because $x^2 = x \cdot x$. The square root of 64, $\sqrt{64}$, is 8 because $64 = 8 \cdot 8$. Therefore, $x = 8$.

6. What positive value of x makes the following equation true: $x^3 = 64$? Explain.

$$x^3 = 64 \qquad \qquad \textit{Check:}$$
$$\sqrt[3]{x^3} = \sqrt[3]{64} \qquad \qquad 4^3 = 64$$
$$x = \sqrt[3]{64} \qquad \qquad 64 = 64$$
$$x = 4$$

To solve the equation, I need to find the positive value of x so that when it is cubed, it is equal to 64. Therefore, I can take the cube root of both sides of the equation. The cube root of x^3, $\sqrt[3]{x^3}$, is x because $x^3 = x \cdot x \cdot x$. The cube root of 64, $\sqrt[3]{64}$, is 4 because $64 = 4 \cdot 4 \cdot 4$. Therefore, $x = 4$.

Discussion (5 minutes)

- Now let's try to identify some roots of some more interesting numbers. Can you determine each of the following?

$$\sqrt{\frac{1}{4}} \qquad \sqrt{\frac{1}{81}} \qquad \sqrt[3]{\frac{8}{27}}$$

Provide students a few minutes to work with a partner to find the roots of the above numbers.

$$\sqrt{\frac{1}{4}} = \frac{1}{2}, \quad \sqrt{\frac{1}{81}} = \frac{1}{9}, \quad \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

- Let's extend this thinking to equations. Consider the equation $x^2 = 25^{-1}$. What is another way to write 25^{-1}?

  □ *The number 25^{-1} is the same as $\frac{1}{25}$.*

- Again, assuming that x is positive, we can solve the equation as before.

$$x^2 = 25^{-1}$$
$$x^2 = \frac{1}{25}$$
$$\sqrt{x^2} = \sqrt{\frac{1}{25}}$$
$$x = \sqrt{\frac{1}{25}}$$
$$x = \frac{1}{5}$$

We know we are correct because $\left(\frac{1}{5}\right)^2 = \frac{1}{25} = 25^{-1}$.

Exercises 7–9 (4 minutes)

Students complete Exercises 7–9 independently. Allow them to use a calculator to check their answers. Also consider showing students how to use the calculator to find the square root of a number.

---

7.   **Find the positive value of x that makes the equation true: $x^2 = 256^{-1}$.**

$$x^2 = 256^{-1}$$ *Check:*

$$\sqrt{x^2} = \sqrt{256^{-1}}$$

$$x = \sqrt{256^{-1}}$$ $$(16^{-1})^2 = 256^{-1}$$

$$16^{-2} = 256^{-1}$$

$$x = \sqrt{\frac{1}{256}}$$ $$\frac{1}{16^2} = 256^{-1}$$

$$\frac{1}{256} = 256^{-1}$$

$$x = \frac{1}{16}$$ $$256^{-1} = 256^{-1}$$

$$x = 16^{-1}$$

8.   **Find the positive value of x that makes the equation true: $x^3 = 343^{-1}$.**

$$x^3 = 343^{-1}$$ *Check:*

$$\sqrt[3]{x^3} = \sqrt[3]{343^{-1}}$$

$$x = \sqrt[3]{343^{-1}}$$ $$(7^{-1})^3 = 343^{-1}$$

$$7^{-3} = 343^{-1}$$

$$x = \sqrt[3]{\frac{1}{343}}$$ $$\frac{1}{7^3} = 343^{-1}$$

$$\frac{1}{343} = 343^{-1}$$

$$x = \frac{1}{7}$$ $$343^{-1} = 343^{-1}$$

$$x = 7^{-1}$$

9.   **Is 6 a solution to the equation $x^2 - 4 = 5x$? Explain why or why not.**

$$6^2 - 4 = 5(6)$$
$$36 - 4 = 30$$
$$32 \neq 30$$

No, 6 is not a solution to the equation $x^2 - 4 = 5x$. When the number is substituted into the equation and simplified, the left side of the equation and the right side of the equation are not equal; in other words, it is not a true number sentence. Since the number 6 does not satisfy the equation, it is not a solution to the equation.

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Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that the positive n^{th} root of a number exists and is unique.
- We know how to solve equations that contain exponents of 2 and 3; we must use square roots and cube roots.

Lesson Summary

The symbol $\sqrt[n]{}$ is called a *radical*. An equation that contains that symbol is referred to as a *radical equation*. So far, we have only worked with square roots (i.e., $n = 2$). Technically, we would denote a positive square root as $\sqrt[2]{}$, but it is understood that the symbol $\sqrt{}$ alone represents a positive square root.

When $n = 3$, then the symbol $\sqrt[3]{}$ is used to denote the cube root of a number. Since $x^3 = x \cdot x \cdot x$, the cube root of x^3 is x (i.e., $\sqrt[3]{x^3} = x$).

The square or cube root of a positive number exists, and there can be only one positive square root or one cube root of the number.

Exit Ticket (5 minutes)

Name _____   Date _____

Lesson 3: Existence and Uniqueness of Square Roots and Cube Roots

Exit Ticket

Find the positive value of x that makes each equation true. Check your solution.

1. $x^2 = 225$

     a. Explain the first step in solving this equation.

     b. Solve and check your solution.

2. $x^3 = 64$

3. $x^2 = 361^{-1}$

4. $x^3 = 1000^{-1}$

Exit Ticket Sample Solutions

Find the positive value of x that makes each equation true. Check your solution.

1. $x^2 = 225$

   a. **Explain the first step in solving this equation.**

   The first step is to take the square root of both sides of the equation.

   b. **Solve and check your solution.**

   $$x^2 = 225$$
   $$\sqrt{x^2} = \sqrt{225}$$
   $$x = \sqrt{225}$$
   $$x = 15$$

   Check:
   $$15^2 = 225$$
   $$225 = 225$$

2. $x^3 = 64$

   $$x^3 = 64$$
   $$\sqrt[3]{x^3} = \sqrt[3]{64}$$
   $$x = \sqrt[3]{64}$$
   $$x = 4$$

   Check:
   $$4^3 = 64$$
   $$64 = 64$$

3. $x^2 = 361^{-1}$

   $$x^2 = 361^{-1}$$
   $$\sqrt{x^2} = \sqrt{361^{-1}}$$
   $$x = \sqrt{361^{-1}}$$
   $$x = \sqrt{\frac{1}{361}}$$
   $$x = \frac{1}{19}$$
   $$x = 19^{-1}$$

   Check:
   $$(19^{-1})^2 = 361^{-1}$$
   $$19^{-2} = 361^{-1}$$
   $$\frac{1}{19^2} = 361^{-1}$$
   $$\frac{1}{361} = 361^{-1}$$
   $$361^{-1} = 361^{-1}$$

4. $x^3 = 1000^{-1}$

   $$x^3 = 1000^{-1}$$
   $$\sqrt[3]{x^3} = \sqrt[3]{1000^{-1}}$$
   $$x = \sqrt[3]{1000^{-1}}$$
   $$x = \sqrt[3]{\frac{1}{1000}}$$
   $$x = \frac{1}{10}$$
   $$x = 10^{-1}$$

   Check:
   $$(10^{-1})^3 = 1000^{-1}$$
   $$10^{-3} = 1000^{-1}$$
   $$\frac{1}{10^3} = 1000^{-1}$$
   $$\frac{1}{1000} = 1000^{-1}$$
   $$1000^{-1} = 1000^{-1}$$

Problem Set Sample Solutions

Find the positive value of x that makes each equation true. Check your solution.

1. What positive value of x makes the following equation true: $x^2 = 289$? Explain.

$$x^2 = 289$$ *Check:*
$$\sqrt{x^2} = \sqrt{289}$$ $$17^2 = 289$$
$$x = \sqrt{289}$$ $$289 = 289$$
$$x = 17$$

   To solve the equation, I need to find the positive value of x so that when it is squared, it is equal to 289. Therefore, I can take the square root of both sides of the equation. The square root of x^2, $\sqrt{x^2}$, is x because $x^2 = x \cdot x$. The square root of 289, $\sqrt{289}$, is 17 because $289 = 17 \cdot 17$. Therefore, $x = 17$.

2. A square-shaped park has an area of 400 yd^2. What are the dimensions of the park? Write and solve an equation.

$$x^2 = 400$$ *Check:*
$$\sqrt{x^2} = \sqrt{400}$$ $$20^2 = 400$$
$$x = \sqrt{400}$$ $$400 = 400$$
$$x = 20$$

   The square park is 20 yd. in length and 20 yd. in width.

3. A cube has a volume of 64 in^3. What is the measure of one of its sides? Write and solve an equation.

$$x^3 = 64$$ *Check:*
$$\sqrt[3]{x^3} = \sqrt[3]{64}$$ $$4^3 = 64$$
$$x = \sqrt[3]{64}$$ $$64 = 64$$
$$x = 4$$

   The cube has a side length of 4 in.

4. What positive value of x makes the following equation true: $125 = x^3$? Explain.

$$125 = x^3$$ *Check:*
$$\sqrt[3]{125} = \sqrt[3]{x^3}$$ $$125 = 5^3$$
$$\sqrt[3]{125} = x$$ $$125 = 125$$
$$5 = x$$

   To solve the equation, I need to find the positive value of x so that when it is cubed, it is equal to 125. Therefore, I can take the cube root of both sides of the equation. The cube root of x^3, $\sqrt[3]{x^3}$, is x because $x^3 = x \cdot x \cdot x$. The cube root of 125, $\sqrt[3]{125}$, is 5 because $125 = 5 \cdot 5 \cdot 5$. Therefore, $x = 5$.

5. Find the positive value of x that makes the equation true: $x^2 = 441^{-1}$.

   a. Explain the first step in solving this equation.

      The first step is to take the square root of both sides of the equation.

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   b.   **Solve and check your solution.**

$$x^2 = 441^{-1}$$

$$\sqrt{x^2} = \sqrt{441^{-1}}$$

$$x = \sqrt{441^{-1}}$$

$$x = \sqrt{\frac{1}{441}}$$

$$x = \frac{1}{21}$$

$$x = 21^{-1}$$

Check:

$$(21^{-1})^2 = 441^{-1}$$

$$21^{-2} = 441^{-1}$$

$$\frac{1}{21^2} = 441^{-1}$$

$$\frac{1}{441} = 441^{-1}$$

$$441^{-1} = 441^{-1}$$

6.   **Find the positive value of x that makes the equation true: $x^3 = 125^{-1}$.**

$$x^3 = 125^{-1}$$

$$\sqrt[3]{x^3} = \sqrt[3]{125^{-1}}$$

$$x = \sqrt[3]{125^{-1}}$$

$$x = \sqrt[3]{\frac{1}{125}}$$

$$x = \frac{1}{5}$$

$$x = 5^{-1}$$

Check:

$$(5^{-1})^3 = 125^{-1}$$

$$5^{-3} = 125^{-1}$$

$$\frac{1}{5^3} = 125^{-1}$$

$$\frac{1}{125} = 125^{-1}$$

$$125^{-1} = 125^{-1}$$

7.   **The area of a square is 196 in^2. What is the length of one side of the square? Write and solve an equation, and then check your solution.**

   Let x in. represent the length of one side of the square.

$$x^2 = 196$$

$$\sqrt{x^2} = \sqrt{196}$$

$$x = \sqrt{196}$$

$$x = 14$$

Check:

$$14^2 = 196$$

$$196 = 196$$

   The length of one side of the square is 14 in.

8.   **The volume of a cube is 729 cm^3. What is the length of one side of the cube? Write and solve an equation, and then check your solution.**

   Let x cm represent the length of one side of the cube.

$$x^3 = 729$$

$$\sqrt[3]{x^3} = \sqrt[3]{729}$$

$$x = \sqrt[3]{729}$$

$$x = 9$$

Check:

$$9^3 = 729$$

$$729 = 729$$

   The length of one side of the cube is 9 cm.

9.   What positive value of x would make the following equation true: $19 + x^2 = 68$?

$$19 + x^2 = 68$$
$$19 - 19 + x^2 = 68 - 19$$
$$x^2 = 49$$
$$x = 7$$

The positive value for x that makes the equation true is 7.

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Lesson 4: Simplifying Square Roots

Student Outcomes

- Students use factors of a number to simplify a square root.

Lesson Notes

This lesson is optional. In this lesson, students learn to simplify square roots by examining the factors of a number and looking specifically for perfect squares. Students must learn how to work with square roots in Grade 8 in preparation for their work in Algebra I and the quadratic formula. Though this lesson is optional, it is strongly recommended that students learn how to work with numbers in radical form in preparation for the work that they do in Algebra I. Throughout the remaining lessons of this module, students work with dimensions in the form of a simplified square root and learn to express answers as a simplified square root to increase their fluency in working with numbers in this form.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

a.

   i.   **What does $\sqrt{16}$ equal?**

       4

   ii.   **What does 4×4 equal?**

       16

   iii.   **Does $\sqrt{16} = \sqrt{4 \times 4}$?**

       Yes

b.

   i.   **What does $\sqrt{36}$ equal?**

       6

   ii.   **What does 6×6 equal?**

       36

   iii.   **Does $\sqrt{36} = \sqrt{6 \times 6}$?**

       Yes

c.

   i.   **What does $\sqrt{121}$ equal?**

       11

   ii.   **What does 11×11 equal?**

       121

   iii.   **Does $\sqrt{121} = \sqrt{11 \times 11}$?**

       Yes

d.

   i.   **What does $\sqrt{81}$ equal?**

       9

   ii.   **What does 9×9 equal?**

       81

   iii.   **Does $\sqrt{81} = \sqrt{9 \times 9}$?**

       Yes

> **e.** Rewrite $\sqrt{20}$ using at least one perfect square factor.
>
> $$\sqrt{20} = \sqrt{4 \times 5}$$
>
> **f.** Rewrite $\sqrt{28}$ using at least one perfect square factor.
>
> $$\sqrt{28} = \sqrt{4 \times 7}$$

Discussion (7 minutes)

- We know from the last lesson that square roots can be simplified to a whole number when they are perfect squares. That is,

$$\sqrt{9} = \sqrt{3 \times 3} = \sqrt{3^2} = 3.$$

- Given x^2 (x is a positive integer and x squared is a perfect square), it is easy to see that when $C = \sqrt{x^2}$ and $D = x$, then $C = D$, where C and D are positive numbers. In terms of the previous example, when $C = \sqrt{9} = \sqrt{3^2}$ and $D = 3$, then $3 = 3$.

- We can show that this is true even when we do not have perfect squares. All we need to show is that when C and D are positive numbers and n is a positive integer, that $C^n = D^n$. If we can show that $C^n = D^n$, then we know that $C = D$.

> *Scaffolding:*
>
> If necessary, remind students about numbers and their squares using the visual below.
>
>

Ask students to explain why $C^n = D^n$ implies $C = D$. They should reference the definition of exponential notation that they learned in Module 1. For example, since $C^n = \underbrace{C \times C \times \cdots \times C}_{n \text{ times}}$ and $D^n = \underbrace{D \times D \times \cdots \times D}_{n \text{ times}}$, and we are given that

$\underbrace{C \times C \times \cdots \times C}_{n \text{ times}} = \underbrace{D \times D \times \cdots \times D}_{n \text{ times}}$, then C must be the same number as D.

- Now, for the proof that the nth root of a number can be expressed as a product of the nth root of its factors:

Let $C = \sqrt[n]{ab}$ and $D = \sqrt[n]{a} \times \sqrt[n]{b}$, where a and b are positive integers and n is a positive integer greater than or equal to 2. We want to show that $C^n = D^n$.
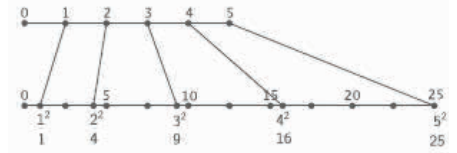
> *Scaffolding:*
>
> Students may know the term *product* in other contexts. For that reason, students may need to be reminded that it refers to multiplication in this context.

$$C^n = \left(\sqrt[n]{ab}\right)^n$$
$$= \underbrace{\sqrt[n]{ab} \times \sqrt[n]{ab} \times \cdots \times \sqrt[n]{ab}}_{n \text{ times}}$$
$$= ab$$

$$D^n = \left(\sqrt[n]{a} \times \sqrt[n]{b}\right)^n$$
$$= \underbrace{\left(\sqrt[n]{a} \times \sqrt[n]{b}\right) \times \left(\sqrt[n]{a} \times \sqrt[n]{b}\right) \times \cdots \times \left(\sqrt[n]{a} \times \sqrt[n]{b}\right)}_{n \text{ times}}$$
$$= \underbrace{\sqrt[n]{a} \times \sqrt[n]{a} \times \cdots \times \sqrt[n]{a}}_{n \text{ times}} \times \underbrace{\sqrt[n]{b} \times \sqrt[n]{b} \times \cdots \times \sqrt[n]{b}}_{n \text{ times}}$$
$$= ab$$

- Since $C^n = D^n$ implies $C = D$, then $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

- Let's look again at some concrete numbers. What is $\sqrt{36}$?

  □ $\sqrt{36} = 6$

Lesson 4:   Simplifying Square Roots

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- Now, consider the factors of 36. Specifically, consider those that are perfect squares. We want to rewrite $\sqrt{36}$ as a product of perfect squares. What will that be?
    - $\sqrt{36} = \sqrt{4 \times 9}$
- Based on what we just learned, we can write $\sqrt{36} = \sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9}$. What does the last expression simplify to? How does it compare to our original statement that $\sqrt{36} = 6$?
    - $\sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$; *the answers are the same, so $\sqrt{36} = \sqrt{4} \times \sqrt{9}$.*
- Rewrite $\sqrt{64}$ in the form of $\sqrt{a} \times \sqrt{b}$ in two different ways. Explain your work to a partner.
    - $\sqrt{64} = \sqrt{8 \times 8} = \sqrt{8^2} = 8$; *the number 64 is a product of 8 multiplied by itself, which is the same as 8^2. Since the square root symbol asks for the number that when multiplied by itself is 64, then $\sqrt{64} = 8$.*
    - $\sqrt{64} = \sqrt{16 \times 4} = \sqrt{16} \times \sqrt{4} = \sqrt{4^2} \times \sqrt{2^2} = 4 \times 2 = 8$; *the number 64 is a product of 16 and 4. We can first rewrite $\sqrt{64}$ as a product of its factors, $\sqrt{16 \times 4}$, and then as $\sqrt{16} \times \sqrt{4}$. Each of the numbers 16 and 4 are perfect squares that can be simplified as before, so $\sqrt{16} \times \sqrt{4} = \sqrt{4^2} \times \sqrt{2^2} = 4 \times 2 = 8$. Therefore, $\sqrt{64} = 8$. This means that $\sqrt{64} = \sqrt{16} \times \sqrt{4}$.*

Example 1 (4 minutes)

> **Example 1**
>
> **Simplify the square root as much as possible.**
>
> $\sqrt{50} =$

- Is the number 50 a perfect square? Explain.
    - *The number 50 is not a perfect square because there is no integer squared that equals 50.*
- Since 50 is not a perfect square, when we need to simplify $\sqrt{50}$, we write the factors of the number 50 looking specifically for those that are perfect squares. What are the factors of 50?
    - $50 = 2 \times 5^2$
- Since $50 = 2 \times 5^2$, then $\sqrt{50} = \sqrt{2 \times 5^2}$. We can rewrite $\sqrt{50}$ as a product of its factors:
$$\sqrt{50} = \sqrt{2} \times \sqrt{5^2}.$$
Obviously, 5^2 is a perfect square. Therefore, $\sqrt{5^2} = 5$, so $\sqrt{50} = 5 \times \sqrt{2} = 5\sqrt{2}$. Since $\sqrt{2}$ is not a perfect square, we leave it as it is. We have simplified this expression *as much as possible* because there are no other perfect square factors remaining in the square root.
- The number $\sqrt{50}$ is said to be in its simplified form when all perfect square factors have been simplified. Therefore, $5\sqrt{2}$ is the simplified form of $\sqrt{50}$.
- Now that we know $\sqrt{50}$ can be expressed as a product of its factors, we also know that we can multiply expressions containing square roots. For example, if we are given $\sqrt{2} \times \sqrt{5^2}$, we can rewrite the expression as $\sqrt{2 \times 5^2} = \sqrt{50}$.

Example 2 (3 minutes)

> **Example 2**
>
> Simplify the square root as much as possible.
>
> $\sqrt{28} =$

- Is the number 28 a perfect square? Explain.
  - *The number* 28 *is not a perfect square because there is no integer squared that equals* 28.
- What are the factors of 28?
  - $28 = 2^2 \times 7$
- Since $28 = 2^2 \times 7$, then $\sqrt{28} = \sqrt{2^2 \times 7}$. We can rewrite $\sqrt{28}$ as a product of its factors:
  $$\sqrt{28} = \sqrt{2^2} \times \sqrt{7}.$$
  Obviously, 2^2 is a perfect square. Therefore, $\sqrt{2^2} = 2$, and $\sqrt{28} = 2 \times \sqrt{7} = 2\sqrt{7}$. Since $\sqrt{7}$ is not a perfect square, we leave it as it is.
- The number $\sqrt{28}$ is said to be in its simplified form when all perfect square factors have been simplified. Therefore, $2\sqrt{7}$ is the simplified form of $\sqrt{28}$.

Exercises 1–4 (5 minutes)

Students complete Exercises 1–4 independently.

> **Exercises 1–4**
>
> Simplify the square roots as much as possible.
>
> 1. $\sqrt{18}$ $\qquad \begin{aligned} \sqrt{18} &= \sqrt{2 \times 3^2} \\ &= \sqrt{2} \times \sqrt{3^2} \\ &= 3\sqrt{2} \end{aligned}$
>
> 2. $\sqrt{44}$ $\qquad \begin{aligned} \sqrt{44} &= \sqrt{2^2 \times 11} \\ &= \sqrt{2^2} \times \sqrt{11} \\ &= 2\sqrt{11} \end{aligned}$
>
> 3. $\sqrt{169}$ $\qquad \begin{aligned} \sqrt{169} &= \sqrt{13^2} \\ &= 13 \end{aligned}$
>
> 4. $\sqrt{75}$ $\qquad \begin{aligned} \sqrt{75} &= \sqrt{3 \times 5^2} \\ &= \sqrt{3} \times \sqrt{5^2} \\ &= 5\sqrt{3} \end{aligned}$

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Example 3 (4 minutes)

> **Example 3**
>
> Simplify the square root as much as possible.
>
> $\sqrt{128} =$

In this example, students may or may not recognize 128 as 64×2. The work below assumes that they do not. Consider showing students the solution below, as well as this alternative solution:

$$\sqrt{128} = \sqrt{64 \times 2} = \sqrt{64} \times \sqrt{2} = 8 \times \sqrt{2} = 8\sqrt{2}.$$

- ▪ Is the number 128 a perfect square? Explain.
  - ▫ *The number* 128 *is not a perfect square because there is no integer squared that equals* 128.
- ▪ What are the factors of 128?
  - ▫ $128 = 2^7$
- ▪ Since $128 = 2^7$, then $\sqrt{128} = \sqrt{2^7}$. We know that we can simplify perfect squares, so we can rewrite 2^7 as $2^2 \times 2^2 \times 2^2 \times 2$ because of what we know about the laws of exponents. Then, $\sqrt{128} = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2}$.

  Each 2^2 is a perfect square. Therefore, $\sqrt{128} = 2 \times 2 \times 2 \times \sqrt{2} = 8\sqrt{2}$.

Example 4 (4 minutes)

> **Example 4**
>
> Simplify the square root as much as possible.
>
> $\sqrt{288} =$

In this example, students may or may not recognize 288 as 144×2. The work below assumes that they do not. Consider showing students the solution below, as well as this alternative solution:

$$\sqrt{288} = \sqrt{144 \times 2} = \sqrt{144} \times \sqrt{2} = 12 \times \sqrt{2} = 12\sqrt{2}.$$

- ▪ Is the number 288 a perfect square? Explain.
  - ▫ *The number* 288 *is not a perfect square because there is no integer squared that equals* 288.
- ▪ What are the factors of 288?
  - ▫ $288 = 2^5 \times 3^2$
- ▪ Since $288 = 2^5 \times 3^2$, then $\sqrt{288} = \sqrt{2^5 \times 3^2}$. What do we do next?
  - ▫ *Use the laws of exponents to rewrite* 2^5 *as* $2^2 \times 2^2 \times 2$.
- ▪ Then, $\sqrt{288}$ is equivalent to
  - ▫ $\sqrt{288} = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2}$.
- ▪ What does this simplify to?
  - ▫ $\sqrt{288} = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2} = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{2} = 2 \times 2 \times 3 \times \sqrt{2} = 12\sqrt{2}$

Exercises 5–8 (5 minutes)

Students work independently or in pairs to complete Exercises 5–8.

Exercises 5–8

5. Simplify $\sqrt{108}$.

$$\sqrt{108} = \sqrt{2^2 \times 3^3}$$
$$= \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{3}$$
$$= 2 \times 3\sqrt{3}$$
$$= 6\sqrt{3}$$

6. Simplify $\sqrt{250}$.

$$\sqrt{250} = \sqrt{2 \times 5^3}$$
$$= \sqrt{2} \times \sqrt{5^2} \times \sqrt{5}$$
$$= 5\sqrt{2} \times \sqrt{5}$$
$$= 5\sqrt{10}$$

7. Simplify $\sqrt{200}$.

$$\sqrt{200} = \sqrt{2^3 \times 5^2}$$
$$= \sqrt{2^2} \times \sqrt{2} \times \sqrt{5^2}$$
$$= 2 \times 5\sqrt{2}$$
$$= 10\sqrt{2}$$

8. Simplify $\sqrt{504}$.

$$\sqrt{504} = \sqrt{2^3 \times 3^2 \times 7}$$
$$= \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2} \times \sqrt{7}$$
$$= 2 \times 3 \times \sqrt{2} \times \sqrt{7}$$
$$= 6\sqrt{14}$$

Scaffolding:

Some simpler problems are included here.

- Simplify $\sqrt{12}$.
  - $\sqrt{12} = \sqrt{2^2 \times 3}$
    $$= \sqrt{2^2} \times \sqrt{3}$$
    $$= 2 \times \sqrt{3}$$
    $$= 2\sqrt{3}$$

- Simplify $\sqrt{48}$.
  - $\sqrt{48} = \sqrt{2^4 \times 3}$
    $$= \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{3}$$
    $$= 2 \times 2 \times \sqrt{3}$$
    $$= 4\sqrt{3}$$

- Simplify $\sqrt{350}$.
  - $\sqrt{350} = \sqrt{5^2 \times 2 \times 7}$
    $$= \sqrt{5^2} \times \sqrt{2} \times \sqrt{7}$$
    $$= 5 \times \sqrt{2} \times \sqrt{7}$$
    $$= 5\sqrt{14}$$

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know how to simplify a square root by using the factors of a given number and then simplifying the perfect squares.

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Lesson Summary

Square roots of some non-perfect squares can be simplified by using the factors of the number. Any perfect square factors of a number can be simplified.

For example:

$$\sqrt{72} = \sqrt{36 \times 2}$$
$$= \sqrt{36} \times \sqrt{2}$$
$$= \sqrt{6^2} \times \sqrt{2}$$
$$= 6 \times \sqrt{2}$$
$$= 6\sqrt{2}$$

Exit Ticket (5 minutes)

Name _____   Date _____

Lesson 4: Simplifying Square Roots

Exit Ticket

Simplify the square roots as much as possible.

1. $\sqrt{24}$

2. $\sqrt{338}$

3. $\sqrt{196}$

4. $\sqrt{2420}$

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Exit Ticket Sample Solutions

Simplify the square roots as much as possible.

1. $\sqrt{24}$

$$\sqrt{24} = \sqrt{2^2 \times 6}$$
$$= \sqrt{2^2} \times \sqrt{6}$$
$$= 2\sqrt{6}$$

2. $\sqrt{338}$

$$\sqrt{338} = \sqrt{13^2 \times 2}$$
$$= \sqrt{13^2} \times \sqrt{2}$$
$$= 13\sqrt{2}$$

3. $\sqrt{196}$

$$\sqrt{196} = \sqrt{14^2}$$
$$= 14$$

4. $\sqrt{2420}$

$$\sqrt{2420} = \sqrt{2^2 \times 11^2 \times 5}$$
$$= \sqrt{2^2} \times \sqrt{11^2} \times \sqrt{5}$$
$$= 2 \times 11 \times \sqrt{5}$$
$$= 22\sqrt{5}$$

Problem Set Sample Solutions

Simplify each of the square roots in Problems 1–5 as much as possible.

1. $\sqrt{98}$

$$\sqrt{98} = \sqrt{2 \times 7^2}$$
$$= \sqrt{2} \times \sqrt{7^2}$$
$$= 7\sqrt{2}$$

2. $\sqrt{54}$

$$\sqrt{54} = \sqrt{2 \times 3^3}$$
$$= \sqrt{2} \times \sqrt{3} \times \sqrt{3^2}$$
$$= 3\sqrt{6}$$

3. $\sqrt{144}$

$$\sqrt{144} = \sqrt{12^2}$$
$$= 12$$

4. $\sqrt{512}$

$$\sqrt{512} = \sqrt{2^9}$$
$$= \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2}$$
$$= 2 \times 2 \times 2 \times 2\sqrt{2}$$
$$= 16\sqrt{2}$$

5. $\sqrt{756}$

$$\sqrt{756} = \sqrt{2^2 \times 3^3 \times 7}$$
$$= \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{3} \times \sqrt{7}$$
$$= 2 \times 3 \times \sqrt{21}$$
$$= 6\sqrt{21}$$

6. What is the length of the unknown side of the right triangle? Simplify your answer, if possible.
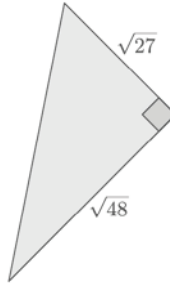
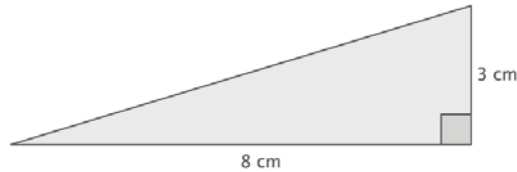*Let c units represent the length of the hypotenuse.*

$$\left(\sqrt{27}\right)^2 + \left(\sqrt{48}\right)^2 = c^2$$
$$27 + 48 = c^2$$
$$75 = c^2$$
$$\sqrt{75} = \sqrt{c^2}$$
$$\sqrt{5^2} \times \sqrt{3} = c$$
$$5\sqrt{3} = c$$

The length of the hypotenuse is $5\sqrt{3}$ units.

7. What is the length of the unknown side of the right triangle? Simplify your answer, if possible.
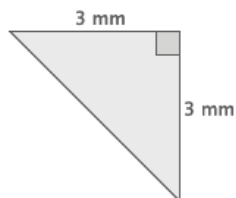

3 cm

8 cm

Let c cm represent the length of the hypotenuse.

$$3^2 + 8^2 = c^2$$
$$9 + 64 = c^2$$
$$73 = c^2$$
$$\sqrt{73} = \sqrt{c^2}$$
$$\sqrt{73} = c$$

The length of the unknown side is $\sqrt{73}$ cm.

8. What is the length of the unknown side of the right triangle? Simplify your answer, if possible.
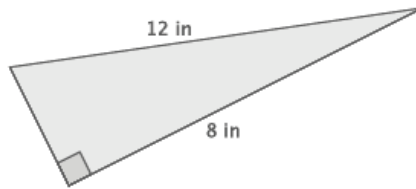
3 mm

3 mm

Let c mm represent the length of the hypotenuse.

$$3^2 + 3^2 = c^2$$
$$9 + 9 = c^2$$
$$18 = c^2$$
$$\sqrt{18} = \sqrt{c^2}$$
$$\sqrt{18} = c$$
$$\sqrt{3^2} \times \sqrt{2} = c$$
$$3\sqrt{2} = c$$

The length of the unknown side is $3\sqrt{2}$ mm.

9. **What is the length of the unknown side of the right triangle? Simplify your answer, if possible.**

Let x in. represent the unknown length.

$$x^2 + 8^2 = 12^2$$
$$x^2 + 64 = 144$$
$$x^2 + 64 - 64 = 144 - 64$$
$$x^2 = 80$$
$$\sqrt{x^2} = \sqrt{80}$$
$$x = \sqrt{80}$$
$$x = \sqrt{2^4 \cdot 5}$$
$$x = \sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{5}$$
$$x = 2 \cdot 2\sqrt{5}$$
$$x = 4\sqrt{5}$$

The length of the unknown side is $4\sqrt{5}$ in.

10. **Josue simplified $\sqrt{450}$ as $15\sqrt{2}$. Is he correct? Explain why or why not.**

$$\sqrt{450} = \sqrt{2 \times 3^2 \times 5^2}$$
$$= \sqrt{2} \times \sqrt{3^2} \times \sqrt{5^2}$$
$$= 3 \times 5 \times \sqrt{2}$$
$$= 15\sqrt{2}$$

Yes, Josue is correct because the number $450 = 2 \times 3^2 \times 5^2$. The factors that are perfect squares simplify to 15 leaving just the factor of 2 that cannot be simplified. Therefore, $\sqrt{450} = 15\sqrt{2}$.

11. **Tiah was absent from school the day that you learned how to simplify a square root. Using $\sqrt{360}$, write Tiah an explanation for simplifying square roots.**

To simplify $\sqrt{360}$, first write the factors of 360. The number $360 = 2^3 \times 3^2 \times 5$. Now, we can use the factors to write $\sqrt{360} = \sqrt{2^3 \times 3^2 \times 5}$, which can then be expressed as $\sqrt{360} = \sqrt{2^3} \times \sqrt{3^2} \times \sqrt{5}$. Because we want to simplify square roots, we can rewrite the factor $\sqrt{2^3}$ as $\sqrt{2^2} \times \sqrt{2}$ because of the laws of exponents. Now, we have

$$\sqrt{360} = \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2} \times \sqrt{5}.$$

Each perfect square can be simplified as follows:

$$\sqrt{360} = 2 \times \sqrt{2} \times 3 \times \sqrt{5}$$
$$= 2 \times 3 \times \sqrt{2} \times \sqrt{5}$$
$$= 6\sqrt{10}.$$

The simplified version of $\sqrt{360} = 6\sqrt{10}$.

Lesson 5: Solving Equations with Radicals

Student Outcomes

- Students find the positive solutions to equations algebraically equivalent to equations of the form $x^2 = p$ and $x^3 = p$.

Lesson Notes

As with previous lessons, students are asked to find only the positive value of x that makes each equation true. However, we have included in the examples and exercises both the positive and the negative values of x so that the teacher can choose whether to provide the rationale for both solutions and whether to require students to give them as part of their answers.

Classwork

Discussion (15 minutes)

- Just recently, we began solving equations that required us to find the square root or cube root of a number. All of those equations were in the form of $x^2 = p$ or $x^3 = p$, where p was a positive rational number.

Example 1

> **Example 1**
>
> $$x^3 + 9x = \frac{1}{2}(18x + 54)$$

- Now that we know about square roots and cube roots, we can combine that knowledge with our knowledge of the properties of equality to begin solving nonlinear equations like $x^3 + 9x = \frac{1}{2}(18x + 54)$. Transform the equation until you can determine the positive value of x that makes the equation true.

Challenge students to solve the equation independently or in pairs. Have students share their strategy for solving the equation. Ask them to explain each step.

- *Sample response:*

$$x^3 + 9x = \frac{1}{2}(18x + 54)$$
$$x^3 + 9x = 9x + 27$$
$$x^3 + 9x - 9x = 9x - 9x + 27$$
$$x^3 = 27$$
$$\sqrt[3]{x^3} = \sqrt[3]{27}$$
$$x = \sqrt[3]{3^3}$$
$$x = 3$$

Scaffolding:

Consider using a simpler version of the equation (line 2, for example):

$$x^3 + 9x = 9x + 27.$$

- Now, we verify our solution is correct.

$$3^3 + 9(3) = \frac{1}{2}(18(3) + 54)$$

$$27 + 27 = \frac{1}{2}(54 + 54)$$

$$54 = \frac{1}{2}(108)$$

$$54 = 54$$

- Since the left side is the same as the right side, our solution is correct.

Example 2

Example 2

$$x(x - 3) - 51 = -3x + 13$$

- Let's look at another nonlinear equation. Find the positive value of x that makes the equation true: $x(x - 3) - 51 = -3x + 13$.

Provide students with time to solve the equation independently or in pairs. Have students share their strategy for solving the equation. Ask them to explain each step.

□ *Sample response:*

$$x(x - 3) - 51 = -3x + 13$$

$$x^2 - 3x - 51 = -3x + 13$$

$$x^2 - 3x + 3x - 51 = -3x + 3x + 13$$

$$x^2 - 51 = 13$$

$$x^2 - 51 + 51 = 13 + 51$$

$$x^2 = 64$$

$$\sqrt{x^2} = \pm\sqrt{64}$$

$$x = \pm\sqrt{64}$$

$$x = \pm 8$$

- Now we verify our solution is correct.

Provide students time to check their work.

- Let $x = 8$.   Let $x = -8$.

$$8(8 - 3) - 51 = -3(8) + 13 \qquad\qquad -8(-8 - 3) - 51 = -3(-8) + 13$$

$$8(5) - 51 = -24 + 13 \qquad\qquad\qquad -8(-11) - 51 = 24 + 13$$

$$40 - 51 = -11 \qquad\qquad\qquad\qquad 88 - 51 = 37$$

$$-11 = -11 \qquad\qquad\qquad\qquad\qquad 37 = 37$$

- Now it is clear that the left side is exactly the same as the right side, and our solution is correct.

Exercises (20 minutes)

Students complete Exercises 1–7 independently or in pairs. Although we are asking students to find the positive value of x that makes each equation true, we have included in the exercises both the positive and the negative values of x so that the teacher can choose whether to use them.

Exercises

Find the positive value of x that makes each equation true, and then verify your solution is correct.

1.

   a.   **Solve $x^2 - 14 = 5x + 67 - 5x$.**

$$x^2 - 14 = 5x + 67 - 5x$$
$$x^2 - 14 = 67$$
$$x^2 - 14 + 14 = 67 + 14$$
$$x^2 = 81$$
$$\sqrt{x^2} = \pm\sqrt{81}$$
$$x = \pm\sqrt{81}$$
$$x = \pm 9$$

Check:

$$9^2 - 14 = 5(9) + 67 - 5(9)$$
$$81 - 14 = 45 + 67 - 45$$
$$67 = 67$$

$$(-9)^2 - 14 = 5(-9) + 67 - 5(-9)$$
$$81 - 14 = -45 + 67 + 45$$
$$67 = 67$$

   b.   **Explain how you solved the equation.**

   To solve the equation, I had to first use the properties of equality to transform the equation into the form of $x^2 = 81$. Then, I had to take the square root of both sides of the equation to determine that $x = 9$ since the number x is being squared.

2.   **Solve and simplify: $x(x - 1) = 121 - x$.**

$$x(x - 1) = 121 - x$$
$$x^2 - x = 121 - x$$
$$x^2 - x + x = 121 - x + x$$
$$x^2 = 121$$
$$\sqrt{x^2} = \pm\sqrt{121}$$
$$x = \pm\sqrt{121}$$
$$x = \pm 11$$

Check:

$$11(11 - 1) = 121 - 11$$
$$11(10) = 110$$
$$110 = 110$$

$$-11(-11 - 1) = 121 - (-11)$$
$$-11(-12) = 121 + 11$$
$$132 = 132$$

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3. A square has a side length of $3x$ inches and an area of 324 in^2. What is the value of x?

$$(3x)^2 = 324$$
$$3^2x^2 = 324$$
$$9x^2 = 324$$
$$\frac{9x^2}{9} = \frac{324}{9}$$
$$x^2 = 36$$
$$\sqrt{x^2} = \sqrt{36}$$
$$x = 6$$

Check:
$$(3(6))^2 = 324$$
$$18^2 = 324$$
$$324 = 324$$

A negative number would not make sense as a length, so $x = 6$.

4. $-3x^3 + 14 = -67$

$$-3x^3 + 14 = -67$$
$$-3x^3 + 14 - 14 = -67 - 14$$
$$-3x^3 = -81$$
$$\frac{-3x^3}{-3} = \frac{-81}{-3}$$
$$x^3 = 27$$
$$\sqrt[3]{x^3} = \sqrt[3]{27}$$
$$x = 3$$

Check:
$$-3(3)^3 + 14 = -67$$
$$-3(27) + 14 = -67$$
$$-81 + 14 = -67$$
$$-67 = -67$$

5. $x(x + 4) - 3 = 4(x + 19.5)$

$$x(x + 4) - 3 = 4(x + 19.5)$$
$$x^2 + 4x - 3 = 4x + 78$$
$$x^2 + 4x - 4x - 3 = 4x - 4x + 78$$
$$x^2 - 3 = 78$$
$$x^2 - 3 + 3 = 78 + 3$$
$$x^2 = 81$$
$$\sqrt{x^2} = \pm\sqrt{81}$$
$$x = \pm 9$$

Check:
$$9(9 + 4) - 3 = 4(9 + 19.5)$$
$$9(13) - 3 = 4(28.5)$$
$$117 - 3 = 114$$
$$114 = 114$$

$$-9(-9 + 4) - 3 = 4(-9 + 19.5)$$
$$-9(-5) - 3 = 4(10.5)$$
$$45 - 3 = 42$$
$$42 = 42$$

6. $216 + x = x(x^2 - 5) + 6x$

$$216 + x = x(x^2 - 5) + 6x$$
$$216 + x = x^3 - 5x + 6x$$
$$216 + x = x^3 + x$$
$$216 + x - x = x^3 + x - x$$
$$216 = x^3$$
$$\sqrt[3]{216} = \sqrt[3]{x^3}$$
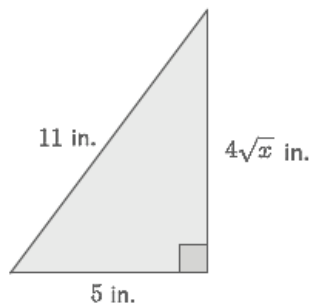$$6 = x$$

Check:
$$216 + 6 = 6(6^2 - 5) + 6(6)$$
$$222 = 6(31) + 36$$
$$222 = 186 + 36$$
$$222 = 222$$

7.

    a.   **What are we trying to determine in the diagram below?**

We need to determine the value of x so that its square root, multiplied by 4, satisfies the equation

$$5^2 + \left(4\sqrt{x}\right)^2 = 11^2.$$

    b.   **Determine the value of x, and check your answer.**

$$5^2 + \left(4\sqrt{x}\right)^2 = 11^2$$
$$25 + 4^2\left(\sqrt{x}\right)^2 = 121$$
$$25 - 25 + 4^2\left(\sqrt{x}\right)^2 = 121 - 25$$
$$16x = 96$$
$$\frac{16x}{16} = \frac{96}{16}$$
$$x = 6$$

The value of x is 6.

Check:

$$5^2 + \left(4\sqrt{6}\right)^2 = 11^2$$
$$25 + 16(6) = 121$$
$$25 + 96 = 121$$
$$121 = 121$$

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know how to solve equations with squared and cubed variables and verify that our solutions are correct.

> **Lesson Summary**
>
> Equations that contain variables that are squared or cubed can be solved using the properties of equality and the definition of square and cube roots.
>
> Simplify an equation until it is in the form of $x^2 = p$ or $x^3 = p$, where p is a positive rational number; then, take the square or cube root to determine the positive value of x.
>
> **Example:**
>
> Solve for x.
>
> $$\frac{1}{2}(2x^2 + 10) = 30$$
> $$x^2 + 5 = 30$$
> $$x^2 + 5 - 5 = 30 - 5$$
> $$x^2 = 25$$
> $$\sqrt{x^2} = \sqrt{25}$$
> $$x = 5$$
>
> Check:
>
> $$\frac{1}{2}(2(5)^2 + 10) = 30$$
> $$\frac{1}{2}(2(25) + 10) = 30$$
> $$\frac{1}{2}(50 + 10) = 30$$
> $$\frac{1}{2}(60) = 30$$
> $$30 = 30$$

Exit Ticket (5 minutes)

Name _____   Date _____

Lesson 5: Solving Equations with Radicals

Exit Ticket

1. Find the positive value of x that makes the equation true, and then verify your solution is correct.

$$x^2 + 4x = 4(x + 16)$$

2. Find the positive value of x that makes the equation true, and then verify your solution is correct.

$$(4x)^3 = 1728$$

Exit Ticket Sample Solutions

1. **Find the positive value of x that makes the equation true, and then verify your solution is correct.**

$$x^2 + 4x = 4(x + 16)$$

$$x^2 + 4x = 4(x + 16)$$
$$x^2 + 4x = 4x + 64$$
$$x^2 + 4x - 4x = 4x - 4x + 64$$
$$x^2 = 64$$
$$\sqrt{x^2} = \sqrt{64}$$
$$x = 8$$

Check:

$$8^2 + 4(8) = 4(8 + 16)$$
$$64 + 32 = 4(24)$$
$$96 = 96$$

2. **Find the positive value of x that makes the equation true, and then verify your solution is correct.**

$$(4x)^3 = 1728$$

$$(4x)^3 = 1728$$
$$64x^3 = 1728$$
$$\frac{1}{64}(64x^3) = (1728)\frac{1}{64}$$
$$x^3 = 27$$
$$\sqrt[3]{x^3} = \sqrt[3]{27}$$
$$x = 3$$

Check:

$$\left(4(3)\right)^3 = 1728$$
$$12^3 = 1728$$
$$1728 = 1728$$

Problem Set Sample Solutions

Find the positive value of x that makes each equation true, and then verify your solution is correct.

1. $x^2(x + 7) = \frac{1}{2}(14x^2 + 16)$

$$x^2(x + 7) = \frac{1}{2}(14x^2 + 16)$$
$$x^3 + 7x^2 = 7x^2 + 8$$
$$x^3 + 7x^2 - 7x^2 = 7x^2 - 7x^2 + 8$$
$$x^3 = 8$$
$$\sqrt[3]{x^3} = \sqrt[3]{8}$$
$$x = 2$$

Check:

$$2^2(2 + 7) = \frac{1}{2}(14(2^2) + 16)$$
$$4(9) = \frac{1}{2}(56 + 16)$$
$$36 = \frac{1}{2}(72)$$
$$36 = 36$$

2. $x^3 = 1331^{-1}$

$$x^3 = 1331^{-1}$$

Check:

$$\sqrt[3]{x^3} = \sqrt[3]{1331^{-1}}$$

$$x = \sqrt[3]{\frac{1}{1331}}$$

$$x = \sqrt[3]{\frac{1}{11^3}}$$

$$x = \frac{1}{11}$$

$$\left(\frac{1}{11}\right)^3 = 1331^{-1}$$

$$\frac{1}{11^3} = 1331^{-1}$$

$$\frac{1}{1331} = 1331^{-1}$$

$$1331^{-1} = 1331^{-1}$$

3. Determine the positive value of x that makes the equation true, and then explain how you solved the equation.

$$\frac{x^9}{x^7} - 49 = 0$$

$$\frac{x^9}{x^7} - 49 = 0$$

Check:

$$x^2 - 49 = 0$$

$$x^2 - 49 + 49 = 0 + 49$$

$$x^2 = 49$$

$$\sqrt{x^2} = \sqrt{49}$$

$$x = 7$$

$$7^2 - 49 = 0$$

$$49 - 49 = 0$$

$$0 = 0$$

To solve the equation, I first had to simplify the expression $\frac{x^9}{x^7}$ to x^2. Next, I used the properties of equality to transform the equation into $x^2 = 49$. Finally, I had to take the square root of both sides of the equation to solve for x.

4. Determine the positive value of x that makes the equation true.

$$(8x)^2 = 1$$

$$(8x)^2 = 1$$

$$64x^2 = 1$$

$$\sqrt{64x^2} = \sqrt{1}$$

$$8x = 1$$

$$\frac{8x}{8} = \frac{1}{8}$$

$$x = \frac{1}{8}$$

Check:

$$\left(8\left(\frac{1}{8}\right)\right)^2 = 1$$

$$1^2 = 1$$

$$1 = 1$$

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5. $\left(9\sqrt{x}\right)^2 - 43x = 76$

$$\left(9\sqrt{x}\right)^2 - 43x = 76$$

$$9^2\left(\sqrt{x}\right)^2 - 43x = 76$$

$$81x - 43x = 76$$

$$38x = 76$$

$$\frac{38x}{38} = \frac{76}{38}$$

$$x = 2$$

Check:

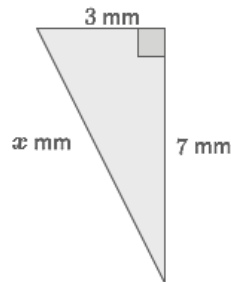$$\left(9(\sqrt{2})\right)^2 - 43(2) = 76$$

$$9^2\left(\sqrt{2}\right)^2 - 86 = 76$$

$$81(2) - 86 = 76$$

$$162 - 86 = 76$$

$$76 = 76$$

6. Determine the length of the hypotenuse of the right triangle below.

$$3^2 + 7^2 = x^2$$

$$9 + 49 = x^2$$

$$58 = x^2$$

$$\sqrt{58} = \sqrt{x^2}$$
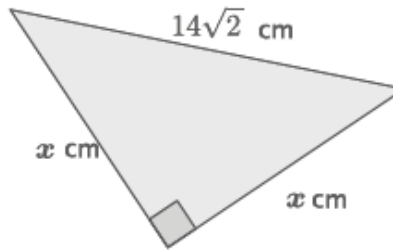
$$\sqrt{58} = x$$

Check:

$$3^2 + 7^2 = \sqrt{58}^2$$

$$9 + 49 = 58$$

$$58 = 58$$

Since $x = \sqrt{58}$, the length of the hypotenuse is $\sqrt{58}$ mm.

7.   Determine the length of the legs in the right triangle below.

$14\sqrt{2}$ cm

x cm

x cm

$$x^2 + x^2 = \left(14\sqrt{2}\right)^2$$
$$2x^2 = 14^2\left(\sqrt{2}\right)^2$$
$$2x^2 = 196(2)$$
$$\frac{2x^2}{2} = \frac{196(2)}{2}$$
$$x^2 = 196$$
$$\sqrt{x^2} = \sqrt{196}$$
$$x = \sqrt{14^2}$$
$$x = 14$$

Check:

$$14^2 + 14^2 = \left(14\sqrt{2}\right)^2$$
$$196 + 196 = 14^2\left(\sqrt{2}\right)^2$$
$$392 = 196(2)$$
$$392 = 392$$

Since $x = 14$, the length of each of the legs of the right triangle is 14 cm.

8.   An equilateral triangle has side lengths of 6 cm. What is the height of the triangle? What is the area of the triangle?

Note: This problem has two solutions, one with a simplified root and one without. Choose the appropriate solution for your classes based on how much simplifying you have taught them.

Let h cm represent the height of the triangle.

6 cm   6 cm

3 cm   3 cm
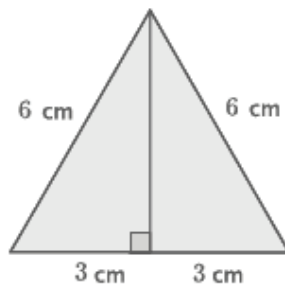
$$3^2 + h^2 = 6^2$$
$$9 + h^2 = 36$$
$$9 - 9 + h^2 = 36 - 9$$
$$h^2 = 27$$
$$\sqrt{h^2} = \sqrt{27}$$
$$h = \sqrt{27}$$
$$h = \sqrt{3^3}$$
$$h = \sqrt{3^2} \times \sqrt{3}$$
$$h = 3\sqrt{3}$$

Let A represent the area of the triangle.

$$A = \frac{6\left(3\sqrt{3}\right)}{2}$$
$$A = 3\left(3\sqrt{3}\right)$$
$$A = 9\sqrt{3}$$

Simplified: The height of the triangle is $3\sqrt{3}$ cm, and the area is $9\sqrt{3}$ cm^2.

Unsimplified: The height of the triangle is $\sqrt{27}$ cm, and the area is $3\sqrt{27}$ cm^2.

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9. Challenge: Find the positive value of x that makes the equation true.

$$\left(\frac{1}{2}x\right)^2 - 3x = 7x + 8 - 10x$$

$$\left(\frac{1}{2}x\right)^2 - 3x = 7x + 8 - 10x$$

$$\frac{1}{4}x^2 - 3x = -3x + 8$$

$$\frac{1}{4}x^2 - 3x + 3x = -3x + 3x + 8$$

$$\frac{1}{4}x^2 = 8$$

$$4\left(\frac{1}{4}\right)x^2 = 8(4)$$

$$x^2 = 32$$

$$\sqrt{x^2} = \sqrt{32}$$

$$x = \sqrt{2^5}$$

$$x = \sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{2}$$

$$x = 4\sqrt{2}$$

Check:

$$\left(\frac{1}{2}\left(4\sqrt{2}\right)\right)^2 - 3\left(4\sqrt{2}\right) = 7\left(4\sqrt{2}\right) + 8 - 10\left(4\sqrt{2}\right)$$

$$\frac{1}{4}(16)(2) - 3\left(4\sqrt{2}\right) = 7\left(4\sqrt{2}\right) - 10\left(4\sqrt{2}\right) + 8$$

$$\frac{32}{4} - 3\left(4\sqrt{2}\right) = 7\left(4\sqrt{2}\right) - 10\left(4\sqrt{2}\right) + 8$$

$$8 - 3\left(4\sqrt{2}\right) = (7 - 10)\left(4\sqrt{2}\right) + 8$$

$$8 - 3\left(4\sqrt{2}\right) = -3\left(4\sqrt{2}\right) + 8$$

$$8 - 8 - 3\left(4\sqrt{2}\right) = -3\left(4\sqrt{2}\right) + 8 - 8$$

$$-3\left(4\sqrt{2}\right) = -3\left(4\sqrt{2}\right)$$

10. Challenge: Find the positive value of x that makes the equation true.

$$11x + x(x - 4) = 7(x + 9)$$

$$11x + x(x - 4) = 7(x + 9)$$

$$11x + x^2 - 4x = 7x + 63$$

$$7x + x^2 = 7x + 63$$

$$7x - 7x + x^2 = 7x - 7x + 63$$

$$x^2 = 63$$

$$\sqrt{x^2} = \sqrt{63}$$

$$x = \sqrt{(3^2)(7)}$$

$$x = \sqrt{3^2} \cdot \sqrt{7}$$

$$x = 3\sqrt{7}$$

Check:

$$11\left(3\sqrt{7}\right) + 3\sqrt{7}\left(3\sqrt{7} - 4\right) = 7\left(3\sqrt{7} + 9\right)$$

$$33\sqrt{7} + 3^2\left(\sqrt{7}\right)^2 - 4\left(3\sqrt{7}\right) = 21\sqrt{7} + 63$$

$$33\sqrt{7} - 4\left(3\sqrt{7}\right) + 9(7) = 21\sqrt{7} + 63$$

$$33\sqrt{7} - 12\sqrt{7} + 63 = 21\sqrt{7} + 63$$

$$(33 - 12)\sqrt{7} + 63 = 21\sqrt{7} + 63$$

$$21\sqrt{7} + 63 = 21\sqrt{7} + 63$$

$$21\sqrt{7} + 63 - 63 = 21\sqrt{7} + 63 - 63$$

$$21\sqrt{7} = 21\sqrt{7}$$

Topic B

Decimal Expansions of Numbers

Focus Standards:	■ Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.
	■ Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue to get better approximations.*
	■ Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
Instructional Days:	9
Lesson 6:	Finite and Infinite Decimals (P)[1]
Lesson 7:	Infinite Decimals (S)
Lesson 8:	The Long Division Algorithm (E)
Lesson 9:	Decimal Expansions of Fractions, Part 1 (P)
Lesson 10:	Converting Repeating Decimals to Fractions (P)
Lesson 11:	The Decimal Expansion of Some Irrational Numbers (S)
Lesson 12:	Decimal Expansions of Fractions, Part 2 (S)
Lesson 13:	Comparing Irrational Numbers (E)
Lesson 14:	Decimal Expansion of π (S)

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[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Throughout this topic, the terms *expanded form of a decimal* and *decimal expansion* are used. The expanded form of a decimal refers to the value of a number written as a sum. For example, the expanded form of the decimal 0.125 is $\frac{1}{10} + \frac{2}{10^2} + \frac{5}{10^3}$, which is closely related to the notion of expanded form used at the elementary level. When students are asked to determine the decimal expansion of a number such as $\sqrt{2}$, we expect them to write the number in decimal form. For example, the decimal expansion of $\sqrt{2}$ begins with 1.4142. The examination of the decimal expansion leads to an understanding of irrational numbers. Numbers with decimal expansions that are infinite (i.e., non-terminating) and that do not have a repeating block are called *irrational numbers*. Numbers with finite (i.e., terminating) decimal expansions, as well as those numbers that are infinite with repeating blocks, are called *rational numbers*. Students spend significant time engaging with finite and infinite decimals before the notion of an irrational number is fully explored in Lesson 11.

In Lesson 6, students learn that every number has a decimal expansion that is finite or infinite. Finite and infinite decimals are defined, and students learn a strategy for writing a fraction as a finite decimal that focuses on the denominator and its factors. That is, a fraction can be written as a finite decimal if the denominator is a product of twos, a product of fives, or a product of twos and fives. In Lesson 7, students learn that numbers that cannot be expressed as finite decimals are infinite decimals. Students write the expanded form of infinite decimals and show on the number line their decimal representation in terms of intervals of tenths, hundredths, thousandths, and so on. This work with infinite decimals prepares students for understanding how to approximate the decimal expansion of an irrational number. In Lesson 8, students use the long division algorithm to determine the decimal form of a number and can relate the work of the algorithm to why digits in a decimal expansion repeat. Students engage in a discussion about numbers that have an infinite decimal expansion with no discernable pattern in the digits, leading to the idea that numbers can be irrational. It is in these first few lessons of Topic B that students recognize that rational numbers have a decimal expansion that repeats eventually, either in zeros or in a repeating block of digits. The discussion of infinite decimals continues with Lesson 9, where students learn how to use what they know about powers of 10 and equivalent fractions to make sense of why the long division algorithm can be used to convert a fraction to a decimal. Students know that multiplying the numerator and denominator of a fraction by a power of 10 is similar to putting zeros after the decimal point when doing long division.

In Lesson 10, students learn that a number with a decimal expansion that repeats can be expressed as a fraction. Students learn a strategy for writing repeating decimals as fractions that relies on their knowledge of multiplying by powers of 10 and solving linear equations. Lesson 11 introduces students to the method of rational approximation using a series of rational numbers to get closer and closer to a given number. Students write the approximate decimal expansion of irrational numbers in Lesson 11, and it is in this lesson that irrational numbers are defined as numbers that are not equal to rational numbers. Students realize that irrational numbers are different because they have infinite decimal expansions that do not repeat. Therefore, irrational numbers are those that are not equal to rational numbers. Rational approximation is used again in Lesson 12 to verify the decimal expansions of rational numbers. Students then compare the method of rational approximation to long division. In Lesson 13, students compare the value of rational and irrational numbers. Students use the method of rational approximation to determine the decimal expansion of an irrational number. Then, they compare that value to the decimal expansion of rational numbers in the form of a fraction, decimal, perfect square, or perfect cube. Students can now place irrational numbers on a number line with more accuracy than they did in Lesson 2. In Lesson 14, students approximate π using the area of a quarter circle that is drawn on grid paper. Students estimate the area of the quarter circle using inner and outer boundaries. As with the method of rational approximation, students continue to refine their

estimates of the area, which improves their estimate of the value of π. Students then determine the approximate values of expressions involving π.

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 Lesson 6: Finite and Infinite Decimals

Student Outcomes

- Students prove that those real numbers with a finite decimal expansion are precisely the fractions that can be written with a denominator that is a power of 10.
- Students realize that any fraction with a denominator that is a product of 2's and/or 5's can be written in an equivalent form with a denominator that is a power of 10.

Lesson Notes

In this lesson, students show that a real number possessing a finite decimal expansion is a fraction with a denominator that is a power of 10. (For example, $0.045 = \dfrac{45}{10^3}$.) And, conversely, every fraction with a denominator that is a power of 10 has a finite decimal expansion. As fractions can be written in many equivalent forms, students realize that any fraction with a denominator that is a product of solely 2's and 5's is equivalent to a fraction with a denominator that is a power of 10, and so has a finite decimal expansion.

Classwork

Opening Exercise (7 minutes)

Provide students time to work, and then share their responses to part (e) with the class.

Opening Exercise

a. Use long division to determine the decimal expansion of $\dfrac{54}{20}$.

$$\frac{54}{20} = 2.7$$

b. Use long division to determine the decimal expansion of $\dfrac{7}{8}$.

$$\frac{7}{8} = 0.875$$

c. Use long division to determine the decimal expansion of $\dfrac{8}{9}$.

$$\frac{8}{9} = 0.8888\ldots$$

d. Use long division to determine the decimal expansion of $\dfrac{22}{7}$.

$$\frac{22}{7} = 3.142857\ldots$$

---

e. **What do you notice about the decimal expansions of parts (a) and (b) compared to the decimal expansions of parts (c) and (d)?**

The decimal expansions of parts (a) and (b) ended. That is, when I did the long division, I was able to stop after a few steps. That was different from the work I had to do in parts (c) and (d). In part (c), I noticed that the same number kept coming up in the steps of the division, but it kept going on. In part (d), when I did the long division, it did not end. I stopped dividing after I found a few decimal digits of the decimal expansion.

Discussion (5 minutes)

Use the discussion below to elicit a dialogue about finite and infinite decimals that may not have come up in the debrief of the Opening Exercise and to prepare students for what is covered in this lesson in particular (i.e., writing fractions as finite decimals without using long division).

Scaffolding:

Students may benefit from a graphic organizer, shown below, that shows the key information regarding finite and infinite decimals.

Finite Decimals	Infinite Decimals
Definition:	Definition:
Examples:	Examples:

- How would you classify the decimal expansions of parts (a)–(d)?
  - *Parts (a) and (b) are finite decimals, and parts (c) and (d) are infinite decimals.*
- Every real number sits somewhere on the number line and so has a decimal expansion. That is, every number is equal to a decimal. In this lesson, we focus on the decimal expansions of fractions, that is, numbers like $\frac{17}{125}$, and work to understand when their decimal expansions will be finite (like those of $\frac{54}{20}$ and $\frac{7}{8}$) or continue indefinitely (like those of $\frac{8}{9}$ and $\frac{22}{7}$).
- Certainly, every finite decimal can be written as a fraction, one with a power of 10 as its denominator. For example, $0.3 = \frac{3}{10}$, $0.047 = \frac{47}{1000}$, and $6.513467 = \frac{6513467}{1000000}$. (Invite students to practice converting their own examples of finite decimals into fractions with denominators that are a power of 10.)
- And conversely, we can write any fraction with a denominator that is a power of 10 as a finite decimal. For example, $\frac{3}{100} = 0.03$, $\frac{345}{10} = 34.5$, and $\frac{50105}{1000} = 50.105$. (Invite students to practice converting their own examples of fractions with denominators that are a power of 10 into finite decimals.)
- Of course, we can write fractions in equivalent forms. This might cause us to lose sight of any denominator that is a power of 10. For example, $0.5 = \frac{5}{10} = \frac{1}{2}$ and $2.04 = \frac{204}{100} = \frac{51}{25}$.
- Our job today is to see if we can recognize when a fraction can be written in an equivalent form with a denominator that is a power of 10 and so recognize it as a fraction with a finite decimal expansion.
- Return to the Opening Exercise. We know that the decimals in parts (a) and (b) are finite, while the decimals in parts (c) and (d) are not. What can you now observe about the fractions in each example?
  - *In part (a), the fraction is $\frac{54}{20}$, which is equivalent to $\frac{27}{10}$, a fraction with a denominator that is a power of 10.*

    In part (b), the faction is $\frac{7}{8}$, which is equivalent to $\frac{875}{1000}$, a fraction with a denominator that is a power of 10.

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In part (c), the fraction is $\frac{8}{9}$. As no power of 10 has 9 as a factor, this fraction is not equivalent to one with a denominator that is a power of 10. Thus, its decimal expansion cannot be finite.

In part (d), the fraction is $\frac{22}{7}$. As no power of 10 has 7 as a factor, this fraction is not equivalent to one with a denominator that is a power of 10. Thus, its decimal expansion cannot be finite.

▪ As $10 = 2 \times 5$, any power of 10 has factors composed only of 2's and 5's. Thus, for a (simplified) fraction to be equivalent to one with a denominator that is a power of 10, it must be the case that its denominator is a product of only 2's and 5's. For example, the fraction $\frac{13}{2\times2\times5}$ is equivalent to the fraction $\frac{13\times5}{2\times2\times5\times5}$ which is equal to $\frac{65}{10^2}$, and $\frac{1}{2^3\times5^5}$ is equivalent to the fraction $\frac{2^2}{2^5\times5^5}$ which is equal to $\frac{4}{10^5}$. If a (simplified) fraction has a denominator involving factors different from 2's and 5's, then it cannot be equivalent to one with a denominator that is a power of 10.

▪ For example, the denominator of $\frac{5}{14}$ has a factor of 7. Thus, $\frac{5}{14}$ is not equivalent to a fraction with a denominator that is a power of 10, and so it must have an infinite decimal expansion.

▪ Why do you think I am careful to talk about *simplified* fractions in our thinking about the denominators of fractions?

  ▫ *It is possible for non-simplified fractions to have denominators containing factors other than those composed of 2's and 5's and still be equivalent to a fraction that with a denominator that is a power of 10. For example, $\frac{14}{35}$ has a denominator containing a factor of 7. But this is an unnecessary factor since the simplified version of $\frac{14}{35}$ is $\frac{2}{5}$, and this is equivalent to $\frac{4}{10}$. We only want to consider the essential factors of the denominators in a fraction.*

Example 1 (4 minutes)

> **Example 1**
>
> Consider the fraction $\frac{5}{8}$. Write an equivalent form of this fraction with a denominator that is a power of 10, and write the decimal expansion of this fraction.

▪ Consider the fraction $\frac{5}{8}$. Is it equivalent to one with a denominator that is a power of 10? How do you know?

  ▫ *Yes. The fraction $\frac{5}{8}$ has denominator 8 and so has factors that are products of 2's only.*

▪ Write $\frac{5}{8}$ as an equivalent fraction with a denominator that is a power of 10.

  ▫ *We have $\frac{5}{8} = \frac{5}{2\times2\times2} = \frac{5\times5\times5\times5}{2\times2\times2\times5\times5\times5} = \frac{625}{10\times10\times10} = \frac{625}{10^3}$.*

- What is $\dfrac{5}{8}$ as a finite decimal?

  □ $\dfrac{5}{8} = \dfrac{625}{1000} = 0.625$

Example 2 (4 minutes)

> **Example 2**
>
> Consider the fraction $\dfrac{17}{125}$. Is it equal to a finite or an infinite decimal? How do you know?

- Let's consider the fraction $\dfrac{17}{125}$. We want the decimal value of this number. Will it be a finite or an infinite decimal? How do you know?

  □ *We know that the fraction $\dfrac{17}{125}$ is equal to a finite decimal because the denominator 125 is a product of 5's, specifically, 5^3, and so we can write the fraction as one with a denominator that is a power of 10.*

- What will we need to multiply 5^3 by to obtain a power of 10?

  □ *We will need to multiply by 2^3. Then, $5^3 \times 2^3 = (5 \times 2)^3 = 10^3$.*

- Write $\dfrac{17}{125}$ or its equivalent $\dfrac{17}{5^3}$ as a finite decimal.

  □ $\dfrac{17}{125} = \dfrac{17}{5^3} = \dfrac{17 \times 2^3}{5^3 \times 2^3} = \dfrac{17 \times 8}{(5 \times 2)^3} = \dfrac{136}{10^3} = 0.136$

  (If the above two points are too challenging for some students, have them write out:
  $\dfrac{17}{125} = \dfrac{17}{5 \times 5 \times 5} = \dfrac{17 \times 2 \times 2 \times 2}{5 \times 5 \times 5 \times 2 \times 2 \times 2} = \dfrac{136}{1000} = 0.136.$)

Exercises 1–5 (5 minutes)

Students complete Exercises 1–5 independently.

> **Exercises 1–5**
>
> You may use a calculator, but show your steps for each problem.
>
> 1. Consider the fraction $\dfrac{3}{8}$.
>
>   a. Write the denominator as a product of 2's and/or 5's. Explain why this way of rewriting the denominator helps to find the decimal representation of $\dfrac{3}{8}$.
>
>   *The denominator is equal to 2^3. It is helpful to know that $8 = 2^3$ because it shows how many factors of 5 will be needed to multiply the numerator and denominator by so that an equivalent fraction is produced with a denominator that is a multiple of 10. When the denominator is a multiple of 10, the fraction can easily be written as a decimal using what I know about place value.*

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b. Find the decimal representation of $\frac{3}{8}$. Explain why your answer is reasonable.

$$\frac{3}{8} = \frac{3}{2^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{375}{10^3} = 0.375$$

The answer is reasonable because the decimal value, 0.375, is less than $\frac{1}{2}$, just like the fraction $\frac{3}{8}$.

2. Find the first four places of the decimal expansion of the fraction $\frac{43}{64}$.

The denominator is equal to 2^6.

$$\frac{43}{64} = \frac{43}{2^6} = \frac{43 \times 5^6}{2^6 \times 5^6} = \frac{671\,875}{10^6} = 0.671875$$

The decimal expansion to the first four decimal places is 0.6718.

3. Find the first four places of the decimal expansion of the fraction $\frac{29}{125}$.

The denominator is equal to 5^3.

$$\frac{29}{125} = \frac{29}{5^3} = \frac{29 \times 2^3}{5^3 \times 2^3} = \frac{232}{10^3} = 0.232$$

The decimal expansion to the first four decimal places is 0.2320.

4. Find the first four decimal places of the decimal expansion of the fraction $\frac{19}{34}$.

Using long division, the decimal expansion to the first four places is $0.5588\ldots$.

5. Identify the type of decimal expansion for each of the numbers in Exercises 1–4 as finite or infinite. Explain why their decimal expansion is such.

We know that the number $\frac{7}{8}$ had a finite decimal expansion because the denominator 8 is a product of 2's and so is equivalent to a fraction with a denominator that is a power of 10. We know that the number $\frac{43}{64}$ had a finite decimal expansion because the denominator 64 is a product of 2's and so is equivalent to a fraction with a denominator that is a power of 10. We know that the number $\frac{29}{125}$ had a finite decimal expansion because the denominator 125 is a product of 5's and so is equivalent to a fraction with a denominator that is a power of 10. We know that the number $\frac{19}{34}$ had an infinite decimal expansion because the denominator was not a product of 2's or 5's; it had a factor of 17 and so is not equivalent to a fraction with a denominator that is a power of 10.

Example 3 (4 minutes)

Example 3

Will the decimal expansion of $\frac{7}{80}$ be finite or infinite? If it is finite, find it.

- Will $\dfrac{7}{80}$ have a finite or infinite decimal expansion?

  □ *We know that the fraction $\dfrac{7}{80}$ is equal to a finite decimal because the denominator 80 is a product of 2's and 5's. Specifically, $2^4 \times 5$. This means the fraction is equivalent to one with a denominator that is a power of 10.*

- What will we need to multiply $2^4 \times 5$ by so that it is equal to $(2 \times 5)^n = 10^n$ for some n?

  □ *We will need to multiply by 5^3 so that $2^4 \times 5^4 = (2 \times 5)^4 = 10^4$.*

- Begin with $\dfrac{7}{80}$ or $\dfrac{7}{2^4 \times 5}$. Use what you know about equivalent fractions to rewrite $\dfrac{7}{80}$ in the form $\dfrac{k}{10^n}$ and then write the decimal form of the fraction.

  □ $\dfrac{7}{80} = \dfrac{7}{2^4 \times 5} = \dfrac{7 \times 5^3}{2^4 \times 5 \times 5^3} = \dfrac{7 \times 125}{(2 \times 5)^4} = \dfrac{875}{10^4} = 0.0875$

Example 4 (4 minutes)

> **Example 4**
>
> **Will the decimal expansion of $\dfrac{3}{160}$ be finite or infinite? If it is finite, find it.**

- Will $\dfrac{3}{160}$ have a finite or infinite decimal expansion?

  □ *We know that the fraction $\dfrac{3}{160}$ is equal to a finite decimal because the denominator 160 is a product of 2's and 5's. Specifically, $2^5 \times 5$. This means the fraction is equivalent to one with a denominator that is a power of 10.*

- What will we need to multiply $2^5 \times 5$ by so that it is equal to $(2 \times 5)^n = 10^n$ for some n?

  □ *We will need to multiply by 5^4 so that $2^5 \times 5^5 = (2 \times 5)^5 = 10^5$.*

- Begin with $\dfrac{3}{160}$ or $\dfrac{3}{2^5 \times 5}$. Use what you know about equivalent fractions to rewrite $\dfrac{3}{160}$ in the form $\dfrac{k}{10^n}$ and then write the decimal form of the fraction.

  □ $\dfrac{3}{160} = \dfrac{3}{2^5 \times 5} = \dfrac{3 \times 5^4}{2^5 \times 5 \times 5^4} = \dfrac{3 \times 625}{(2 \times 5)^5} = \dfrac{1875}{10^5} = 0.01875$

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Exercises 6–8 (5 minutes)

Students complete Exercises 6–8 independently.

Exercises 6–8

You may use a calculator, but show your steps for each problem.

6. **Convert the fraction $\dfrac{37}{40}$ to a decimal.**

   a. **Write the denominator as a product of 2's and/or 5's. Explain why this way of rewriting the denominator helps to find the decimal representation of $\dfrac{37}{40}$.**

   The denominator is equal to $2^3 \times 5$. It is helpful to know that 40 is equal to $2^3 \times 5$ because it shows by how many factors of 5 the numerator and denominator will need to be multiplied to produce an equivalent fraction with a denominator that is a power of 10. When the denominator is a power of 10, the fraction can easily be written as a decimal using what I know about place value.

   b. **Find the decimal representation of $\dfrac{37}{40}$. Explain why your answer is reasonable.**

   $$\frac{37}{40} = \frac{37}{2^3 \times 5} = \frac{37 \times 5^2}{2^3 \times 5 \times 5^2} = \frac{925}{10^3} = 0.925$$

   The answer is reasonable because the decimal value, 0.925, is less than 1, just like the fraction $\dfrac{37}{40}$. Also, it is reasonable and correct because the fraction $\dfrac{925}{1000} = \dfrac{37}{40}$; therefore, it has the decimal expansion 0.925.

7. **Convert the fraction $\dfrac{3}{250}$ to a decimal.**

   The denominator is equal to 2×5^3.

   $$\frac{3}{250} = \frac{3}{2 \times 5^3} = \frac{3 \times 2^2}{2 \times 2^2 \times 5^3} = \frac{12}{10^3} = 0.012$$

8. **Convert the fraction $\dfrac{7}{1250}$ to a decimal.**

   The denominator is equal to 2×5^4.

   $$\frac{7}{1250} = \frac{7}{2 \times 5^4} = \frac{7 \times 2^3}{2 \times 2^3 \times 5^4} = \frac{56}{10^4} = 0.0056$$

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that finite decimals are fractions with denominators that are a power of 10. Any fraction with a denominator that can be expressed as products of 2's and/or 5's can be expressed as one with a denominator that is a power of 10.
- We know how to use equivalent fractions to convert a fraction to its decimal equivalent.

Lesson Summary

Fractions with denominators that can be expressed as products of 2's and/or $5's$ are equivalent to fractions with denominators that are a power of 10. These are precisely the fractions with finite decimal expansions.

Example:

Does the fraction $\frac{1}{8}$ have a finite or an infinite decimal expansion?

Since $8 = 2^3$, then the fraction has a finite decimal expansion. The decimal expansion is found as

$$\frac{1}{8} = \frac{1}{2^3} = \frac{1 \times 5^3}{2^3 \times 5^3} = \frac{125}{10^3} = 0.125.$$

If the denominator of a (simplified) fraction cannot be expressed as a product of 2's and/or 5's, then the decimal expansion of the number will be infinite.

Exit Ticket (4 minutes)

Name _____   Date _____

Lesson 6: Finite and Infinite Decimals

Exit Ticket

Convert each fraction to a finite decimal if possible. If the fraction cannot be written as a finite decimal, then state how you know. You may use a calculator, but show your steps for each problem.

1. $\dfrac{9}{16}$

2. $\dfrac{8}{125}$

3. $\dfrac{4}{15}$

4. $\dfrac{1}{200}$

Exit Ticket Sample Solutions

Convert each fraction to a finite decimal if possible. If the fraction cannot be written as a finite decimal, then state how you know. You may use a calculator, but show your steps for each problem.

1. $\dfrac{9}{16}$

The denominator is equal to 2^4.

$$\frac{9}{16} = \frac{9}{2^4} = \frac{9 \times 5^4}{2^4 \times 5^4} = \frac{9 \times 625}{10^4} = \frac{5625}{10^4} = 0.5625$$

2. $\dfrac{8}{125}$

The denominator is equal to 5^3.

$$\frac{8}{125} = \frac{8}{5^3} = \frac{8 \times 2^3}{5^3 \times 2^3} = \frac{8 \times 8}{10^3} = \frac{64}{10^3} = 0.064$$

3. $\dfrac{4}{15}$

The fraction $\dfrac{4}{15}$ is not a finite decimal because the denominator is equal to 5×3. Since the denominator cannot be expressed as a product of 2's and 5's, then $\dfrac{4}{15}$ is not a finite decimal.

4. $\dfrac{1}{200}$

The denominator is equal to $2^3 \times 5^2$.

$$\frac{1}{200} = \frac{1}{2^3 \times 5^2} = \frac{1 \times 5}{2^3 \times 5^2 \times 5} = \frac{5}{2^3 \times 5^3} = \frac{5}{10^3} = 0.005$$

Problem Set Sample Solutions

Convert each fraction given to a finite decimal, if possible. If the fraction cannot be written as a finite decimal, then state how you know. You may use a calculator, but show your steps for each problem.

1. $\dfrac{2}{32}$

The fraction $\dfrac{2}{32}$ simplifies to $\dfrac{1}{16}$.

The denominator is equal to 2^4.

$$\frac{1}{16} = \frac{1}{2^4} = \frac{1 \times 5^4}{2^4 \times 5^4} = \frac{625}{10^4} = 0.0625$$

2. $\dfrac{99}{125}$

The denominator is equal to 5^3.

$$\frac{99}{125} = \frac{99}{5^3} = \frac{99 \times 2^3}{2^3 \times 5^3} = \frac{792}{10^3} = 0.792$$

3. $\dfrac{15}{128}$

The denominator is equal to 2^7.

$$\frac{15}{128} = \frac{15}{2^7} = \frac{15 \times 5^7}{2^7 \times 5^7} = \frac{1171875}{10^7} = 0.1171875$$

4. $\dfrac{8}{15}$

The fraction $\dfrac{8}{15}$ is not a finite decimal because the denominator is equal to 3×5. Since the denominator cannot be expressed as a product of 2's and 5's, then $\dfrac{8}{15}$ is not a finite decimal.

5. $\dfrac{3}{28}$

The fraction $\dfrac{3}{28}$ is not a finite decimal because the denominator is equal to $2^2 \times 7$. Since the denominator cannot be expressed as a product of 2's and 5's, then $\dfrac{3}{28}$ is not a finite decimal.

6. $\dfrac{13}{400}$

The denominator is equal to $2^4 \times 5^2$.

$$\frac{13}{400} = \frac{13}{2^4 \times 5^2} = \frac{13 \times 5^2}{2^4 \times 5^2 \times 5^2} = \frac{325}{10^4} = 0.0325$$

7. $\dfrac{5}{64}$

The denominator is equal to 2^6.

$$\frac{5}{64} = \frac{5}{2^6} = \frac{5 \times 5^6}{2^6 \times 5^6} = \frac{78125}{10^6} = 0.078125$$

8. $\dfrac{15}{35}$

The fraction $\dfrac{15}{35}$ reduces to $\dfrac{3}{7}$. The denominator 7 cannot be expressed as a product of 2's and 5's. Therefore, $\dfrac{3}{7}$ is not a finite decimal.

9. $\dfrac{199}{250}$

The denominator is equal to 2×5^3.

$$\frac{199}{250} = \frac{199}{2 \times 5^3} = \frac{199 \times 2^2}{2 \times 2^2 \times 5^3} = \frac{796}{10^3} = 0.796$$

10. $\dfrac{219}{625}$

The denominator is equal to 5^4.

$$\frac{219}{625} = \frac{219}{5^4} = \frac{219 \times 2^4}{2^4 \times 5^4} = \frac{3504}{10^4} = 0.3504$$

Lesson 6:   Finite and Infinite Decimals

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Lesson 7: Infinite Decimals

Student Outcomes

- Students develop an intuitive understanding of the placement of infinite decimals on the number line.
- Students develop an argument for believing that 0.9999… should equal 1.

Lesson Notes

The purpose of this lesson is to show the linkage between the various forms of a number, specifically, its decimal expansion, the expanded form of its decimal, and the placement of the number on the number line. Given the decimal expansion of a number, students use what they know about place value to write the expanded form of the number. That expanded form is then used to determine the placement of the number on the number line by looking at increasingly smaller intervals of negative powers of 10, beginning with tenths, then hundredths, then thousandths, and so on. This is the strategy students use to determine the beginning decimal expansions of irrational numbers.

Classwork

Opening Exercise (7 minutes)

Opening Exercise

a. **Write the expanded form of the decimal 0.3765 using powers of 10.**

$$0.3765 = \frac{3}{10} + \frac{7}{10^2} + \frac{6}{10^3} + \frac{5}{10^4}$$

b. **Write the expanded form of the decimal $0.3333333…$ using powers of 10.**

$$0.333333… = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \frac{3}{10^6} + \cdots$$

c. **Have you ever wondered about the value of $0.99999…$? Some people say this infinite decimal has a value of 1. Some disagree. What do you think?**

Answers will vary. Have a brief discussion with students about this exercise. The answer will be revisited in the discussion below.

Discussion (20 minutes)

Example 1

Example 1

The number 0.253 is represented on the number line below.
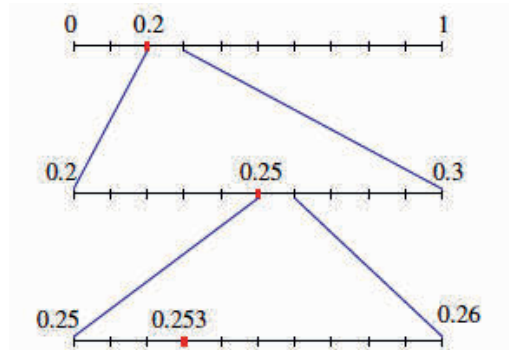
- How can we locate the number $0.253 = \frac{2}{10} + \frac{5}{100} + \frac{3}{1000}$ on the number line?

  We can see that 0.253 is a tad larger than 0.2 and is smaller than 0.3. So if we divide the line segment from 0 to 1 into tenths, 0.253 lies somewhere in the segment between 0.2 and 0.3.
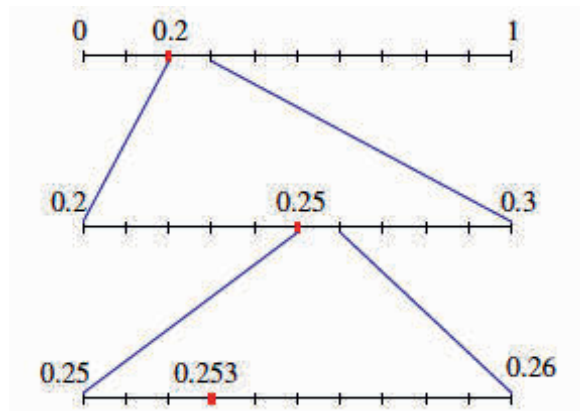
  Now divide this segment into ten equal parts. (Those parts are hundredths of the original unit length.) We know that 0.253 is larger than 0.25 but smaller than 0.26, and so lies in the segment between these two values.

  Now divide this small segment into tenths again. (We are now dealing with thousandths of the original unit length.) We can now see where exactly to pin 0.253 on the number line.

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- Writing 0.253 in its expanded decimal form of $\frac{2}{10} + \frac{5}{100} + \frac{3}{1000}$ illustrates this process:

    The first decimal digit of 0.253 is 0.2, or $\frac{2}{10}$, and this tells us within which tenth we are to place 0.253.

    The first two decimal digits of 0.253 are 0.25 which is equal to $\frac{2}{10} + \frac{5}{10^2}$, or $\frac{25}{10^2}$, and this tells us within which hundredth we are to place 0.253.

    The first three decimal digits of 0.253 are 0.253 which is equal to $\frac{2}{10} + \frac{5}{10^2} + \frac{3}{10^3}$, or $\frac{253}{10^3}$, and this tells us within which thousandth we are to place 0.253. And since the decimal terminates here, we are done.

Have the students explain this process again in their own words, referring to the number line diagram as they do so.

- How do you think this process would change if we tried to locate an infinite decimal on the number line?

    □ *The sequence for an infinite decimal would never end; it would go on infinitely.*

- We need to introduce some notation. If a decimal has a repeating pattern, as for 0.3333… or 7.45454545…, for instance, then a horizontal bar is used to indicate that a block of digits is being repeated. For example, 0.3333… is written as $0.\overline{3}$ and 7.45454545… as $7.\overline{45}$. It is conceivable that an infinite decimal could have no repeating pattern.

Example 2

Example 2

The number $\frac{5}{6}$, which is equal to $0.833333…$, or $0.8\overline{3}$, is partially represented on the number line below.
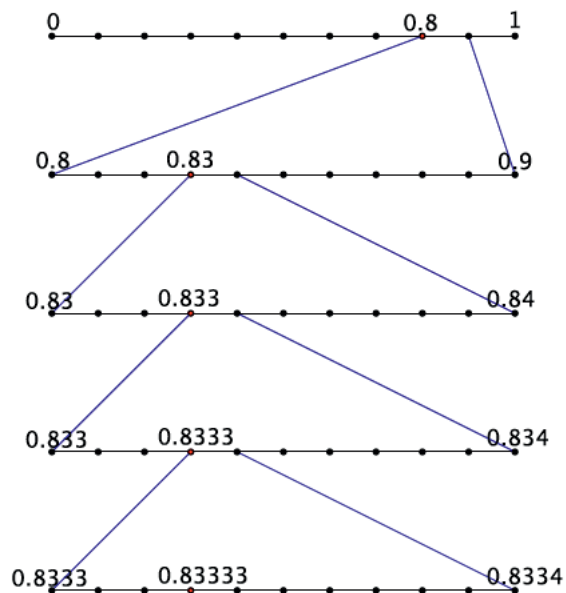
- Now, consider the equality $\frac{5}{6} = 0.833333\ldots = 0.8\overline{3}$. Notice that at the second step, the work feels as though it repeats, which coincides with the fact that the decimal digit of 3 repeats.
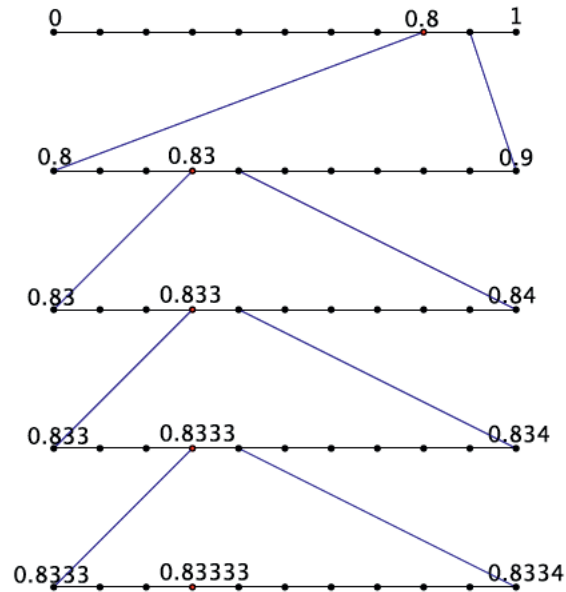
- What is the expanded form of the decimal $0.833333\ldots$?
  - $0.833333\ldots = \frac{8}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \frac{3}{10^6} + \cdots$

- We see again that at the second step the work begins to repeat.

- Each step can be represented by increasing powers of 10 in the denominator: $\frac{8}{10}, \frac{83}{10^2}, \frac{833}{10^3}, \frac{8333}{10^4}, \frac{83333}{10^5}, \frac{833333}{10^6}$, and so on. When will it end? Explain.
  - *It will never end because the decimal is infinite.*

- As we step through this process we are pinning the exact location of $0.8333\ldots$ into smaller and smaller intervals, intervals with sizes shrinking to zero: an interval of a tenth, and then a hundredth, then a thousandth, and, if we kept going, an interval of size $\frac{1}{10^{20}}$, and later on to an interval of size $\frac{1}{10^{100}}$, and so on, supposedly forever!

> *Scaffolding:*
> The words *finite* and *infinite* would appear to have similar pronunciations, when in fact the stresses and vowel sounds are different. Helping students make the text or speech connection will be useful so that they recognize the words when written and when spoken.

- Okay. Let's now think deeply about $0.9999\ldots$. Where do we find this number on the number line?

- Draw on the board a sequence of number-line segments akin to those in Example 2, and have students give instructions on how to pin down the location of $0.9999\ldots$. In which tenth does it lie? In which hundredth? In which thousandth? And so on. Ask: At any stage of this process is $0.9999\ldots$ right at the number 1 on the number line?
  - *No. We are always just to the left of 1.*

- Right. And that makes sense as 0.9 is smaller than 1, and 0.99 is smaller than 1, as are 0.999 and 0.9999. At every stage of the process we are just shy of the number 1.

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- But here is the tricky part: this is an *infinite* process. I, as a human being, cannot actually conduct our process forever. And whenever I stop, it is true that I'll be just shy of 1. But if I could go on forever, would I see that the actual location of 0.9999… is right at 1, do you think?
  - *This is philosophically challenging, and many students might well argue that if every stage of the process gives a finite decimal just to the left of 1 on the number line, then the infinite decimal will be to the left of 1 as well.*
- Okay. Let me turn the thinking around a bit. If 0.9999… is not 1, then there must be some space between 0.9999… and 1 on the number line. Could that space be as big as one-hundredth of a unit?
  - *Looking at the second stage of the number line diagram drawn on the board, we see that 0.9999… is closer than one-hundredth of a unit to 1.*
- Could the space between 0.9999… and 1 be as much as one-thousandth of a unit?
  - *No. Looking at the next stage of the number-line diagram, we see that 0.9999… is certainly within a distance of one-thousandth of a unit from 1.*
- Could it be that the space between 0.9999… and 1 is as much as a millionth of a unit? A billionth of a unit? A quadrillionth of a unit?
  - *No. The number-line diagram, if we kept going, would show that 0.9999… is within all these ranges from 1.*
- Could there be any space between 0.9999… and 1 on the number line at all?
  - *No. The number-line diagrams would show that 0.9999… is within any range of 1 we care to identify.*
- So if there is no space between 0.9999… and 1 on the number line, what must you conclude?
  - 0.9999 … *and* 1 *must be the same number. That is,* 0.9999… *equals* 1.
- This is wonderfully surprising!

Exercises 1–5 (8 minutes)

Students complete Exercises 1–5 independently or in pairs.

Exercises 1–5

1.

    a.   Write the expanded form of the decimal 0.125 using powers of 10.

$$0.125 = \frac{1}{10} + \frac{2}{10^2} + \frac{5}{10^3}$$

    b.   Show on the number line the placement of the decimal 0.125.

2.

a. Write the expanded form of the decimal 0.3875 using powers of 10.

$$0.3875 = \frac{3}{10} + \frac{8}{10^2} + \frac{7}{10^3} + \frac{5}{10^4}$$

b. Show on the number line the placement of the decimal 0.3875.

3.
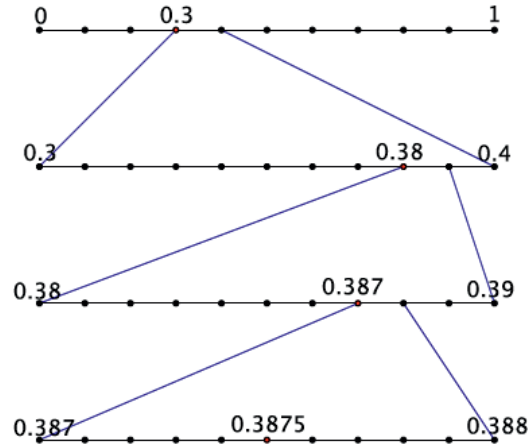
a. Write the expanded form of the decimal $0.777777\ldots$ using powers of 10.

$$0.777777\ldots = \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \frac{7}{10^5} + \frac{7}{10^6} + \cdots$$

b. Show the first few stages of placing the decimal $0.777777\ldots$ on the number line.

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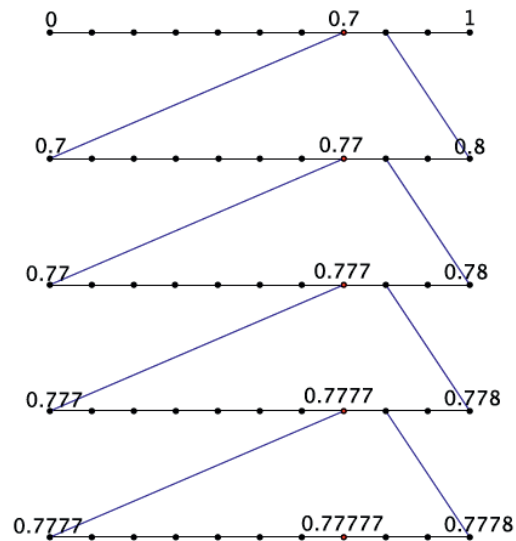
4.

a.   Write the expanded form of the decimal $0.\overline{45}$ using powers of 10.

$$0.\overline{45} = \frac{4}{10} + \frac{5}{10^2} + \frac{4}{10^3} + \frac{5}{10^4} + \frac{4}{10^5} + \frac{5}{10^6} + \cdots$$

b.   Show the first few stages of placing the decimal $0.\overline{45}$ on the number line.
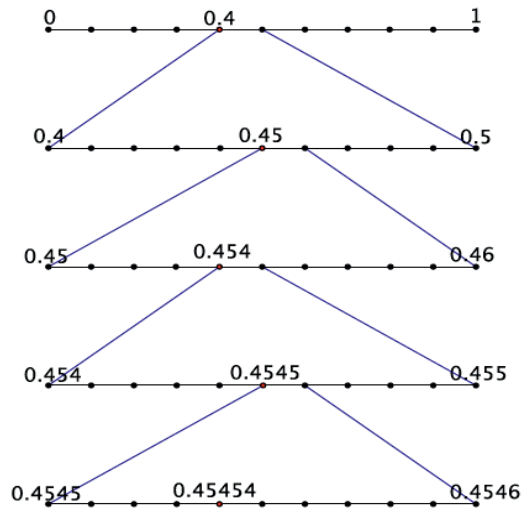
5.

a.   Order the following numbers from least to greatest: 2.121212, 2.1, 2.2, and $2.\overline{12}$.

$2.1, 2.121212, 2.\overline{12}, and\ 2.2$

b.   Explain how you knew which order to put the numbers in.

Each number is the sum of the whole number 2 and a decimal. When you write each number in this manner, you get the following expansions.

$$2.121212 = 2 + \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{2}{10^6}$$

$$2.1 = 2 + \frac{1}{10}$$

$$2.2 = 2 + \frac{2}{10}$$

$$2.\overline{12} = 2 + \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{2}{10^6} + \frac{1}{10^7} + \frac{2}{10^8} + \cdots$$

In this form, it is clear that 2.1 is the least of the four numbers, followed by the finite decimal 2.121212, then the infinite decimal $2.\overline{12}$, and finally 2.2.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We know how to place finite decimals on the number line.
- For an infinite decimal, we know that our process of pinning down its location on the number line places it in smaller and smaller ranges of intervals; within a tenth, within a hundredth, within a thousandth, and so on.
- We know that the infinite decimal $0.\overline{9}$ equals 1, and we can explain why this must be so.

Lesson Summary

An infinite decimal is a decimal whose expanded form is infinite.

Example:

The expanded form of the decimal $0.8\overline{3} = 0.83333\ldots$ is $\dfrac{8}{10} + \dfrac{3}{10^2} + \dfrac{3}{10^3} + \dfrac{3}{10^4} + \cdots$.

To pin down the placement of an infinite decimal on the number line, we first identify within which tenth it lies, then within which hundredth it lies, then within which thousandth, and so on. These intervals have widths getting closer and closer to a width of zero.
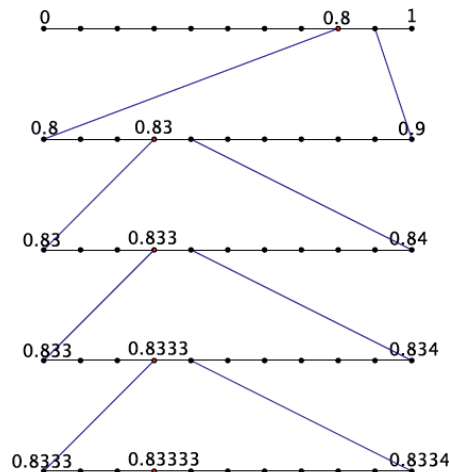
This reasoning allows us to deduce that the infinite decimal $0.9999\ldots$ and 1 have the same location on the number line. Consequently, $0.\overline{9} = 1$.

Exit Ticket (5 minutes)

There are three items as part of the Exit Ticket. Just the first two might be sufficient to assess students' understanding.

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Name _____   Date _____

Lesson 7: Infinite Decimals

Exit Ticket

1.

    a.   Write the expanded form of the decimal 0.829 using powers of 10.

    b.   Show the placement of the decimal 0.829 on the number line.

2.

    a.   Write the expanded form of the decimal 0.55555… using powers of 10.

    b.   Show the first few stages of placing the decimal 0.555555… on the number line.

3.
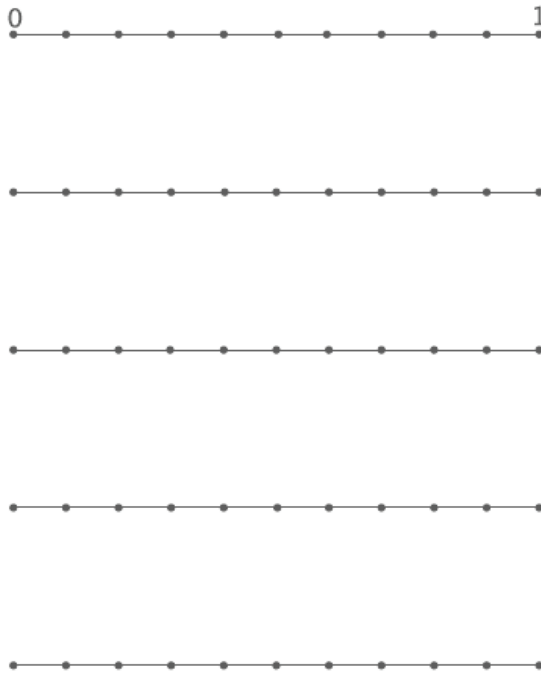
    a.   Write the expanded form of the decimal $0.\overline{573}$ using powers of 10.

    b.   Show the first few stages of placing the decimal $0.\overline{573}$ on the number line.

Exit Ticket Sample Solutions

1.
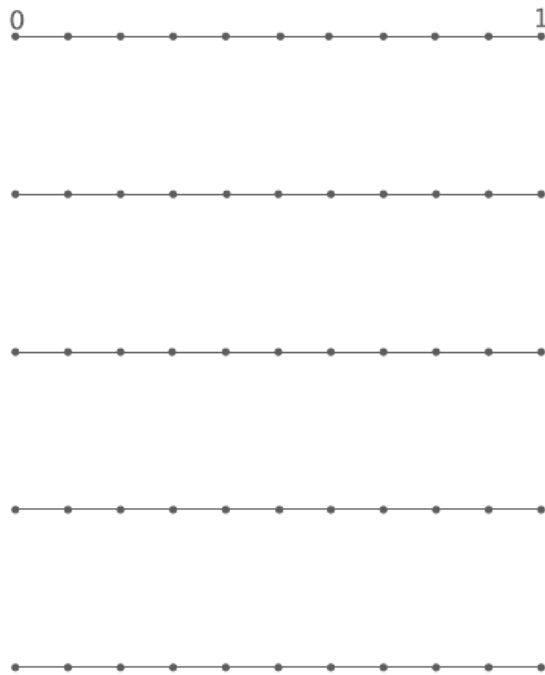
    a.   **Write the expanded form of the decimal 0.829 using powers of 10.**

$$0.829 = \frac{8}{10} + \frac{2}{10^2} + \frac{9}{10^3}$$

    b.   **Show the placement of the decimal 0.829 on the number line.**

2.
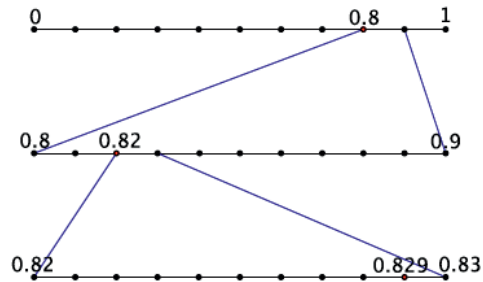
    a.   **Write the expanded form of the decimal $0.55555\ldots$ using powers of 10.**

$$0.55555\ldots = \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \frac{5}{10^5} + \frac{5}{10^6} + \cdots$$

    b.   **Show the first few stages of placing the decimal $0.555555\ldots$ on the number line.**

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3.
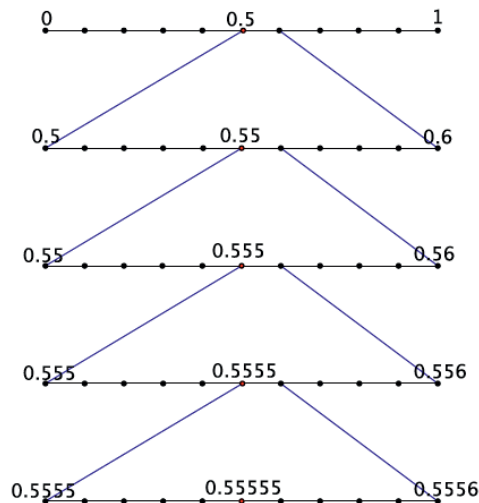
    a.   Write the expanded form of the decimal $0.\overline{573}$ using powers of 10.

$$0.\overline{573} = \frac{5}{10} + \frac{7}{10^2} + \frac{3}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \frac{3}{10^6} + \cdots$$

    b.   Show the first few stages of placing the decimal $0.\overline{573}$ on the number line.

Problem Set Sample Solutions

1.
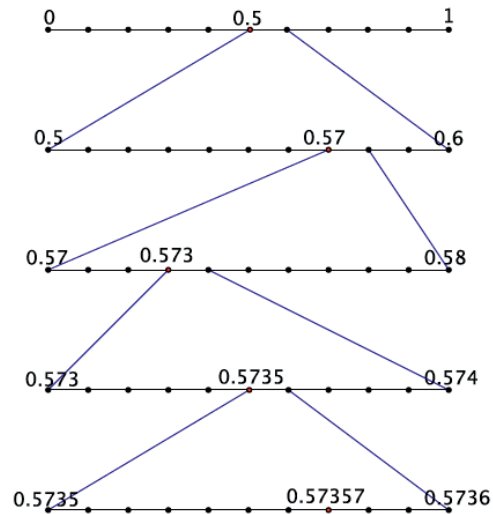
    a.   Write the expanded form of the decimal 0.625 using powers of 10.

$$0.625 = \frac{6}{10} + \frac{2}{10^2} + \frac{5}{10^3}$$

    b.   Place the decimal 0.625 on the number line.

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Lesson 7:   Infinite Decimals

2.
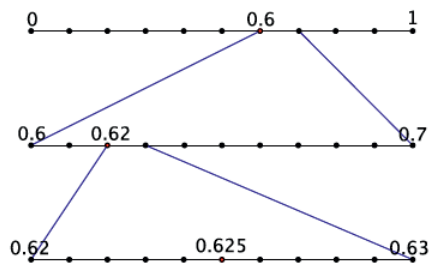
    a. Write the expanded form of the decimal $0.\overline{370}$ using powers of 10.

$$0.\overline{370} = \frac{3}{10} + \frac{7}{10^2} + \frac{0}{10^3} + \frac{3}{10^4} + \frac{7}{10^5} + \frac{0}{10^6} + \cdots$$

    b. Show the first few stages of placing the decimal $0.370370\ldots$ on the number line.
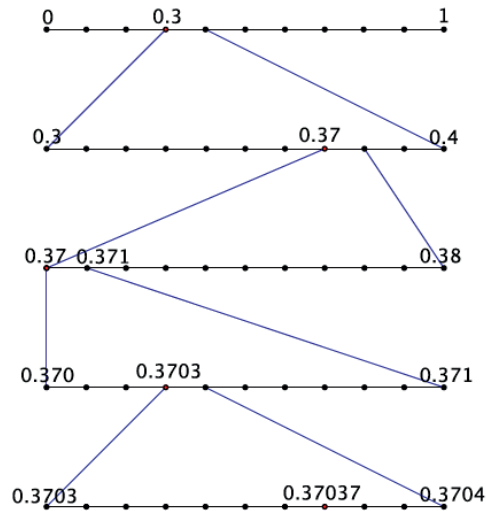
3. Which is a more accurate representation of the fraction $\frac{2}{3}$: 0.6666 or $0.\overline{6}$? Explain. Which would you prefer to compute with?

The number $\frac{2}{3}$ is more accurately represented by the decimal $0.\overline{6}$ compared to 0.6666. The long division algorithm with $\frac{2}{3}$ shows that the digit 6 repeats. Then, the expanded form of the decimal $0.\overline{6}$ is $\frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \frac{6}{10^5} + \frac{6}{10^6} + \cdots$, and the expanded form of the decimal 0.6666 is $\frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4}$. For this reason, $0.\overline{6}$ is precise, but 0.6666 is an approximation. For computations, I would prefer to use 0.6666. My answer would be less precise, but at least I would be able to compute with it. When attempting to compute with an infinite number, you would never finish writing it; thus, you could never compute with it.

4. Explain why we shorten infinite decimals to finite decimals to perform operations. Explain the effect of shortening an infinite decimal on our answers.

We often shorten infinite decimals to finite decimals to perform operations because it would be impossible to represent an infinite decimal precisely since the sequence that describes infinite decimals has an infinite number of steps. Our answers are less precise; however, they are not that much less precise because with each additional digit we include in the sequence, we are adding a very small amount to the value of the number. The more decimals we include, the closer the value we add approaches zero. Therefore, it does not make that much of a difference with respect to our answer.

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5. A classmate missed the discussion about why $0.\overline{9} = 1$. Convince your classmate that this equality is true.

Answers will vary. Accept any reasonable explanation. One is provided below.

Ask: Could there be any space between the locations of $0.9999\ldots$ and 1 on the number line? We have that $0.9999\ldots$ is larger than 0.9 and so is within one-tenth of 1 on the number line. We also have that $0.9999\ldots$ is larger than 0.99 and so is within one-hundredth of 1 on the number line. And $0.9999\ldots$ is larger than 0.999 and so is within one-thousandth of 1 on the number line, and so on. There can be no space between $0.9999\ldots$ and 1 on the number line, as we can always argue that $0.9999\ldots$ must be within any given distance from 1. Thus, $0.9999\ldots$ and 1 must sit at the same location on the number line and so are the same number.

6. Explain why $0.3333 < 0.33333$.

$0.3333 = \dfrac{3}{10} + \dfrac{3}{10^2} + \dfrac{3}{10^3} + \dfrac{3}{10^4}$, *and* $0.33333 = \dfrac{3}{10} + \dfrac{3}{10^2} + \dfrac{3}{10^3} + \dfrac{3}{10^4} + \dfrac{3}{10^5}$. *That means that 0.33333 is exactly $\dfrac{3}{10^5}$ larger than 0.3333. If we examined the numbers on the number line, 0.33333 is to the right of 0.3333, meaning that it is larger than 0.3333.*

Lesson 8: The Long Division Algorithm

Student Outcomes

- Students explore a variation of the long division algorithm.
- Students discover that every rational number has a repeating decimal expansion.

Lesson Notes

In this lesson, students move toward being able to define an irrational number by first noting the decimal structure of rational numbers.

Classwork

Example 1 (5 minutes)

> **Example 1**
>
> Show that the decimal expansion of $\dfrac{26}{4}$ is 6.5.

Scaffolding:

There is no single long division algorithm. The algorithm commonly taught and used in the U.S. is rarely used elsewhere. Students may come with earlier experiences with other division algorithms that make more sense to them. Consider using formative assessment to determine how different students approach long division.

Use the example with students so they have a model to complete Exercises 1–5.

- Show that the decimal expansion of $\dfrac{26}{4}$ is 6.5.
  - *Students might use the long division algorithm, or they might simply observe $\dfrac{26}{4} = \dfrac{13}{2} = 6.5$.*
- Here is another way to see this: What is the greatest number of groups of 4 that are in 26?
  - *There are 6 groups of 4 in 26.*
- Is there a remainder?
  - *Yes, there are 2 left over.*
- This means we can write 26 as

$$26 = 6 \times 4 + 2.$$

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- This means we could also compute $\dfrac{26}{4}$ as follows:

$$\dfrac{26}{4} = \dfrac{6 \times 4 + 2}{4}$$

$$\dfrac{26}{4} = \dfrac{6 \times 4}{4} + \dfrac{2}{4}$$

$$\dfrac{26}{4} = 6 + \dfrac{2}{4}$$

$$\dfrac{26}{4} = 6\dfrac{2}{4} = 6\dfrac{1}{2}.$$

(Some students might note we are simply rewriting the fraction as a mixed number.)

- The fraction $\dfrac{26}{4}$ is equal to the finite decimal 6.5. When the fraction is not equal to a finite decimal, then we need to use the long division algorithm to determine the decimal expansion of the number.

Exploratory Challenge/Exercises 1–5 (15 minutes)

Students complete Exercises 1–5 independently or in pairs. The discussion that follows is related to the concepts in the exercises.

Exploratory Challenge/Exercises 1–5

1.

   a. Use long division to determine the decimal expansion of $\dfrac{142}{2}$.

$$\begin{array}{r} 71.0 \\ 2\overline{)142.0} \end{array}$$

   b. Fill in the blanks to show another way to determine the decimal expansion of $\dfrac{142}{2}$.

$$142 = \underline{\ 71\ } \times 2 + \underline{\ 0\ }$$

$$\dfrac{142}{2} = \dfrac{\underline{\ 71\ } \times 2 + \underline{\ 0\ }}{2}$$

$$\dfrac{142}{2} = \dfrac{\underline{\ 71\ } \times 2}{2} + \dfrac{\underline{\ 0\ }}{2}$$

$$\dfrac{142}{2} = \underline{\ 71\ } + \dfrac{\underline{\ 0\ }}{2}$$

$$\dfrac{142}{2} = \underline{\ 71.0\ }$$

   c. Does the number $\dfrac{142}{2}$ have a finite or an infinite decimal expansion?

   The decimal expansion of $\dfrac{142}{2}$ is 71.0 and is finite.

2.

a.   Use long division to determine the decimal expansion of $\frac{142}{4}$.

$$4\overline{)142.0}^{\,35.5}$$

b.   Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{4}$.

$$142 = \underline{\ 35\ } \times 4 + \underline{\ 2\ }$$

$$\frac{142}{4} = \frac{\underline{\ 35\ } \times 4 + \underline{\ 2\ }}{4}$$

$$\frac{142}{4} = \frac{\underline{\ 35\ } \times 4}{4} + \frac{\underline{\ 2\ }}{4}$$

$$\frac{142}{4} = \underline{\ 35\ } + \frac{\underline{\ 2\ }}{4}$$

$$\frac{142}{4} = \underline{\ 35\frac{2}{4} = 35.5\ }$$

c.   Does the number $\frac{142}{4}$ have a finite or an infinite decimal expansion?

The decimal expansion of $\frac{142}{4}$ is 35.5 and is finite.

3.

a.   Use long division to determine the decimal expansion of $\frac{142}{6}$.

$$
\begin{array}{r}
23.666 \\
6\overline{)142.000} \\
\underline{12} \\
22 \\
\underline{18} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
4
\end{array}
$$

b.   Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{6}$.

$$142 = \underline{\ 23\ } \times 6 + \underline{\ 4\ }$$

$$\frac{142}{6} = \frac{\underline{\ 23\ } \times 6 + \underline{\ 4\ }}{6}$$

$$\frac{142}{6} = \frac{\underline{\ 23\ } \times 6}{6} + \frac{\underline{\ 4\ }}{6}$$

$$\frac{142}{6} = \underline{\ 23\ } + \frac{\underline{\ 4\ }}{6}$$

$$\frac{142}{6} = \underline{\ 23\frac{4}{6} = 23.666\ldots\ }$$

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c.   Does the number $\frac{142}{6}$ have a finite or an infinite decimal expansion?

The decimal expansion of $\frac{142}{6}$ is $23.666...$ and is infinite.

4.

a.   Use long division to determine the decimal expansion of $\frac{142}{11}$.

```
        12.90909
   11)142.00000
        11
        32
        22
       100
        99
        10
        00
       100
        99
        10
        00
       100
        99
        10
```

b.   Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{11}$.

$$142 = \underline{\ 12\ } \times 11 + \underline{\ 10\ }$$

$$\frac{142}{11} = \frac{\underline{\ 12\ } \times 11 + \underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \frac{\underline{\ 12\ } \times 11}{11} + \frac{\underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \underline{\ 12\ } + \frac{\underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \underline{\ 12\frac{10}{11}\ } = 12.90909...$$

c.   Does the number $\frac{142}{11}$ have a finite or an infinite decimal expansion?

The decimal expansion of $\frac{142}{11}$ is $12.90909...$ and is infinite.

5.   In general, which fractions produce infinite decimal expansions?

We discovered in Lesson 6 that fractions equivalent to ones with denominators that are a power of 10 are precisely the fractions with finite decimal expansions. These fractions, when written in simplified form, have denominators with factors composed of 2's and 5's. Thus any fraction, in simplified form, whose denominator contains a factor different from 2 or 5 must yield an infinite decimal expansion.

Discussion (10 minutes)

- What is the decimal expansion of $\frac{142}{2}$?

If students respond 71, ask them what decimal digits they could include without changing the value of the number.

  □ *The fraction $\frac{142}{2}$ is equal to the decimal 71.00000….*

- Did you need to use the long division algorithm to determine your answer? Why or why not?
  □ *No, the long division algorithm was not necessary because there was a whole number of 2's in 142.*

- What is the decimal expansion of $\frac{142}{4}$?

  □ *The fraction $\frac{142}{4}$ is equal to the decimal 35.5.*

- What decimal digits could we include to the right of the ". 5" in 35.5 without changing the value of the number?

  □ *We could write the decimal as 35.500000….*

- Did you need to use the long division algorithm to determine your answer? Why or why not?

  □ *No, the long division algorithm was not necessary because $\frac{142}{4} = 35 + \frac{2}{4}$, and $\frac{2}{4}$ is a finite decimal. We can write $\frac{2}{4}$ as 0.5.*

- What is the decimal expansion of $\frac{142}{6}$?

  □ *The fraction $\frac{142}{6}$ is equal to the decimal 23.66666….*

- Did you need to use the long division algorithm to determine your answer? Why or why not?

  □ *Yes, the long division algorithm was necessary because $\frac{142}{6} = 23 + \frac{2}{3}$, and $\frac{2}{3}$ is not a finite decimal.*

    Note: Some students may have recognized the fraction $\frac{2}{3}$ as 0.6666… and not used the long division algorithm to determine the decimal expansion.

- How did you know when you could stop dividing?

  □ *I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, the number 40 kept appearing, and there are 6 groups of 6 in 40, leaving 4 as a remainder each time, which became 40 when I brought down another 0.*

- We represent the decimal expansion of $\frac{142}{6}$ as $23.\overline{6}$, where the line above the 6 is the *repeating block*; that is, the digit 6 repeats as we saw in the long division algorithm.

- What is the decimal expansion of $\frac{142}{11}$?

  □ *The fraction $\frac{142}{11}$ is equal to the decimal 12.90909090….*

- Did you need to use the long division algorithm to determine your answer? Why or why not?

  □ *Yes, the long division algorithm was necessary because $\frac{142}{11} = 12 + \frac{10}{11}$, and $\frac{10}{11}$ is not a finite decimal.*

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- How did you know when you could stop dividing?
  - *I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, I kept getting the number 10, which is not divisible by 11, so I had to bring down another 0, making the number 100. This kept happening, so I knew to stop once I noticed the work I was doing was the same.*

- Which block of digits kept repeating?
  - *The block of digits that kept repeating was 90.*

- How do we represent the decimal expansion of $\dfrac{142}{11}$?
  - *The decimal expansion of $\dfrac{142}{11}$ is $12.\overline{90}$.*

- In general, the long division algorithm shows that decimal expansion of any fraction $\dfrac{a}{b}$ is either finite or is an infinitely long decimal with a repeating pattern.

- Actually, even finite decimals can be thought of as infinitely long decimals with a repeating pattern. Can you see how to view the decimal expansion of $\dfrac{1}{4}$ this way? How about the decimal expansions of $\dfrac{142}{2}$ and $\dfrac{142}{4}$ too?
  - *Have students discuss this.*

- We have $\dfrac{1}{4} = 0.25 = 0.2500000\ldots = 0.25\overline{0}$, $\dfrac{142}{2} = 71.\overline{0}$, and $\dfrac{142}{4} = 35.5\overline{0}$.

- A definition: A number is called *rational* if it can be written in the form $\dfrac{a}{b}$ for two integers a and b with b non-zero. Thus all fractions, like $\dfrac{3}{11}$, for instance, are rational numbers, as is the answer to $9.2 \div 3$ (the answer is equivalent to the fraction $\dfrac{92}{30}$).

- We have argued that the long-division algorithm shows that every rational number has a decimal expansion that eventually falls into a repeating pattern. (And if that pattern is one of repeating zeros, then it is really just a finite decimal expansion.)

- Repeat this: Every rational number has an infinite decimal expansion with a repeating pattern. (It could be a repeating pattern of zeros.)

- Okay, so this means that if a number has an infinite decimal expansion that does <u>not</u> fall into a repeating pattern, then that number cannot be rational. That is, that number cannot be written as a fraction.

- Have a look at this number:

$$0.10100100010000100000100000 1\ldots$$

(Assume the pattern you see continues.) There is certainly a pattern to this decimal expansion, but is it a repeating pattern? Can this number be rational?
  - *Have students discuss this. It does not have a repeating pattern.*

- This infinite decimal does not have a repeating pattern and so cannot come from the process of long division. This is an example of a real number that is <u>not</u> rational. We call any number that is not rational *irrational*, and we have just established that irrational numbers exist—we found one!

- Can you write down another example of an infinite decimal that must represent an irrational number?
  - *Students might develop examples such as $0.10203040506070809010011012013014 0\ldots$ or $0.717117111711117\ldots.$*

- For millennia scholars suspected that the number π might be irrational. Looking at its decimal expansion doesn't help settle the question. For example, here are the first 25 decimal digits of π:

$$\pi = 3.1415926535897932384626433\ldots$$

  We don't see a repeating pattern, but that doesn't mean there isn't one. Maybe the pattern repeats after the 26th decimal place, or the 100th, or the fourteen-quadrillion-and-thirteen place? Scholars computed more and more decimal places of π and never saw a repeating pattern, but they always wondered if they computed a few more places whether one might later appear. It wasn't until the mid-1700s that Swiss mathematician Johann Lambert finally managed to give a mathematical proof that the number π is irrational and so will have an infinitely long decimal expansion with no repeating pattern.

- Scholars also managed to prove that the numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}$, and most other square roots are irrational too, but this is not at all obvious. (Why are $\sqrt{4}$ and $\sqrt{9}$ considered rational numbers?) Their decimal expansions must be infinitely long without any repeating patterns.

Exercises 6–10 (5 minutes)

Students complete Exercises 6–10 independently.

Exercises 6–10

6. Does the number $\dfrac{65}{13}$ have a finite or an infinite decimal expansion? Does its decimal expansion have a repeating pattern?

   The number $\dfrac{65}{13}$ is rational and so has a decimal expansion with a repeating pattern. Actually, $\dfrac{65}{13} = \dfrac{5 \times 13}{13} = 5$, so it is a finite decimal. Viewed as an infinite decimal, $\dfrac{65}{13}$ is $5.0000\ldots$ with a repeat block of 0.

7. Does the number $\dfrac{17}{11}$ have a finite or an infinite decimal expansion? Does its decimal expansion have a repeating pattern?

   The rational $\dfrac{17}{11}$ is in simplest form, and we see that it is not equivalent to a fraction with a denominator that is a power of 10. Thus, the rational has an infinite decimal expansion with a repeating pattern.

8. Is the number $0.212112111211112111112\ldots$ rational? Explain. (Assume the pattern you see in the decimal expansion continues.)

   Although the decimal expansion of this number has a pattern, it is not a repeating pattern. The number cannot be rational. It is irrational.

9. Does the number $\dfrac{860}{999}$ have a finite or an infinite decimal expansion? Does its decimal expansion have a repeating pattern?

   The number is rational and so has a decimal expansion with a repeating pattern. Since the fraction is not equivalent to one with a denominator that is a power of 10, it is an infinite decimal expansion.

10. Is the number $0.123456789101112131415161718192021223\ldots$ rational? Explain. (Assume the pattern you see in the decimal expansion continues.)

    Although the decimal expansion of this number has a pattern, it is not a repeating pattern. The number cannot be rational. It is irrational.

Lesson 8:   The Long Division Algorithm

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Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We can use the long division algorithm to compute the decimal expansions of numbers.
- We know that every rational number has a decimal expansion that repeats eventually.

Lesson Summary

A rational number is a number that can be written in the form $\frac{a}{b}$ for a pair of integers a and b with b not zero.

The long division algorithm shows that every rational number has a decimal expansion that falls into a repeating pattern. For example, the rational number 32 has a decimal expansion of $32.\overline{0}$, the rational number $\frac{1}{3}$ has a decimal expansion of $0.\overline{3}$, and the rational number $\frac{4}{11}$ has a decimal expansion of $0.\overline{36}$.

Exit Ticket (5 minutes)

Name _____   Date _____

Lesson 8: The Long Division Algorithm

Exit Ticket

1. Will the decimal expansion of $\dfrac{125}{8}$ be finite or infinite? Explain. If we were to write the decimal expansion of this rational number as an infinitely long decimal, which block of numbers repeat?

2. Write the decimal expansion of $\dfrac{13}{7}$ as an infinitely long repeating decimal.

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Exit Ticket Sample Solutions

1. Will the decimal expansion of $\frac{125}{8}$ be finite or infinite? Explain. If we were to write the decimal expansion of this rational number as an infinitely long decimal, which block of numbers repeat?

   The decimal expansion of $\frac{125}{8}$ will be finite because $\frac{125}{8}$ is equivalent to a fraction with a denominator that is a power of 10. (Multiply the numerator and denominator each by $5 \times 5 \times 5$.) If we were to write the decimal as an infinitely long decimal, then we'd have a repeating block consisting of 0.

2. Write the decimal expansion of $\frac{13}{7}$ as an infinitely long repeating decimal.

   $$\frac{13}{7} = \frac{1 \times 7}{7} + \frac{6}{7}$$
   $$= 1\frac{6}{7}$$

$$
\begin{array}{r}
1.857142857142 \\
7\overline{)13.0000000000000} \\
\underline{7} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
10 \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50
\end{array}
$$

   The decimal expansion of $\frac{13}{7}$ is $1.\overline{857142}$.

Problem Set Sample Solutions

1. Write the decimal expansion of $\dfrac{7000}{9}$ as an infinitely long repeating decimal.

$$\dfrac{7000}{9} = \dfrac{777 \times 9}{9} + \dfrac{7}{9}$$
$$= 777 \dfrac{7}{9}$$

$$
\begin{array}{r}
777.77 \\
9\overline{)7000.00} \\
63 \\
\overline{70} \\
63 \\
\overline{70} \\
63 \\
\overline{70} \\
63 \\
\overline{70} \\
63 \\
\overline{70} \\
\end{array}
$$

The decimal expansion of $\dfrac{7000}{9}$ is $777.\overline{7}$.

2. Write the decimal expansion of $\dfrac{6\,555\,555}{3}$ as an infinitely long repeating decimal.

$$\dfrac{6\,555\,555}{3} = \dfrac{2\,185\,185 \times 3}{3} + \dfrac{0}{3}$$
$$= 2\,185\,185$$

The decimal expansion of $\dfrac{6\,555\,555}{3}$ is $2,185,185.\overline{0}$.

3. Write the decimal expansion of $\dfrac{350\,000}{11}$ as an infinitely long repeating decimal.

$$\dfrac{350\,000}{11} = \dfrac{31\,818 \times 11}{11} + \dfrac{2}{11}$$
$$= 31\,818 \dfrac{2}{11}$$

$$
\begin{array}{r}
31818.18 \\
11\overline{)350,000.00} \\
33 \\
\overline{20} \\
11 \\
\overline{90} \\
88 \\
\overline{20} \\
11 \\
\overline{90} \\
88 \\
\overline{20} \\
11 \\
\overline{90} \\
88 \\
\overline{20} \\
\end{array}
$$

The decimal expansion of $\dfrac{350\,000}{11}$ is $31,818.\overline{18}$.

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4. Write the decimal expansion of $\dfrac{12\,000\,000}{37}$ as an infinitely long repeating decimal.

$$\dfrac{12\,000\,000}{37} = \dfrac{324\,324 \times 37}{37} + \dfrac{12}{37}$$
$$= 324\,324\dfrac{12}{37}$$

$$
\begin{array}{r}
324324.324 \\
37\overline{)12{,}000{,}000.000} \\
\underline{111} \\
90 \\
\underline{74} \\
160 \\
\underline{148} \\
120 \\
\underline{111} \\
90 \\
\underline{74} \\
160 \\
\underline{148} \\
120 \\
\underline{111} \\
90 \\
\underline{74} \\
160 \\
\underline{148} \\
120 \\
\end{array}
$$

The decimal expansion of $\dfrac{12\,000\,000}{37}$ *is* $324{,}324.\overline{324}$.

5. Someone notices that the long division of $2{,}222{,}222$ by 6 has a quotient of $370{,}370$ and a remainder of 2 and wonders why there is a repeating block of digits in the quotient, namely 370. Explain to the person why this happens.

$$\dfrac{2\,222\,222}{6} = \dfrac{370\,370 \times 6}{6} + \dfrac{2}{6}$$
$$= 370\,370\dfrac{2}{6}$$

$$
\begin{array}{r}
370370 \\
6\overline{)2222222} \\
\underline{18} \\
42 \\
\underline{42} \\
022 \\
\underline{18} \\
42 \\
\underline{42} \\
02 \\
\end{array}
$$

The block of digits 370 *keeps repeating because the long division algorithm leads us to perform the same division over and over again. In the algorithm shown above, we see that there are three groups of* 6 *in* 22*, leaving a remainder of* 4*. When we bring down the next* 2*, we see that there are exactly seven groups of* 6 *in* 42*. When we bring down the next* 2*, we see that there are zero groups of* 6 *in* 2*, leaving a remainder of* 2*. It is then that the process starts over because the next step is to bring down another* 2*, giving us* 22*, which is what we started with. Since the division repeats, then the digits in the quotient will repeat.*

6. Is the answer to the division problem $10 \div 3.2$ a rational number? Explain.

    Yes. This is equivalent to the division problem $100 \div 32$, which can be written as $\dfrac{100}{32}$, and so it is a rational number.

7. Is $\dfrac{3\pi}{77\pi}$ a rational number? Explain.

    Yes. $\dfrac{3\pi}{77\pi}$ is equal to $\dfrac{3}{77}$ and so it is a rational number.

8. The decimal expansion of a real number x has every digit 0 except the first digit, the tenth digit, the hundredth digit, the thousandth digit, and so on, are each 1. Is x a rational number? Explain.

    No. Although there is a pattern to this decimal expansion, it is not a repeating pattern. Thus, x cannot be rational.

 Lesson 9: Decimal Expansions of Fractions, Part 1

Student Outcomes

- Students identify the size in error when truncating an infinite decimal to a finite number of decimal places.

Classwork

Opening Exercise (8 minutes)

Opening Exercise

a. **Compute the decimal expansions of $\frac{5}{6}$ and $\frac{7}{9}$.**

$$\frac{5}{6} = 0.8333\ldots \text{ and } \frac{7}{9} = 0.7777\ldots$$

b. **What is $\frac{5}{6} + \frac{7}{9}$ as a fraction? What is the decimal expansion of this fraction?**

$$\frac{5}{6} + \frac{7}{9} = \frac{15 + 14}{18} = \frac{29}{18}$$

$$\frac{29}{18} = 1.61111\ldots$$

c. **What is $\frac{5}{6} \times \frac{7}{9}$ as a fraction? According to a calculator, what is the decimal expansion of the answer?**

$$\frac{5}{6} \times \frac{7}{9} = \frac{35}{54} = 0.648\,148\,148\,1\ldots$$

d. **If you were given just the decimal expansions of $\frac{5}{6}$ and $\frac{7}{9}$, without knowing which fractions produced them, do you think you could easily add the two decimals to find the decimal expansion of their sum? Could you easily multiply the two decimals to find the decimal expansion of their product?**

No. To add $0.8333\ldots$ and $0.777\ldots$, we need to start by adding together their rightmost digits. But these decimals are infinitely long, and there are no rightmost digits. It is not clear how we can start the addition.

Thinking about how to multiply the two decimals, $0.8333\ldots \times 0.77777\ldots$, is even more confusing!

Discussion (10 minutes)

- In the opening exercise, we saw that $\frac{5}{6} = 0.8333\ldots$ and $\frac{7}{9} = 0.7777\ldots$. We certainly know how to add and multiply fractions, but it is not at all clear how we can add and multiply infinitely long decimals.

- But we can approximate infinitely long decimals as finite ones. For example, 0.83 does approximate $0.833333…$ The error in the approximation is $0.003333…$, which is a number smaller than 0.01, a hundredth. If we approximate $0.7777…$ as 0.77, is the error also smaller than a hundredth?

  - *The error is $0.007777…$, which is smaller than 0.01. Yes, the error is smaller than a hundredth.*

- We know that $\dfrac{5}{6} + \dfrac{7}{9} = \dfrac{29}{18}$. Compute $0.83 + 0.77$. Does the answer approximate the decimal expansion of $\dfrac{29}{18}$?

  - $0.83 + 0.77 = 1.60$, *which does approximate $1.6111…$*

- And we know $\dfrac{5}{6} \times \dfrac{7}{9} = \dfrac{35}{54}$. Compute 0.83×0.77. Does the answer approximate the decimal expansion of $\dfrac{35}{54}$?

  - $0.83 \times 0.77 = 0.6391$. *It is not as clear if this is a good approximation of $0.6\overline{481}$.*

- If we use the approximations 0.833 and 0.777 for the decimal expansions of $\dfrac{5}{6}$ and $\dfrac{7}{9}$, and compute $0.833 + 0.777$ and 0.833×0.777, do we obtain better approximations to the decimal expansions of $\dfrac{5}{6} + \dfrac{7}{9}$, which is $\dfrac{29}{18}$, and $\dfrac{5}{6} \times \dfrac{7}{9}$, which is $\dfrac{35}{54}$?

  - $0.833 + 0.777 = 1.610$ *is a better approximation to $1.61111…$, and $0.833 \times 0.777 = 0.647241$ is a better approximation to $0.6\overline{481}$.*

- Do matters improve still if we use the approximations 0.8333 and 0.7777?

  - $0.8333 + 0.7777$, *or 1.6110, is an even better approximation to $1.61111…$ and 0.8333×0.7777, or 0.64805741, is an even better approximation to $0.6\overline{481}$.*

- The point is that working with infinite decimals is challenging. But we can approximate real numbers with infinitely long decimal expansions by truncating their decimal expansions and working with the finite decimal approximations instead as we compute sums and products. The answers we obtain will approximate the true sum or product of the real numbers. We can improve the approximations by working with longer finite decimals that approximate the original numbers.

Exercise 1 (6 minutes)

Students complete Exercise 1 in pairs. Allow students to use calculators. Part (c) of the exercise might challenge students.

Exercise 1

Two irrational numbers x and y have infinite decimal expansions that begin $0.67035267…$ for x and $0.84991341…$ for y.

  a. Explain why 0.670 is an approximation for x with an error of less than one thousandth. Explain why 0.849 is an approximation for y with an error of less than one thousandth.

  The difference between 0.670 and $0.67035267…$ is $0.00035267…$, which is less than 0.001, a thousandth.

  The difference between 0.849 and $0.84991341…$ is $0.00091341…$, which is less than 0.001, a thousandth.

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> b.   Using the approximations given in part (a), what is an approximate value for $x + y$, for $x \times y$, and for $x^2 + 7y^2$?
>
> *$x + y$ is approximately 1.519 because $0.670 + 0.849 = 1.519$.*
>
> *$x \times y$ is approximately 0.56883 because $0.670 \times 0.849 = 0.56883$.*
>
> *$x^2 + 7y^2$ is approximately 5.494507 because $(0.670)^2 + 7(0.849)^2 = 5.494\,507$.*
>
> c.   Repeat part (b), but use approximations for x and y that have errors less than $\frac{1}{10^5}$.
>
> *We want the error in the approximation to be less than 0.00001.*
>
> *If we approximate x by truncating to five decimal places, that is, as 0.67035, then the error is $0.00000267...$, which is indeed less than 0.00001.*
>
> *Truncating y to five decimal places, that is, as 0.84991, gives an error of $0.00000341...$, which is indeed less than 0.00001.*
>
> *Now:*
>
> *$x + y$ is approximately 1.52026 because $0.67035 + 0.84991 = 1.52026$.*
>
> *$x \times y$ is approximately 0.5697371685 because $0.67035 \times 0.84991 = 0.569\,737\,168\,5$.*
>
> *$x^2 + 7y^2$ is approximately 5.505798179 because $(0.67035)^2 + 7(0.84991)^2 = 5.505\,798\,179$.*

Discussion (5 minutes)

▪ If we approximate an infinite decimal $0.abcdef...$ by truncating the decimal to two decimal places, explain why the error in the approximation is less than $\frac{1}{100}$.

   ▫ *Approximating $0.abcdef...$ as $0.ab$ has an error of $0.00cdef$, which is smaller than 0.01.*

▪ If we approximate an infinite decimal $0.abcdef...$ by truncating the decimal to three decimal places, explain why the error in the approximation is less than $\frac{1}{10^3}$.

   ▫ *Approximating $0.abcdef...$ as $0.abc$ has an error of $0.000def$, which is smaller than 0.001, or $\frac{1}{10^3}$.*

▪ If we approximate an infinite decimal $0.abcdef...$ by truncating the decimal to five decimal places, explain why the error in the approximation is less than $\frac{1}{10^5}$.

   ▫ *Approximating $0.abcdef...$ as $0.abcde$ has an error of $0.00000f$, which is smaller than 0.00001, or $\frac{1}{10^5}$.*

▪ We see that, in general, if we truncate an infinite decimal to n decimal places, the resulting decimal approximation has an error of less than $\frac{1}{10^n}$.

Exercise 2 (9 minutes)

Allow students to use calculators to perform every operation except division.

Exercise 2

Two real numbers have decimal expansions that begin with the following:

$$x = 0.1538461\ldots$$
$$y = 0.3076923\ldots$$

a. Using approximations for x and y that are accurate within a measure of $\dfrac{1}{10^3}$, find approximate values for $x + y$ and $y - 2x$.

Using $x \approx 0.153$ and $y \approx 0.307$, we obtain $x + y \approx 0.460$ and $y - 2x \approx 0.001$.

b. Using approximations for x and y that are accurate within a measure of $\dfrac{1}{10^7}$, find approximate values for $x + y$ and $y - 2x$.

Using $x \approx 0.153\,846\,1$ and $y \approx 0.307\,692\,3$, we obtain $x + y \approx 0.461\,538\,4$ and $y - 2x \approx 0.000\,000\,1$.

c. We now reveal that $x = \dfrac{2}{13}$ and $y = \dfrac{4}{13}$. How accurate is your approximate value to $y - 2x$ from part (a)? From part (b)?

The error in part (a) is 0.001. The error in part (b) is 0.0000001.

d. Compute the first seven decimal places of $\dfrac{6}{13}$. How accurate is your approximate value to $x + y$ from part (a)? From part (b)?

$$\frac{6}{13} = 0.461\,538\,4\ldots$$

The error in part (a) is $0.461\,538\,4\ldots - 0.460 = 0.001\,538\,4\ldots$, which is less than 0.01.

Our approximate answer in part (b) and the exact answer match in the first seven decimal places. There is likely a mismatch from the eighth decimal place onward. This means that the error is no larger than 0.0000001, or $\dfrac{1}{10^7}$.

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- It is not clear how to perform arithmetic on numbers given as infinitely long decimals.
- If we approximate numbers by truncating their infinitely long decimal expansions to a finite number of decimal places, then we can perform arithmetic on the approximate values to estimate answers.
- Truncating a decimal expansion to n decimal places gives an approximation with an error of less than $\dfrac{1}{10^n}$.

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Lesson Summary

It is not clear how to perform arithmetic on numbers given as infinitely long decimals. If we approximate these numbers by truncating their infinitely long decimal expansions to a finite number of decimal places, then we can perform arithmetic on the approximate values to estimate answers.

Truncating a decimal expansion to n decimal places gives an approximation with an error of less than $\frac{1}{10^n}$. For example, 0.676 is an approximation for $0.676767\ldots$ with an error of less than 0.001.

Exit Ticket (4 minutes)

Name _____   Date _____

Lesson 9: Decimal Expansions of Fractions, Part 1

Exit Ticket

Suppose $x = \frac{2}{3} = 0.6666\ldots$ and $y = \frac{5}{9} = 0.5555\ldots$

a. Using 0.666 as an approximation for x and 0.555 as an approximation for y, find an approximate value for $x + y$.

b. What is the true value of $x + y$ as an infinite decimal?

c. Use approximations for x and y, each accurate to within an error of $\frac{1}{10^5}$, to estimate a value of the product $x \times y$.

Exit Ticket Sample Solutions

Suppose $x = \frac{2}{3} = 0.6666...$ **and** $y = \frac{5}{9} = 0.5555...$

    a.   **Using** 0.666 **as an approximation for** x **and** 0.555 **as an approximation for** y**, find an approximate value for** $x + y$**.**

$$x + y \approx 0.666 + 0.555 = 1.221$$

    b.   **What is the true value of** $x + y$ **as an infinite decimal?**

$$x + y = \frac{2}{3} + \frac{5}{9} = \frac{11}{9} = 1 + \frac{2}{9} = 1.22222...$$

    c.   **Use approximations for** x **and** y**, each accurate to within an error of** $\frac{1}{10^5}$**, to estimate a value of the product** $x \times y$**.**

$$x \times y \approx 0.66666 \times 0.55555 = 0.370\,362\,963\,0$$

Problem Set Sample Solutions

1.   **Two irrational numbers** x **and** y **have infinite decimal expansions that begin** $0.3338117...$ **for** x **and** $0.9769112...$ **for** y**.**

    a.   **Explain why** 0.33 **is an approximation for** x **with an error of less than one hundredth. Explain why** 0.97 **is an approximation for** y **with an error of less than one hundredth.**

        The difference between 0.33 *and* $0.3338117...$ *is* $0.0038117...$, *which is less than* 0.01, *a hundredth.*

        The difference between 0.97 *and* $0.9769112...$ *is* $0.0069112...$, *which is less than* 0.01, *a hundredth.*

    b.   **Using the approximations given in part (a), what is an approximate value for** $2x(y + 1)$**?**

        $2x(y + 1)$ *is approximately* 1.3002 *because* $2 \times 0.33 \times 1.97 = 1.3002$.

    c.   **Repeat part (b), but use approximations for** x **and** y **that have errors less than** $\frac{1}{10^6}$**.**

        We want the error in the approximation to be less than 0.000001.

        If we approximate x *by truncating to six decimal places, that is, as* 0.333811, *then the error is* $0.0000007...$, *which is indeed less than* 0.000001.

        Truncating y *to six decimal places, that is, as* 0.976911, *gives an error of* $0.0000002...$, *which is indeed less than* 0.000001.

        Now:

        $2x(y + 1)$ *is approximately* 1.319829276, *which is a rounding of* $2 \times 0.333\,811 \times 1.976\,911$.

2. Two real numbers have decimal expansions that begin with the following:

$$x = 0.70588\ldots$$
$$y = 0.23529\ldots$$

a. Using approximations for x and y that are accurate within a measure of $\frac{1}{10^2}$, find approximate values for $x + 1.25y$ and $\frac{x}{y}$.

Using $x \approx 0.70$ and $y \approx 0.23$, we obtain $x + 1.25y \approx 0.9875$ and $\frac{x}{y} \approx 3.0434\ldots$.

b. Using approximations for x and y that are accurate within a measure of $\frac{1}{10^4}$, find approximate values for $x + 1.25y$ and $\frac{x}{y}$.

Using $x \approx 0.7058$ and $y \approx 0.2352$, we obtain $x + 1.25y \approx 0.9998$ and $\frac{x}{y} \approx 3.000\,850\ldots$.

c. We now reveal that x and y are rational numbers with the property that each of the values $x + 1.25y$ and $\frac{x}{y}$ is a whole number. Make a guess as to what whole numbers these values are, and use your guesses to find what fractions x and y might be.

It looks like $x + 1.25y = 1$ and $\frac{x}{y} = 3$. Thus, we guess $x = 3y$ and so $3y + 1.25y = 1$, that is, $4.25y = 1$, so $y = \frac{1}{4.25} = \frac{100}{425} = \frac{4}{17}$ and $x = 3y = \frac{12}{17}$.

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 Lesson 10: Converting Repeating Decimals to Fractions

Student Outcomes

- Students develop a convincing argument establishing that every real number with a repeating decimal is a rational number.

Classwork

Discussion (4 minutes)

- In Lesson 8, we say that every rational number, that is, every fraction, has a decimal expansion that falls into a repeating pattern. A natural question now is the converse: If a real number has an infinitely long decimal expansion with a repeating pattern, must it be a rational number?

- We begin by observing the effect of multiplying decimals by powers of 10. Consider, for example, the finite decimal 1.2345678. If we multiply it by 10^5, we get the following:

$$10^5 \times 1.2345678 = 10^5 \times \left(1 + \frac{2}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000} + \frac{6}{100000} + \frac{7}{1000000} + \frac{8}{10000000}\right)$$
$$= 100000 + 20000 + 3000 + 400 + 50 + 6 + \frac{7}{10} + \frac{8}{100}$$
$$= 123\,456.78$$

This example illustrates how to think through such products.

Example 1 (10 minutes)

> **Example 1**
>
> There is a fraction with an infinite decimal expansion of $0.\overline{81}$. Find the fraction.

- We want to find the fraction that is equal to the infinite decimal $0.\overline{81}$. Why might we want to write an infinite decimal as a fraction?
  - *Maybe we want to use $0.\overline{81}$ in some calculation. It is unclear how to do arithmetic with infinitely long decimals. But if we recognize the decimal as a fraction, then we can do the arithmetic with the fraction.*
- Let's start by giving the decimal a name. Let $x = 0.\overline{81} = 0.8181818181\ldots$. Any thoughts on what we might do to this number x? (Of course, our previous discussion was probably a hint!)

Allow students time to work in pairs or small groups to attempt to find the fraction equal to $0.\overline{81}$. Students should guess that multiplying x by some powers of 10 might yield something informative.

- Let's try multiplying $x = 0.\overline{81}$ by some powers of 10.

$$x = 0.8181818181\ldots$$
$$10x = 8.1818181\ldots$$
$$100x = 81.81818181\ldots$$
$$1000x = 818.1818181\ldots$$

(Perhaps have students write x as $\dfrac{8}{10} + \dfrac{1}{100} + \dfrac{8}{1000} + \dfrac{1}{10000} + \cdots$ to help with this process.)

Ask students to pause over the expression $100x$. Can they observe anything interesting about it?

- We see

$$100x = 81.81818181\ldots = 81 + 0.818181\ldots = 81 + x.$$

This now gives an equation for x students can solve.

$$100x = 81 + x$$
$$100x - x = 81 + x - x$$
$$(100 - 1)x = 81$$
$$99x = 81$$
$$\frac{99x}{99} = \frac{81}{99}$$
$$x = \frac{81}{99}$$
$$x = \frac{9}{11}$$

- Therefore, the repeating decimal $0.\overline{81} = \dfrac{9}{11}$.

Have students use calculators to verify that this is correct.

Exercises 1–2 (5 minutes)

Students complete Exercises 1–2 in pairs. Allow them to use calculators to check their work.

> **Exercises 1–2**
>
> 1. There is a fraction with an infinite decimal expansion of $0.\overline{123}$. Let $x = 0.\overline{123}$.
>   a. Explain why looking at $1000x$ helps us find the fractional representation of x.
>
>   *We have $x = 0.123123123\ldots$, and we see that $1000x = 123.123123123\ldots$. This is the same as $123 + 0.123123123\ldots$, which is $123 + x$. So we have the equation $1000x = 123 + x$, which we can use to solve for x.*

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b. **What is x as a fraction?**

$$1000x - x = 123 + x - x$$
$$999x = 123$$
$$\frac{999x}{999} = \frac{123}{999}$$
$$x = \frac{123}{999}$$
$$x = \frac{41}{333}$$

c. **Is your answer reasonable? Check your answer using a calculator.**

Yes, my answer is reasonable and correct. It is reasonable because the denominator cannot be expressed as a product of 2's and 5's; therefore, I know that the fraction must represent an infinite decimal. It is also reasonable because the decimal value is closer to 0 than to 0.5, and the fraction $\frac{41}{333}$ is also closer to 0 than to $\frac{1}{2}$. It is correct because the division of $\frac{41}{333}$ using a calculator is $0.123123….$

2. **There is a fraction with a decimal expansion of $0.\overline{4}$. Find the fraction, and check your answer using a calculator.**

Let $x = 0.\overline{4}$

$$x = 0.\overline{4}$$
$$10x = (10)0.\overline{4}$$
$$10x = 4.\overline{4}$$
$$10x = 4 + x$$
$$10x - x = 4 + x - x$$
$$9x = 4$$
$$\frac{9x}{9} = \frac{4}{9}$$
$$x = \frac{4}{9}$$

Example 2 (10 minutes)

Example 2

Could it be that $2.13\overline{8}$ is also a fraction?

- We want to see if there is a fraction that is equal to the infinite decimal $2.13\overline{8}$. Notice that this time there is just one digit that repeats, but it is three places to the right of the decimal point.

- Let's multiply $x = 2.13\overline{8}$ by various powers of 10 and see if any of the results seem helpful.

$$x = 2.138888…$$
$$10x = 21.38888…$$
$$100x = 213.8888…$$
$$1000x = 2138.888…$$

- Do any of these seem helpful?

  □ *Students might not have any direct thoughts in response to this.*

- What if I asked as a separate question: Is 0.8888… the decimal expansion of a fraction? If knowing that 0.888… is a fraction, would any one of the equations we have then be of use to us?

    - *If we know that* $0.888… = \frac{a}{b}$, *then we would see that* $100x = 213 + 0.888… = 213 + \frac{a}{b}$. *We could work out what* x *is from that.*

- Okay. As a side problem: Is 0.8888… the decimal expansion of some fraction?

    - *Let* $y = 0.\overline{8}$.

$$y = 0.\overline{8}$$
$$10y = 8.\overline{8}$$
$$10y = 8 + 0.\overline{8}$$
$$10y = 8 + y$$
$$10y - y = 8 + y - y$$
$$9y = 8$$
$$\frac{9y}{9} = \frac{8}{9}$$
$$y = \frac{8}{9}$$

- Now that we know that $0.\overline{8} = \frac{8}{9}$, we will go back to our original problem.

$$100x = 213 + 0.\overline{8}$$
$$100x = 213 + \frac{8}{9}$$
$$100x = \frac{213 \cdot 9}{9} + \frac{8}{9}$$
$$100x = \frac{213 \cdot 9 + 8}{9}$$
$$100x = \frac{1925}{9}$$
$$\frac{1}{100}(100x) = \frac{1}{100}\left(\frac{1925}{9}\right)$$
$$x = \frac{1925}{900}$$
$$x = \frac{77}{36}$$

- We can see that this technique applies to any infinite repeating decimal, even if there is a delay before the repeat begins, to show that every real number that has a repeating decimal expansion is, for sure, a rational number, that is, can be expressed as a fraction. And, conversely, we saw in Lesson 8 that every rational number has a repeating decimal expansion. So we have proven that the set of real numbers with repeating decimal expansions precisely matches the set of all rational numbers. Any number that has an infinitely long decimal expansion with no repeating pattern cannot be rational; that is, it must be an irrational number.

Exercises 3–4 (6 minutes)

Students complete Exercises 3–4 independently or in pairs. Allow students to use calculators to check their work.

Exercises 3–4

3. Find the fraction equal to $1.6\overline{23}$. Check your answer using a calculator.

$Let\ x = 1.6\overline{23}$

$$x = 1.6\overline{23}$$
$$10x = 10(1.6\overline{23})$$
$$10x = 16.\overline{23}$$

$Let\ y = 0.\overline{23}$

$$y = 0.\overline{23}$$
$$10^2 y = 10^2 (0.\overline{23})$$
$$100y = 23.\overline{23}$$
$$100y = 23 + y$$
$$100y - y = 23 + y - y$$
$$99y = 23$$
$$\frac{99y}{99} = \frac{23}{99}$$
$$y = \frac{23}{99}$$

$$10x = 16.\overline{23}$$
$$10x = 16 + y$$
$$10x = 16 + \frac{23}{99}$$
$$10x = \frac{16 \cdot 99}{99} + \frac{23}{99}$$
$$10x = \frac{16 \cdot 99 + 23}{99}$$
$$10x = \frac{1607}{99}$$
$$\frac{1}{10}(10x) = \frac{1}{10}\left(\frac{1607}{99}\right)$$
$$x = \frac{1607}{990}$$

$$1.6\overline{23} = \frac{1607}{990}$$

4. Find the fraction equal to $2.9\overline{60}$. Check your answer using a calculator.

$Let\ x = 2.9\overline{60}$

$$x = 2.9\overline{60}$$
$$10x = 10(2.9\overline{60})$$
$$10x = 29.\overline{60}$$

$Let\ y = 0.\overline{60}$

$$y = 0.\overline{60}$$
$$10^2 y = 10^2 (0.\overline{60})$$
$$100y = 60.\overline{60}$$
$$100y = 60 + y$$
$$100y - y = 60 + y - y$$
$$99y = 60$$
$$\frac{99y}{99} = \frac{60}{99}$$
$$y = \frac{60}{99}$$
$$y = \frac{20}{33}$$

$$10x = 29.\overline{60}$$
$$10x = 29 + y$$
$$10x = 29 + \frac{20}{33}$$
$$10x = \frac{29 \cdot 33}{33} + \frac{20}{33}$$
$$10x = \frac{29 \cdot 33 + 20}{33}$$
$$10x = \frac{977}{33}$$
$$\frac{1}{10}(10x) = \frac{1}{10}\left(\frac{977}{33}\right)$$
$$x = \frac{977}{330}$$

$$2.9\overline{60} = \frac{977}{330}$$

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We know that every decimal that has a repeating pattern is a rational number.
- We know how to find the fraction that has a given repeating decimal expansion.

> **Lesson Summary**
>
> **Every decimal with a repeating pattern is a rational number, and we have the means to determine the fraction that has a given repeating decimal expansion.**
>
> **Example: Find the fraction that is equal to the number $0.\overline{567}$.**
>
> **Let x represent the infinite decimal $0.\overline{567}$.**
>
> $$x = 0.\overline{567}$$
> $$10^3 x = 10^3 (0.\overline{567}) \qquad \text{Multiply by } 10^3 \text{ because there are 3 digits that repeat.}$$
> $$1000x = 567.\overline{567} \qquad \text{Simplify}$$
> $$1000x = 567 + 0.\overline{567} \qquad \text{By addition}$$
> $$1000x = 567 + x \qquad \text{By substitution; } x = 0.\overline{567}$$
> $$1000x - x = 567 + x - x \qquad \text{Subtraction property of equality}$$
> $$999x = 567 \qquad \text{Simplify}$$
> $$\frac{999}{999} x = \frac{567}{999} \qquad \text{Division property of equality}$$
> $$x = \frac{567}{999} = \frac{63}{111} \qquad \text{Simplify}$$
>
> **This process may need to be used more than once when the repeating digits, as in numbers such as $1.2\overline{6}$, do not begin immediately after the decimal.**
>
> **Irrational numbers are numbers that are not rational. They have infinite decimal expansions that do not repeat and they cannot be expressed as $\frac{p}{q}$ for integers p and q with $q \neq 0$.**

Exit Ticket (5 minutes)

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Name _____   Date _____

Lesson 10: Converting Repeating Decimals to Fractions

Exit Ticket

1. Find the fraction equal to $0.\overline{534}$.

2. Find the fraction equal to $3.0\overline{15}$.

Exit Ticket Sample Solutions

1. Find the fraction equal to $0.\overline{534}$.

   Let $x = 0.\overline{534}$.

   $$x = 0.\overline{534}$$
   $$10^3 x = 10^3(0.\overline{534})$$
   $$1000x = 534.\overline{534}$$
   $$1000x = 534 + x$$
   $$1000x - x = 534 + x - x$$
   $$999x = 534$$
   $$\frac{999x}{999} = \frac{534}{999}$$
   $$x = \frac{534}{999}$$
   $$x = \frac{178}{333}$$

   $$0.\overline{534} = \frac{178}{333}$$

2. Find the fraction equal to $3.0\overline{15}$.

   Let $x = 3.0\overline{15}$.

   $$x = 3.0\overline{15}$$
   $$10x = 10(3.0\overline{15})$$
   $$10x = 30.\overline{15}$$

   Let $y = 0.\overline{15}$.

   $$y = 0.\overline{15}$$
   $$10^2 y = 10^2(0.\overline{15})$$
   $$100y = 15.\overline{15}$$
   $$100y = 15 + y$$
   $$100y - y = 15 + y - y$$
   $$99y = 15$$
   $$\frac{99y}{99} = \frac{15}{99}$$
   $$y = \frac{5}{33}$$

   $$10x = 30.\overline{15}$$
   $$10x = 30 + y$$
   $$10x = 30 + \frac{5}{33}$$
   $$10x = \frac{30 \cdot 33}{33} + \frac{5}{33}$$
   $$10x = \frac{30 \cdot 33 + 5}{33}$$
   $$10x = \frac{995}{33}$$
   $$\frac{1}{10}(10x) = \frac{1}{10}\left(\frac{995}{33}\right)$$
   $$x = \frac{995}{330}$$
   $$x = \frac{199}{66}$$

   $$3.0\overline{15} = \frac{199}{66}$$

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Problem Set Sample Solutions

1.

a. Let $x = 0.\overline{631}$. Explain why multiplying both sides of this equation by 10^3 will help us determine the fractional representation of x.

When we multiply both sides of the equation by 10^3, on the right side we will have $631.631631\ldots$. This is helpful because we will now see this as $631 + x$.

b. What fraction is x?

$$1000x = 631.\overline{631}$$
$$1000x = 631 + 0.631\,631\ldots$$
$$1000x = 631 + x$$
$$1000x - x = 631 + x - x$$
$$999x = 631$$
$$\frac{999x}{999} = \frac{631}{999}$$
$$x = \frac{631}{999}$$

c. Is your answer reasonable? Check your answer using a calculator.

Yes, my answer is reasonable and correct. It is reasonable because the denominator cannot be expressed as a product of 2's and 5's; therefore, I know that the fraction must represent an infinite decimal. Also, the number 0.631 is closer to 0.5 than 1, and the fraction is also closer to $\frac{1}{2}$ than 1. It is correct because the division of $\frac{631}{999}$ using the calculator is $0.631631\ldots$.

2. Find the fraction equal to $3.40\overline{8}$. Check your answer using a calculator.

Let $x = 3.40\overline{8}$.

$$x = 3.40\overline{8}$$
$$10^2 x = 10^2(3.40\overline{8})$$
$$100x = 340.\overline{8}$$

Let $y = 0.\overline{8}$.

$$y = 0.\overline{8}$$
$$10y = 10(0.\overline{8})$$
$$10y = 8.\overline{8}$$
$$10y = 8 + y$$
$$10y - y = 8 + y - y$$
$$9y = 8$$
$$\frac{9y}{9} = \frac{8}{9}$$
$$y = \frac{8}{9}$$

$$100x = 340.\overline{8}$$
$$100x = 340 + y$$
$$100x = 340 + \frac{8}{9}$$
$$100x = \frac{340 \cdot 9}{9} + \frac{8}{9}$$
$$100x = \frac{340 \cdot 9 + 8}{9}$$
$$100x = \frac{3068}{9}$$
$$\frac{1}{100}(100x) = \frac{1}{100}\left(\frac{3068}{9}\right)$$
$$x = \frac{3068}{900}$$
$$x = \frac{767}{225}$$

$$3.40\overline{8} = \frac{767}{225}$$

3. **Find the fraction equal to $0.\overline{5923}$. Check your answer using a calculator.**

    Let $x = 0.\overline{5923}$.

    $$x = 0.\overline{5923}$$
    $$10^4 x = 10^4 (0.\overline{5923})$$
    $$10000x = 5923.\overline{5923}$$
    $$10000x = 5923 + x$$
    $$10000x - x = 5923 + x - x$$
    $$9999x = 5923$$
    $$\frac{9999x}{9999} = \frac{5923}{9999}$$
    $$x = \frac{5923}{9999}$$

4. **Find the fraction equal to $2.3\overline{82}$. Check your answer using a calculator.**

    Let $x = 2.3\overline{82}$.   *Let $y = 0.\overline{82}$.*

    $$x = 2.3\overline{82}$$ $$y = 0.\overline{82}$$ $$10x = 23.\overline{82}$$
    $$10x = 10(2.3\overline{82})$$ $$10^2 y = 10^2 (0.\overline{82})$$ $$10x = 23 + \frac{82}{99}$$
    $$10x = 23.\overline{82}$$ $$100y = 82.\overline{82}$$ $$10x = \frac{23 \cdot 99}{99} + \frac{82}{99}$$
    $$100y = 82 + y$$ $$10x = \frac{23 \cdot 99 + 82}{99}$$
    $$100y - y = 82 + y - y$$ $$10x = \frac{2359}{99}$$
    $$99y = 82$$ $$\frac{1}{10}(10x) = \frac{1}{10}\left(\frac{2359}{99}\right)$$
    $$\frac{99y}{99} = \frac{82}{99}$$ $$x = \frac{2359}{990}$$
    $$y = \frac{82}{99}$$

    $$2.3\overline{82} = \frac{2359}{990}$$

5. **Find the fraction equal to $0.\overline{714285}$. Check your answer using a calculator.**

    Let $x = 0.\overline{714\,285}$.

    $$x = 0.\overline{714\,285}$$
    $$10^6 x = 10^6 (0.\overline{714\,285})$$
    $$1\,000\,000x = 714\,825.\overline{714\,285}$$
    $$1\,000\,000x = 714\,285 + x$$
    $$1\,000\,000x - x = 714\,285 + x - x$$
    $$999\,999x = 714\,285$$
    $$\frac{999\,999x}{999\,999} = \frac{714\,285}{999\,999}$$
    $$x = \frac{714\,285}{999\,999}$$
    $$x = \frac{5}{7}$$

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6. Explain why an infinite decimal that is not a repeating decimal cannot be rational.

We proved in Lesson 8 that the decimal expansion of any rational number must fall into a repeating cycle. (This came from performing the long division algorithm.) So any number that has an infinitely long decimal expansion that does not repeat cannot be a rational number; that is, it must be irrational.

7. In a previous lesson, we were convinced that it is acceptable to write $0.\overline{9} = 1$. Use what you learned today to show that it is true.

Let $x = 0.\overline{9}$

$$x = 0.\overline{9}$$
$$10x = 10(0.\overline{9})$$
$$10x = 9.\overline{9}$$
$$10x = 9 + x$$
$$10x - x = 9 + x - x$$
$$9x = 9$$
$$\frac{9x}{9} = \frac{9}{9}$$
$$x = \frac{9}{9}$$
$$x = 1$$

8. Examine the following repeating infinite decimals and their fraction equivalents. What do you notice? Why do you think what you observed is true?

$$0.\overline{81} = \frac{81}{99} \qquad 0.\overline{4} = \frac{4}{9} \qquad 0.\overline{123} = \frac{123}{999} \qquad 0.\overline{60} = \frac{60}{99}$$

$$0.\overline{4311} = \frac{4311}{9999} \qquad 0.\overline{01} = \frac{1}{99} \qquad 0.\overline{3} = \frac{1}{3} = \frac{3}{9} \qquad 0.\overline{9} = 1.0$$

In each case, the fraction that represents the infinite decimal has a numerator that is exactly the repeating part of the decimal and a denominator comprised of 9's. Specifically, the denominator has the same number of digits of 9's as the number of digits that repeat. For example, $0.\overline{81}$ has two repeating decimal digits, and the denominator has two 9's.

The pattern is even true for $0.\overline{9}$: According to the pattern, this decimal equals $\frac{9}{9}$, which is 1.

Lesson 11: The Decimal Expansion of Some Irrational Numbers

Student Outcomes

- Students approximate the decimal expansions of roots of integers.

Lesson Notes

In this lesson, students use their calculators to help them determine the decimal expansions of given square roots. This may seem odd to them since the calculator is also capable of computing these roots directly. Make the point, when appropriate, that we don't have the means to compute these roots by pencil and paper alone, but in this lesson they see how to approximate these roots by hand. The method followed involves many long multiplication calculations, which could be done by hand, but the calculator is used here to save time on that computational work.

Classwork

Opening Exercise (5 minutes)

> **Opening Exercise**
>
> Place $\sqrt{28}$ on a number line. Make a guess as to the first few values of the decimal expansion of $\sqrt{28}$. Explain your reasoning.

Lead a discussion where students share their reasoning as to the placement of $\sqrt{28}$ on the number line. Encourage students to critique the reasoning of others while evaluating their own arguments. Consider having students vote on the placement they think is most correct.

Discussion (10 minutes)

- We have seen thus far that numbers whose decimal expansions are infinite and do not repeat (Lesson 8) are irrational. We saw from the last lesson what kinds of numbers (fractions with denominators that are a multiple of 9, or simplified multiple of 9) produce decimals expansions that are infinite but repeat. What kind of number produces a decimal expansion that is both infinite and nonrepeating?

  - *Students may conjecture that square roots of non-perfect squares would have decimal expansions that are infinite and nonrepeating. The lesson investigates the decimal expansions of non-perfect squares.*

- So far, we have been able to estimate the size of a number like $\sqrt{3}$ by stating that it is between the two perfect squares $\sqrt{1}$ and $\sqrt{4}$, meaning that $\sqrt{3}$ is between 1 and 2 but closer to 2. In our work so far, we have found the decimal expansion of fractions by using long division or by noting that the denominator of a fraction is a product of 2's and 5's. Numbers written with a square root symbol require a different method for determining their decimal expansions. The method we will develop gives a sequence of finite decimals that approximate the root with more and more accuracy.

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Example 1

> **Example 1**
>
> Consider the decimal expansion of $\sqrt{3}$.
>
> Find the first two values of the decimal expansion using the following fact: If $c^2 < 3 < d^2$ for positive numbers c and d, then $c < \sqrt{3} < d$.
>
> First approximation: Because $1 < 3 < 4$, we have $1 < \sqrt{3} < 2$.

- We learned in Lesson 3 that if c and d are positive numbers, then $c^2 < d^2$ implies $c < d$ and, conversely, if $c < d$, then $c^2 < d^2$. It follows from this that if $c^2 < N < d^2$, then $c < \sqrt{N} < d$. (And, conversely, if $c < \sqrt{N} < d$, then $c^2 < N < d^2$.)

- Since $1 < 3 < 4$, we get our first approximation: $1 < \sqrt{3} < 2$.

- To get more precise with our estimate of $\sqrt{3}$, we can look at the tenths between the numbers 1 and 2.

> **Second approximation:**
>
>
>
> 1   1.1   1.2   1.3   1.4   1.5   1.6   1.7   1.8   1.9   2.0
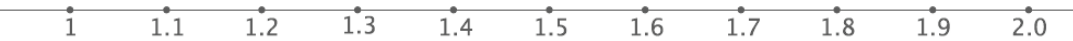
- Is $\sqrt{3}$ between 1.2 and 1.3?

- If $1.2 < \sqrt{3} < 1.3$, then we should have $1.2^2 < 3 < 1.3^2$. But we don't: $1.2^2 = 1.44$ and $1.3^2 = 1.69$. These squares are too small.

- Is $\sqrt{3}$ between 1.8 and 1.9?

- If $1.8 < \sqrt{3} < 1.9$, then we should have $1.8^2 < 3 < 1.9^2$. But we don't: $1.8^2 = 3.24$ and $1.9^2 = 3.81$. These squares are too large.

- Can you find the right tenth interval in which $\sqrt{3}$ belongs?

  - *After some trial and error, students see that $\sqrt{3}$ lies between* 1.7 *and* 1.8 *. We have* $1.7^2 = 2.89$ *and* $1.8^2 = 3.24$*, and so* $2.89 < 3 < 3.24$*.*

- So the decimal expansion of $\sqrt{3}$ begins 1.7.... How could we pin down its next decimal place?

  - *Look for where $\sqrt{3}$ lies in the interval between* 1.7 *and* 1.8*. Divide that interval into ten parts, too.*

- Let's do that!

> **Third approximation:**
>
>
>
> 1.70   1.71   1.72   1.73   1.74   1.75   1.76   1.77   1.78   1.79   1.80

Have students use trial and error to eventually establish that $\sqrt{3}$ lies between 1.73 and 1.74: we have $1.73^2 = 2.9929$ and $1.74^2 = 3.0276$ and $2.9926 < 3 < 3.0276$.

- So what are the first two places of the decimal expansion of $\sqrt{3}$?

  - *We have $\sqrt{3} = 1.73....$*

- What do you think will need to be done to get an even more precise estimate of the number $\sqrt{3}$?
  - *We will need to look at the interval between* 1.73 *and* 1.74 *more closely and repeat the process we did before.*
- Would you like to find the next decimal place for $\sqrt{3}$ just for fun or leave it be for now?
  - *Give students the option to find the next decimal place if they wish.*
- How accurate is our approximation $\sqrt{3} = 1.73\ldots$? (If students computed $\sqrt{3} = 1.732\ldots$, adjust this question and the answer below appropriately.)
  - *We know* $\sqrt{3} = 1.73abc\ldots$ *for some more digits a, b, c, and so on. Now* 1.73 *and* $1.73abc\ldots$ *differ by* $0.00abc\ldots$, *which is less than* 0.01. *A decimal expansion computed to two decimal places gives an approximation that is accurate with an error that is at most* 0.01.

> **Scaffolding:**
>
> Defining the terms *approximate, approximately,* and *approximation* may be useful to English language learners.

Consider using an online calculator, or any calculator that can show more than 8 decimal digits, that gives the decimal expansion of the number $\sqrt{3}$. The calculator located at http://keisan.casio.com/calculator requires you to enter *sqrt(3)* and click *execute* to see the decimal expansion. You can increase the number of digits displayed by using the *Digit* dropdown menu. Once displayed, ask students to examine the decimal expansion for any patterns, or lack thereof.

Discussion (15 minutes)

The following discussion revisits the Opening Exercise. Before you begin, ask students to reevaluate their own reasoning. If they voted, consider asking them to vote again to see if anyone wants to change their mind about the best approximation for $\sqrt{28}$.

Example 2

> **Example 2**
>
> Find the first few places of the decimal expansion of $\sqrt{28}$.
>
> First approximation:

- Between which two integers does $\sqrt{28}$ lie?
  - *Since* $25 < 28 < 36$, *we see* $5 < \sqrt{28} < 6$.
- In which tenth between 5 and 6 does $\sqrt{28}$ lie?

> Second approximation:
>
>

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- How do we determine which interval is correct?

  □ *What if we just square the numbers* 5.0*, * 5.1*, and* 5.2 *and see between which two squares* 28 *lies? After all, we are hoping to see that* $5.3 < \sqrt{28} < 5.4$*, in which case we should have* $5.3^2 < 28 < 5.4^2$*. (This interval is probably not correct, but we can check!)*

Provide students time to determine in which interval the number $\sqrt{28}$ lies. Ask students to share their findings, and then continue the discussion.

- Now that we know that the number $\sqrt{28}$ lies between 5.2 and 5.3, let's look at hundredths.

Third approximation:
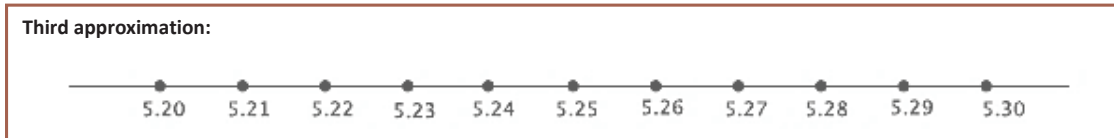
- Can we be efficient? Since $5.20^2 = 27.04$ and $5.30^2 = 28.09$, would an interval to the left, to the middle, or to the right likely contain $\sqrt{28}$?

  □ *We suspect that the interval between* 5.29 *and* 5.30 *might contain* $\sqrt{28}$ *because* 28 *is close to* 28.09*.*

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then continue the discussion.

- Now we know that the number $\sqrt{28}$ is between 5.29 and 5.30. Let's go one step further and examine intervals of thousandths.

Fourth approximation:
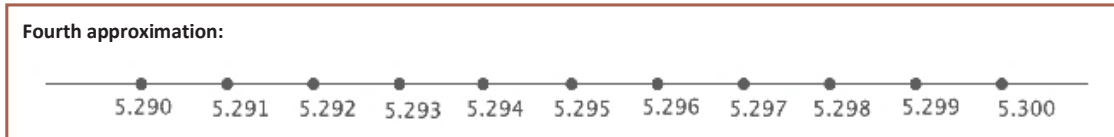
- Since $5.290^2 = 27.9841$ and $5.300^2 = 28.09$, where should we begin our search?

  □ *We should begin with the intervals closer to* 5.290 *and* 5.291 *because* 28 *is closer to* 27.9841 *than to* 28.09*.*

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then finish the discussion.

- The number $\sqrt{28}$ lies between 5.291 and 5.292 because $5.291^2 = 27.994681$ and $5.292^2 = 28.005264$. At this point, we have a fair approximation of the value of $\sqrt{28}$. It is between 5.291 and 5.292 on the number line:

- We could continue this process of rational approximation to see that $\sqrt{28} = 5.291502622\ldots$.

As before, use an online calculator to show the decimal expansion of $\sqrt{28}$. Once displayed, ask students to examine the decimal expansion for any patterns, or lack thereof.

Consider going back to the Opening Exercise to determine whose approximation was the closest.

- Can we conduct this work to also pin down the location of $\sqrt{121}$ on the number line?
  - *No need! $\sqrt{121} = 11$, so we know where it sits!*

Exercise 1 (5 minutes)

Students work in pairs to complete Exercise 1.

Exercise 1

In which interval of hundredths does $\sqrt{14}$ lie? Show your work.

The number $\sqrt{14}$ is between integers 3 and 4 because $9 < 14 < 16$. Then, $\sqrt{14}$ must be checked for the interval of tenths between 3 and 4. Since $\sqrt{14}$ is closer to 4, we will begin with the interval from 3.9 to 4.0. The number $\sqrt{14}$ is between 3.7 and 3.8 because $3.7^2 = 13.69$ and $3.8^2 = 14.44$. Now, we must look at the interval of hundredths between 3.7 and 3.8. Since $\sqrt{14}$ is closer to 3.7, we will begin with the interval 3.70 to 3.71. The number $\sqrt{14}$ is between 3.74 and 3.75 because $3.74^2 = 13.9876$ and $3.75^2 = 14.0625$.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We have a method of finding the first few decimal places of square roots of non-perfect squares.

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Lesson Summary

To find the first few decimal places of the decimal expansion of the square root of a non-perfect square, first determine between which two integers the square root lies, then in which interval of a tenth the square root lies, then in which interval of a hundredth it lies, and so on.

Example: Find the first few decimal places of $\sqrt{22}$.

Begin by determining between which two integers the number would lie.
$\sqrt{22}$ is between the integers 4 and 5 because $16 < 22 < 25$.

Next, determine between which interval of tenths the number belongs.
$\sqrt{22}$ is between 4.6 and 4.7 because $4.6^2 = 21.16 < 22 < 4.7^2 = 22.09$.

Next, determine between which interval of hundredths the number belongs.
$\sqrt{22}$ is between 4.69 and 4.70 because $4.69^2 = 21.9961 < 22 < 4.70^2 = 22.0900$.

A good estimate of the value of $\sqrt{22}$ is 4.69. It is correct to two decimal places and so has an error no larger than 0.01.

Notice that with each step of this process we are getting closer and closer to the actual value $\sqrt{22}$. This process can continue using intervals of thousandths, ten-thousandths, and so on.

Exit Ticket (5 minutes)

EUREKA MATH™   **Lesson 11:**   The Decimal Expansion of Some Irrational Numbers   **141**

Name _____   Date _____

Lesson 11: The Decimal Expansion of Some Irrational Numbers

Exit Ticket

Determine the three-decimal digit approximation of the number $\sqrt{17}$.

Exit Ticket Sample Solutions

Determine the three-decimal digit approximation of the number $\sqrt{17}$.

The number $\sqrt{17}$ is between integers 4 and 5 because $16 < 17 < 25$. Since $\sqrt{17}$ is closer to 4, I will start checking the tenths intervals closer to 4. $\sqrt{17}$ is between 4.1 and 4.2 since $4.1^2 = 16.81$ and $4.2^2 = 17.64$. Checking the hundredths interval, $\sqrt{17}$ is between 4.12 and 4.13 since $4.12^2 = 16.9744$ and $4.13^2 = 17.0569$. Checking the thousandths interval, $\sqrt{17}$ is between 4.123 and 4.124 since $4.123^2 = 16.999129$ and $4.124^2 = 17.007376$.

The three-decimal digit approximation is 4.123.

Problem Set Sample Solutions

1. **In which hundredth interval of the number line does $\sqrt{84}$ lie?**

    The number $\sqrt{84}$ is between 9 and 10 but closer to 9. Looking at the interval of tenths, beginning with 9.0 to 9.1, the number $\sqrt{84}$ lies between 9.1 and 9.2 because $9.1^2 = 82.81$ and $9.2^2 = 84.64$ but is closer to 9.2. In the interval of hundredths, the number $\sqrt{84}$ lies between 9.16 and 9.17 because $9.16^2 = 83.9056$ and $9.17^2 = 84.0889$.

2. **Determine the three-decimal digit approximation of the number $\sqrt{34}$.**

    The number $\sqrt{34}$ is between 5 and 6 but closer to 6. Looking at the interval of tenths, beginning with 5.9 to 6.0, the number $\sqrt{34}$ lies between 5.8 and 5.9 because $5.8^2 = 33.64$ and $5.9^2 = 34.81$ and is closer to 5.8. In the interval of hundredths, the number $\sqrt{34}$ lies between 5.83 and 5.84 because $5.83^2 = 33.9889$ and $5.84^2 = 34.1056$ and is closer to 5.83. In the interval of thousandths, the number $\sqrt{34}$ lies between 5.830 and 5.831 because $5.830^2 = 33.9889$ and $5.831^2 = 34.000561$ but is closer to 5.831. Since 34 is closer to 5.831^2 than 5.830^2, then the three-decimal digit approximation of the number is 5.831.

3. **Write the decimal expansion of $\sqrt{47}$ to at least two-decimal digits.**

    The number $\sqrt{47}$ is between 6 and 7 but closer to 7 because $6^2 < 47 < 7^2$. In the interval of tenths, the number $\sqrt{47}$ is between 6.8 and 6.9 because $6.8^2 = 46.24$ and $6.9^2 = 47.61$. In the interval of hundredths, the number $\sqrt{47}$ is between 6.85 and 6.86 because $6.85^2 = 46.9225$ and $6.86^2 = 47.0596$. Therefore, to two-decimal digits, the number $\sqrt{47}$ is approximately 6.85

4. **Write the decimal expansion of $\sqrt{46}$ to at least two-decimal digits.**

    The number $\sqrt{46}$ is between integers 6 and 7 because $6^2 < 46 < 7^2$. Since $\sqrt{46}$ is closer to 7, I will start checking the tenths intervals between 6.9 and 7. $\sqrt{46}$ is between 6.7 and 6.8 since $6.7^2 = 44.89$ and $6.8^2 = 46.24$. Checking the hundredths interval, $\sqrt{46}$ is between 6.78 and 6.79 since $6.78^2 = 45.9684$ and $6.79^2 = 46.1041$. The two-decimal approximation $\sqrt{46}$ is 6.78.

5. Explain how to improve the accuracy of the decimal expansion of an irrational number.

    In order to improve the accuracy of the decimal expansion of an irrational number, you must examine increasingly smaller increments on the number line. Specifically, examine increments of decreasing powers of 10. The basic inequality allows us to determine which interval a number is between. We begin by determining which two integers the number lies between and then decreasing the power of 10 to look at the interval of tenths. Again using the basic inequality, we can narrow down the approximation to a specific interval of tenths. Then, we look at the interval of hundredths and use the basic inequality to determine which interval of hundredths the number would lie between. Then, we examine the interval of thousandths. Again, the basic inequality allows us to narrow down the approximation to thousandths. The more intervals we examine, the more accurate the decimal expansion of an irrational number will be.

6. Is the number $\sqrt{144}$ rational or irrational? Explain.

    The number $\sqrt{144}$ is 12, a rational number.

7. Is the number $0.\overline{64} = 0.646464646\ldots$ rational or irrational? Explain.

    We have seen that every number that has a repeating decimal expansion is a fraction; that is, it is a rational number. In this case, $0.646\,464\,646\ldots = \frac{64}{99}$, and is therefore a rational number.

8. Henri computed the first 100 decimal digits of the number $\frac{352}{541}$ and got

    $0.65064695009242144177449168207024029574861367837338262476 8946$
    $395563770794824399260628465804066543438077634011090573012 94\ldots.$

    He saw no repeating pattern to the decimal and so concluded that the number is irrational. Do you agree with Henri's conclusion? If not, what would you say to Henri?

    The fraction $\frac{352}{541}$ is certainly a rational number, and so it will have a repeating decimal expansion. One probably has to go beyond 100 decimal places to see the digits repeat.

    (This decimal actually repeats after the 540th decimal place.)

9. Use a calculator to determine the decimal expansion of $\sqrt{35}$. Does the number appear to be rational or irrational? Explain.

    Based on the decimal expansion, the number $\sqrt{35}$ appears to be irrational. The decimal expansion is infinite and does not appear to have a repeating pattern.

10. Use a calculator to determine the decimal expansion of $\sqrt{101}$. Does the number appear to be rational or irrational? Explain.

    Based on the decimal expansion, the number $\sqrt{101}$ appears to be irrational. The decimal expansion is infinite and does not appear to have a repeating pattern.

11. Use a calculator to determine the decimal expansion of $\sqrt{7}$. Does the number appear to be rational or irrational? Explain.

    Based on the decimal expansion, the number $\sqrt{7}$ appears to be irrational. The decimal expansion is infinite and does not appear to have a repeating pattern.

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12. Use a calculator to determine the decimal expansion of $\sqrt{8720}$. Does the number appear to be rational or irrational? Explain.

    Based on the decimal expansion, the number $\sqrt{8720}$ appears to be irrational. The decimal expansion is infinite and does not appear to have a repeating pattern.

13. Use a calculator to determine the decimal expansion of $\sqrt{17956}$. Does the number appear to be rational or irrational? Explain.

    Based on the decimal expansion, the number $\sqrt{17956}$ is rational because it is equivalent to 134.

14. Since the number $\frac{3}{5}$ is rational, must the number $\left(\frac{3}{5}\right)^2$ be rational as well? Explain.

    Yes, since $\frac{3}{5}$ is rational it makes sense that $\left(\frac{3}{5}\right)^2$ would also be rational since $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$ is a ratio of integers.

15. If a number x is rational, must the number x^2 be rational as well? Explain.

    If x is rational, then we can write $x = \frac{a}{b}$ for some integers a and b. This means that $x^2 = \frac{a^2}{b^2}$ and so is necessarily rational as well.

16. Challenge: Determine the two-decimal digit approximation of the number $\sqrt[3]{9}$.

    The number $\sqrt[3]{9}$ is between integers 2 and 3 because $2^3 < 9 < 3^3$. Since $\sqrt[3]{9}$ is closer to 2, I will start checking the tenths intervals between 2 and 3. $\sqrt[3]{9}$ is between 2 and 2.1 since $2^3 = 8$ and $2.1^3 = 9.261$. Checking the hundredths interval, $\sqrt[3]{9}$ is between 2.08 and 2.09 since $2.08^3 = 8.998912$ and $2.09^3 = 9.129329$. The two-decimal digit approximation $\sqrt[3]{9}$ is 2.08.

Lesson 12: Decimal Expansions of Fractions, Part 2

Student Outcomes

- Students develop an alternative method for computing the decimal expansion of a rational number.

Lesson Notes

In this lesson, students use the ideas developed in the previous lesson for finding decimal approximations to quantities and apply them to computing the decimal expansion of rational numbers. This produces a method that allows one to compute decimal expansions of fractions without resorting to the long-division algorithm. The general strategy is to compare a rational number, written as a mixed number, with a decimal: $3\frac{5}{11} = 3\frac{2}{11} = 3.1 + something\ smaller\ than\ a$ *tenth*. The process continues until a repeating pattern emerges, as it must.

This lesson also includes a Rapid White Board Exchange fluency activity on the side topic of volume. It takes approximately 10 minutes to complete and can be found at the end of this lesson.

Classwork

Discussion (20 minutes)

Example 1

Find the decimal expansion of $\frac{35}{11}$.

- For fun, let's see if we can find the decimal expansion of $\frac{35}{11}$ without using long division.

- To start, can we say between which two integers this number lies?

  □ *The number $\frac{35}{11}$ would lie between 3 and 4 on the number line because $\frac{35}{11} = \frac{33}{11} + \frac{2}{11} = 3 + \frac{2}{11}$.*

- Could we say in which tenth between 3 and 4 the number $3 + \frac{2}{11}$ lies? Is this tricky?

Provide time for students to discuss strategies in small groups; then, share their ideas with the class. Encourage students to critique the reasoning of their classmates.

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▪ We know that $\frac{35}{11}$ has a decimal expansion beginning with 3 in the ones place because $\frac{35}{11} = 3 + \frac{2}{11}$. Now we want to determine the tenths digit, the hundredths digit, and then the thousandths digit.

3.

Ones	Tenths	Hundredths	Thousandths

▪ To figure out the tenths digit, we will use an inequality based on tenths. We are looking for the consecutive integers, m and $m + 1$, so that

$$3 + \frac{m}{10} < \frac{35}{11} < 3 + \frac{m+1}{10}.$$

3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9 4.0

Scaffolding:

An alternative way of asking this question is "In which interval could we place the fraction $\frac{2}{11}$?" Show students the number line labeled with tenths.

We can rewrite the middle term:

$$3 + \frac{m}{10} < 3 + \frac{2}{11} < 3 + \frac{m+1}{10}.$$

This means we're looking at

$$\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}.$$

Give students time to make sense of the inequalities $3 + \frac{m}{10} < \frac{35}{11} < 3 + \frac{m+1}{10}$ and $\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}$.

Since the intervals of tenths are represented by $\frac{m}{10}$ and $\frac{m+1}{10}$, consider using concrete numbers. The chart below may help students make sense of the intervals and the inequalities.

Integer	Next Integer
1	2
3	4
5	6
12	13
114	115
m	$m+1$

Tenth	Next Tenth
$0.1 = \frac{1}{10}$	$0.2 = \frac{2}{10}$
$0.3 = \frac{3}{10}$	$0.4 = \frac{4}{10}$
$0.5 = \frac{5}{10}$	$0.6 = \frac{6}{10}$
$1.2 = \frac{12}{10}$	$1.3 = \frac{13}{10}$
$11.4 = \frac{114}{10}$	$11.5 = \frac{115}{10}$
$\frac{m}{10}$	$\frac{m+1}{10}$

- Multiplying through by 10, we get

$$m < 10\left(\frac{2}{11}\right) < m + 1$$

$$m < \frac{20}{11} < m + 1.$$

- So now we are asking the following: Between which two integers does $\frac{20}{11}$ lie?

  □ *We have $\frac{20}{11}$ is between $\frac{11}{11} = 1$ and $\frac{22}{11} = 2$. That is, $\frac{20}{11}$ lies between 1 and 2.*

  (Some students might observe $\frac{20}{11} = 1 + \frac{9}{11}$, which again shows that $\frac{20}{11}$ lies between 1 and 2.)

- So we have that $1 < \frac{20}{11} < 2$. This means that $\frac{1}{10} < \frac{2}{11} < \frac{2}{10}$, and consequently $3 + \frac{1}{10} < 3 + \frac{2}{11} < 3 + \frac{2}{10}$, which is what we were first looking for.

- So what does this say about the location of $\frac{35}{11} = 3 + \frac{2}{11}$ on the number line?

  □ *It means that $\frac{35}{11}$ lies between 3.1 and 3.2 and that we now know the decimal expansion of $\frac{35}{11}$ has a 1 in the tenths place.*

<div align="center">

3. 1

Ones	Tenths	Hundredths	Thousandths

</div>

- Now can we pin down in which hundredth interval $\frac{35}{11}$ lies?

- This time we are looking for the integer m with

$$3 + \frac{1}{10} + \frac{m}{100} < \frac{35}{11} < 3 + \frac{1}{10} + \frac{m+1}{100}.$$

Provide time for students to make sense of this.

- Subtracting 3 and $\frac{1}{10}$ throughout leaves us with

$$\frac{m}{100} < \frac{35}{11} - 3 - \frac{1}{10} < \frac{m+1}{100}.$$

So we need to compute $\frac{35}{11} - 3 - \frac{1}{10}$, which is equal to $3 + \frac{2}{11} - 3 - \frac{1}{10}$, which is equal to $\frac{2}{11} - \frac{1}{10}$. And we can do that:

$$\frac{2}{11} - \frac{1}{10} = \frac{20}{110} - \frac{11}{110} = \frac{9}{110}.$$

So we are left thinking about

$$\frac{m}{100} < \frac{9}{110} < \frac{m+1}{100}.$$

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- We are now looking for consecutive integers m and $m + 1$ so that

$$\frac{m}{100} < \frac{9}{110} < \frac{m+1}{100}.$$

Let's multiply through by some number to make the integers m and $m + 1$ easier to access. Which number should we multiply through by?

- □ *Multiplying through by* 100 *will eliminate the fractions at the beginning and at the end of the inequality.*

- Multiplying through by 100, we get

$$m < \frac{900}{110} < m + 1.$$

- This is now asking the following: Between which two integers, m and $m + 1$, does $\frac{900}{110}$ lie?

- □ $\frac{900}{110}$ *or* $\frac{90}{11}$ *is between* $\frac{88}{11}$, *which is 8 and* $\frac{99}{11}$, *which is 9. (Or students might observe* $\frac{90}{11} = 8 + \frac{2}{11}$.)

- Now we know that $m = 8$. What was the original inequality we were looking at?

- □ $3 + \frac{1}{10} + \frac{m}{100} < \frac{35}{11} < 3 + \frac{1}{10} + \frac{m+1}{100}$

- So we have $3 + \frac{1}{10} + \frac{8}{100} < \frac{35}{11} < 3 + \frac{1}{10} + \frac{9}{100}$, telling us that $\frac{35}{11}$ lies between 3.18 and 3.19 and so has an 8 in the hundredth's place of its decimal expansion.

<div align="center">

3.　　　　　　**1**　　　　　　**8**

Ones	Tenths	Hundredths	Thousandths

</div>

- Now we wonder in which thousandth $\frac{35}{11}$ or $3 + \frac{2}{11}$ lies. We now seek the integer m where

$$3 + \frac{1}{10} + \frac{8}{100} + \frac{m}{1000} < 3 + \frac{2}{11} < 3 + \frac{1}{10} + \frac{8}{100} + \frac{m+1}{1000}.$$

Subtracting 3 and $\frac{1}{10}$ and $\frac{8}{100}$ throughout gives

$$\frac{m}{1000} < 3 + \frac{2}{11} - 3 - \frac{1}{10} - \frac{8}{100} < \frac{m+1}{1000}$$

$$\frac{m}{1000} < \frac{2}{11} - \frac{1}{10} - \frac{8}{100} < \frac{m+1}{1000}.$$

- We need to work out $\frac{2}{11} - \frac{1}{10} - \frac{8}{100}$:

$$\frac{2}{11} - \left(\frac{1}{10} + \frac{8}{100}\right) = \frac{2}{11} - \frac{18}{100}$$
$$= \frac{200}{1100} - \frac{198}{1100}$$
$$= \frac{2}{1100}.$$

So we are looking for the integer m that fits the inequality

$$\frac{m}{1000} < \frac{2}{1100} < \frac{m+1}{1000}.$$

- Multiplying through by 1000 gives

$$m < \frac{20}{11} < m + 1.$$

- Now we are asking the following: Between which two integers does $\frac{20}{11}$ lie? What value of m do we need?

    □ *We have that $\frac{20}{11}$ lies between $\frac{11}{11}$, which is 1 and $\frac{22}{11}$, which is 2. We also see this by writing*
    $\frac{20}{11} = 1 + \frac{9}{11}$. *We need $m = 1$.*

- Have we asked and answered this very question before?

    □ *Yes. It came up when we were looking for the tenths decimal digit.*

- Back to our original equation, substituting $m = 1$ now shows

$$3 + \frac{1}{10} + \frac{8}{100} + \frac{1}{1000} < 3 + \frac{2}{11} < 3 + \frac{1}{10} + \frac{8}{100} + \frac{2}{1000}.$$

Therefore, the next digit in the decimal expansion of $\frac{35}{11}$ is a 1:

<table>
<tr><td align="center">**3.**</td><td align="center">**1**</td><td align="center">**8**</td><td align="center">**1**</td></tr>
<tr><td align="center">**Ones**</td><td align="center">**Tenths**</td><td align="center">**Hundredths**</td><td align="center">**Thousandths**</td></tr>
</table>

- We seem to be repeating our work now, so it is natural to ask the following: Have we reached the repeating part of the decimal? That is, does $\frac{35}{11} = 3.18181818\ldots$? There are two ways we could think about this. Let's work out which fraction has a decimal expansion of $3.181818\ldots$ and see if it is $\frac{35}{11}$.

    □ *Have students compute $0.181818\ldots = \frac{18}{99} = \frac{2}{11}$ and so observe that $3.181818\ldots = 3 + \frac{2}{11} = \frac{35}{11}$. We do indeed now have the correct repeating decimal expansion.*

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- Or we could try to argue directly that we must indeed be repeating our work. We do this as follows:

  We first saw the fraction $\frac{2}{11}$ when we asked in which interval of a tenth our decimal lies, $\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}$.

  We next saw the fraction $\frac{2}{11}$ when asking about which thousandth the decimal lies, $\frac{m}{1000} < \frac{2}{1100} < \frac{m+1}{1000}$,

  which after multiplying through by 100 gives $\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}$. In each scenario, we are two-elevenths along an interval, one at the tenths scale and one at the thousandths scale. The situations are the same, just at different scales, and so the same work applies. We are indeed in a repeating pattern of work.

  (This argument is very subtle. Reassure students that they can always check any repeating decimal expansion they suspect is correct by computing the fraction that goes with that expansion.)

- Of course, we can also compute the decimal expansion of a fraction with the long division algorithm, too.

Exercises 1–3 (5 minutes)

Students work independently or in pairs to complete Exercises 1–3.

Exercises 1–3

1. **Find the decimal expansion of $\frac{5}{3}$ without using long division.**

$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3}$$
$$= 1 + \frac{2}{3}$$

The decimal expansion begins with the integer 1.

Among the intervals of tenths, we are looking for integers m and $m + 1$ so that

$$1 + \frac{m}{10} < 1 + \frac{2}{3} < 1 + \frac{m+1}{10},$$

which is the same as

$$\frac{m}{10} < \frac{2}{3} < \frac{m+1}{10}$$
$$m < \frac{20}{3} < m + 1$$

and

$$\frac{20}{3} = \frac{18}{3} + \frac{2}{3}$$
$$= 6 + \frac{2}{3}.$$

The tenths digit is 6.

Among the intervals of hundredths we are looking for integers m and $m + 1$ so that

$$1 + \frac{6}{10} + \frac{m}{100} < \frac{5}{3} < 1 + \frac{6}{10} + \frac{m+1}{100},$$

which is equivalent to

$$\frac{m}{100} < \frac{2}{3} - \frac{6}{10} < \frac{m+1}{100}.$$

Now

$$\frac{2}{3} - \frac{6}{10} = \frac{2}{30}.$$

So we are looking for integers m and $m + 1$ where

$$\frac{m}{100} < \frac{2}{30} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{20}{3} < m + 1.$$

But we already know that $\frac{20}{3} = 6 + \frac{2}{3}$; therefore, the hundredths digit is 6. We feel like we are repeating our work, so we suspect $\frac{5}{3} = 1.666\ldots$. To check: $0.6666\ldots = \frac{6}{9} = \frac{2}{3}$ and $1.6666\ldots = 1 + \frac{2}{3} = \frac{5}{3}$. We are correct.

2. **Find the decimal expansion of $\dfrac{5}{11}$ without using long division.**

   Its decimal expansion begins with the integer 0.

   In the intervals of tenths, we are looking for integers m and $m + 1$ so that

   $$\frac{m}{10} < \frac{5}{11} < \frac{m+1}{10},$$

   which is the same as

   $$m < \frac{50}{11} < m + 1$$

   $$\frac{50}{11} = \frac{44}{11} + \frac{6}{11}$$
   $$= 4 + \frac{6}{11}$$

   The tenths digit is 4.

   In the intervals of hundredths, we are looking for integers m and $m + 1$ so that

   $$\frac{4}{10} + \frac{m}{100} < \frac{5}{11} < \frac{4}{10} + \frac{m+1}{100}.$$

   This is equivalent to

   $$\frac{m}{100} < \frac{5}{11} - \frac{4}{10} < \frac{m+1}{100}.$$

   Now

   $$\frac{5}{11} - \frac{4}{10} = \frac{6}{110},$$

   so we are looking for integers m and $m + 1$ where

   $$\frac{m}{100} < \frac{6}{110} < \frac{m+1}{100},$$

   which is the same as

   $$m < \frac{60}{11} < m + 1.$$

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As

$$\frac{60}{11} = \frac{55}{11} + \frac{5}{11}$$
$$= 5 + \frac{5}{11}$$

we see that the hundredths digit is 5.

The fraction $\frac{5}{11}$ has reappeared, which makes us suspect we are in a repeating pattern and we have

$\frac{5}{11} = 0.454545....$ *To check:* $0.454545... = \frac{45}{99} = \frac{5}{11}.$ *We are correct.*

3. Find the decimal expansion of the number $\frac{23}{99}$ first without using long division and then again using long division.

    The decimal expansion begins with the integer 0.

    In the interval of tenths, we are looking for integers m and $m + 1$ so that

    $$\frac{m}{10} < \frac{23}{99} < \frac{m+1}{10},$$

    which is the same as

    $$m < \frac{230}{99} < m + 1.$$

    Now

    $$\frac{230}{99} = \frac{198}{99} + \frac{32}{99}$$
    $$= 2 + \frac{32}{99}$$

    showing that the tenths digit is 2.

    In the interval of hundredths, we are looking for integers m and $m + 1$ so that

    $$\frac{2}{10} + \frac{m}{100} < \frac{23}{99} < \frac{2}{10} + \frac{m+1}{100},$$

    which is equivalent to

    $$\frac{m}{100} < \frac{23}{99} - \frac{2}{10} < \frac{m+1}{100}.$$

    Now

    $$\frac{23}{99} - \frac{2}{10} = \frac{32}{990},$$

    so we want

    $$\frac{m}{100} < \frac{32}{990} < \frac{m+1}{100},$$

    which is the same as

    $$m < \frac{320}{99} < m + 1.$$

> *Now*
>
> $$\frac{320}{99} = \frac{297}{99} + \frac{23}{99}$$
>
> $$= 3 + \frac{23}{99}$$
>
> *The hundredths digit is 3. The reappearance of $\frac{23}{99}$ makes us suspect that we're in a repeating pattern and*
> $\frac{23}{99} = 0.232323….$ *We check that $0.232323…$ does indeed equal $\frac{23}{99}$, and we are correct.*

Fluency Exercise (10 minutes): Area and Volume I

Refer to the Rapid White Board Exchanges section in the Module 1 Module Overview for directions to administer a Rapid White Board Exchange.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We have an alternative method for computing the decimal expansions of rational numbers.

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Lesson Summary

For rational numbers, there is no need to guess and check in which interval of tenths, hundredths, or thousandths the number will lie.

For example, to determine where the fraction $\frac{1}{8}$ lies in the interval of tenths, compute using the following inequality:

$$\frac{m}{10} < \frac{1}{8} < \frac{m+1}{10}$$ Use the denominator of 10 because we need to find the tenths digit of $\frac{1}{8}$.

$$m < \frac{10}{8} < m+1$$ Multiply through by 10.

$$m < 1\frac{1}{4} < m+1$$ Simplify the fraction $\frac{10}{8}$.

The last inequality implies that $m = 1$ and $m + 1 = 2$ because $1 < 1\frac{1}{4} < 2$. Then, the tenths digit of the decimal expansion of $\frac{1}{8}$ is 1.

To find in which interval of hundredths $\frac{1}{8}$ lies, we seek consecutive integers m and $m + 1$ so that

$$\frac{1}{10} + \frac{m}{100} < \frac{1}{8} < \frac{1}{10} + \frac{m+1}{100}.$$

This is equivalent to

$$\frac{m}{100} < \frac{1}{8} - \frac{1}{10} < \frac{m+1}{100},$$

so we compute $\frac{1}{8} - \frac{1}{10} = \frac{2}{80} = \frac{1}{40}$. We have

$$\frac{m}{100} < \frac{1}{40} < \frac{m+1}{100}.$$

Multiplying through by 100 gives

$$m < \frac{10}{4} < m+1.$$

This inequality implies that $m = 2$ and $m + 1 = 3$ because $2 < 2\frac{1}{2} < 3$. Then, the hundredths digit of the decimal expansion of $\frac{1}{8}$ is 2.

We can continue the process until the decimal expansion is complete or until we suspect a repeating pattern that we can verify.

Exit Ticket (5 minutes)

Name _____ Date _____

Lesson 12: Decimal Expansions of Fractions, Part 2

Exit Ticket

Find the decimal expansion of $\dfrac{41}{6}$ without using long division.

Exit Ticket Sample Solutions

Find the decimal expansion of $\dfrac{41}{6}$ **without using long division.**

$$\frac{41}{6} = \frac{36}{6} + \frac{5}{6}$$
$$= 6 + \frac{5}{6}$$

The ones digit is 6.

To determine in which interval of tenths the fraction lies, we look for integers m and $m + 1$ so that

$$6 + \frac{m}{10} < 6 + \frac{5}{6} < 6 + \frac{m+1}{10},$$

which is the same as

$$\frac{m}{10} < \frac{5}{6} < \frac{m+1}{10}$$
$$m < \frac{50}{6} < m + 1.$$

We compute

$$\frac{50}{6} = \frac{48}{6} + \frac{2}{6}$$
$$= 8 + \frac{1}{3}$$

The tenths digit is 8.

To determine in which interval of hundredths the fraction lies, we look for integers m and $m + 1$ so that

$$6 + \frac{8}{10} + \frac{m}{100} < 6 + \frac{5}{6} < 6 + \frac{8}{10} + \frac{m+1}{100},$$

which is the same as

$$\frac{m}{100} < \frac{5}{6} - \frac{8}{10} < \frac{m+1}{100}.$$

Now

$$\frac{5}{6} - \frac{8}{10} = \frac{1}{30},$$

so we have

$$\frac{m}{100} < \frac{1}{30} < \frac{m+1}{100}.$$

This is the same as

$$m < \frac{10}{3} < m + 1.$$

Since

$$\frac{10}{3} = 3 + \frac{1}{3},$$

the hundredths digit is 3.

We are a third over a whole number of tenths and a third over a whole number of hundredths. We suspect we are in a repeating pattern and that $\frac{41}{6} = 6.83333\ldots$

To check:

$$x = 6.8333\ldots$$
$$10x = 68.3333\ldots$$
$$10x = 68 + 0.3333\ldots$$
$$10x = 68 + \frac{1}{3}$$
$$10x = \frac{204}{3} + \frac{1}{3}$$
$$10x = \frac{205}{3}$$
$$x = \frac{205}{30}$$
$$x = \frac{41}{6}$$

We are correct.

Problem Set Sample Solutions

1. **Without using long division, explain why the tenths digit of $\frac{3}{11}$ is a 2.**

   In the interval of tenths, we are looking for integers m and $m + 1$ so that

   $$\frac{m}{10} < \frac{3}{11} < \frac{m+1}{10},$$

   which is the same as

   $$m < \frac{30}{11} < m + 1$$

   $$\frac{30}{11} = \frac{22}{11} + \frac{8}{11}$$
   $$= 2 + \frac{8}{11}$$

   In looking at the interval of tenths, we see that the number $\frac{3}{11}$ must be between $\frac{2}{10}$ and $\frac{3}{10}$ because $\frac{2}{10} < \frac{3}{11} < \frac{3}{10}$. For this reason, the tenths digit of the decimal expansion of $\frac{3}{11}$ must be 2.

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2. **Find the decimal expansion of $\frac{25}{9}$ without using long division.**

$$\frac{25}{9} = \frac{18}{9} + \frac{7}{9}$$
$$= 2 + \frac{7}{9}$$

The ones digit is 2. In the interval of tenths, we are looking for integers m and $m + 1$ so that

$$\frac{m}{10} < \frac{7}{9} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{70}{9} < m + 1$$
$$\frac{70}{9} = \frac{63}{9} + \frac{7}{9}$$
$$= 7 + \frac{7}{9}$$

The tenths digit is 7. The difference between $\frac{7}{9}$ and $\frac{7}{10}$ is

$$\frac{7}{9} - \frac{7}{10} = \frac{7}{90}.$$

In the interval of hundredths, we are looking for integers m and $m + 1$ so that

$$\frac{m}{100} < \frac{7}{90} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{70}{9} < m + 1.$$

However, we already know that $\frac{70}{9} = 7 + \frac{7}{9}$; therefore, the hundredths digit is 7. Because we keep getting $\frac{7}{9}$, we can assume the digit of 7 will continue to repeat. Therefore, the decimal expansion of $\frac{25}{9}$ is $2.777\ldots$.

3. **Find the decimal expansion of $\frac{11}{41}$ to at least 5 digits without using long division.**

In the interval of tenths, we are looking for integers m and $m + 1$ so that

$$\frac{m}{10} < \frac{11}{41} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{110}{41} < m + 1$$
$$\frac{110}{41} = \frac{82}{41} + \frac{28}{41} = 2 + \frac{28}{41}.$$

The tenths digit is 2. The difference between $\frac{11}{41}$ and $\frac{2}{10}$ is

$$\frac{11}{41} - \frac{2}{10} = \frac{28}{410}.$$

In the interval of hundredths, we are looking for integers m and $m + 1$ so that

$$\frac{m}{100} < \frac{28}{410} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{280}{41} < m + 1$$

$$\frac{280}{41} = \frac{246}{41} + \frac{34}{41} = 6 + \frac{34}{41}.$$

The hundredths digit is 6. *The difference between* $\frac{11}{41}$ *and* $\left(\frac{2}{10} + \frac{6}{100}\right)$ *is*

$$\frac{11}{41} - \left(\frac{2}{10} + \frac{6}{100}\right) = \frac{11}{41} - \frac{26}{100} = \frac{34}{4100}.$$

In the interval of thousandths, we are looking for integers m *and* $m + 1$ *so that*

$$\frac{m}{1000} < \frac{34}{4100} < \frac{m + 1}{1000},$$

which is the same as

$$m < \frac{340}{41} < m + 1$$

$$\frac{340}{41} = \frac{328}{41} + \frac{12}{41} = 8 + \frac{12}{41}.$$

The thousandths digit is 8. *The difference between* $\frac{11}{41}$ *and* $\left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000}\right)$ *is*

$$\frac{11}{41} - \left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000}\right) = \frac{11}{41} - \frac{268}{1000} = \frac{12}{41000}.$$

In the interval of ten-thousandths, we are looking for integers m *and* $m + 1$ *so that*

$$\frac{m}{10000} < \frac{12}{41000} < \frac{m + 1}{10000},$$

which is the same as

$$m < \frac{120}{41} < m + 1$$

$$\frac{120}{41} = \frac{82}{41} + \frac{38}{41} = 2 + \frac{38}{41}.$$

The ten-thousandths digit is 2. *The difference between* $\frac{11}{41}$ *and* $\left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000} + \frac{2}{10000}\right)$ *is*

$$\frac{11}{41} - \left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000} + \frac{2}{10000}\right) = \frac{11}{41} - \frac{2682}{10000} = \frac{38}{410000}.$$

In the interval of hundred-thousandths, we are looking for integers m *and* $m + 1$ *so that*

$$\frac{m}{100000} < \frac{38}{410000} < \frac{m + 1}{100000},$$

which is the same as

$$m < \frac{380}{41} < m + 1$$

$$\frac{380}{41} = \frac{369}{41} + \frac{11}{41} = 9 + \frac{11}{41}.$$

The hundred-thousandths digit is 9. *We see again the fraction* $\frac{11}{41}$, *so we can expect the decimal digits to repeat at this point. Therefore, the decimal approximation of* $\frac{11}{41}$ *is* $0.2682926829….$

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4. Which number is larger, $\sqrt{10}$ or $\dfrac{28}{9}$? Answer this question without using long division.

The number $\sqrt{10}$ is between 3 and 4. In the sequence of tenths, $\sqrt{10}$ is between 3.1 and 3.2 because $3.1^2 < \left(\sqrt{10}\right)^2 < 3.2^2$. In the sequence of hundredths, $\sqrt{10}$ is between 3.16 and 3.17 because $3.16^2 < \left(\sqrt{10}\right)^2 < 3.17^2$. In the sequence of thousandths, $\sqrt{10}$ is between 3.162 and 3.163 because $3.162^2 < \left(\sqrt{10}\right)^2 < 3.163^2$. The decimal expansion of $\sqrt{10}$ is approximately $3.162\ldots$.

$$\frac{28}{9} = \frac{27}{9} + \frac{1}{9}$$
$$= 3 + \frac{1}{9}$$

In the interval of tenths, we are looking for the integers m and $m+1$ so that

$$\frac{m}{10} < \frac{1}{9} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{10}{9} < m+1$$
$$\frac{10}{9} = \frac{9}{9} + \frac{1}{9}$$
$$= 1 + \frac{1}{9}$$

The tenths digit is 1. Since the fraction $\dfrac{1}{9}$ has reappeared, we can assume that the next digit is also 1, and the work will continue to repeat. Therefore, the decimal expansion of $\dfrac{28}{9}$ is $3.1111\ldots$.

Therefore, $\dfrac{28}{9} < \sqrt{10}$.

Alternatively: $\left(\sqrt{10}\right)^2 = 10$ and $\left(\dfrac{28}{9}\right)^2 = \dfrac{784}{81}$, which is less than $\dfrac{810}{81}$ or 10. Thus, $\dfrac{28}{9}$ is the smaller number.

5. Sam says that $\dfrac{7}{11} = 0.63$, and Jaylen says that $\dfrac{7}{11} = 0.636$. Who is correct? Why?

In the interval of tenths, we are looking for integers m and $m + 1$ so that

$$\frac{m}{10} < \frac{7}{11} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{70}{11} < \frac{m+1}{10}$$

$$\frac{70}{11} = \frac{66}{11} + \frac{4}{11}$$

$$= 6 + \frac{4}{11}$$

The tenths digit is 6. The difference between $\dfrac{7}{11}$ and $\dfrac{6}{10}$ is

$$\frac{7}{11} - \frac{6}{10} = \frac{4}{110}.$$

In the interval of hundredths, we are looking for integers m and $m + 1$ so that

$$\frac{m}{100} < \frac{4}{110} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{40}{11} < m+1$$

$$\frac{40}{11} = \frac{33}{11} + \frac{7}{11}$$

$$= 3 + \frac{7}{11}$$

The hundredths digit is 3. Again, we see the fraction $\dfrac{7}{11}$, which means the next decimal digit will be 6, as it was in the tenths place. This means we will again see the fraction $\dfrac{4}{11}$, meaning we will have another digit of 3. Therefore, the decimal expansion of $\dfrac{7}{11}$ is $0.6363…$.

Technically, Sam and Jaylen are incorrect because the fraction $\dfrac{7}{11}$ is an infinite decimal. However, Sam is correct to the first two decimal digits of the number, and Jaylen is correct to the first three decimal digits of the number.

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Area and Volume I

1. Find the area of the square shown below.

$$A = (4 \text{ m})^2$$
$$= 16 \text{ m}^2$$

4 m

2. Find the volume of the cube shown below.

$$V = (4 \text{ m})^3$$
$$= 64 \text{ m}^3$$

4 m

3. Find the area of the rectangle shown below.

$$A = (8 \text{ cm})(4 \text{ cm})$$
$$= 32 \text{ cm}^2$$

4 cm

8 cm

4. Find the volume of the rectangular prism shown below.

$$V = (32 \text{ cm}^2)(6 \text{ cm})$$
$$= 192 \text{ cm}^3$$

4 cm

6 cm

8 cm

5. Find the area of the circle shown below.

$$A = (7 \text{ m})^2 \pi$$
$$= 49\pi \text{ m}^2$$

14 m

6. Find the volume of the cylinder shown below.

$$V = (49\pi \text{ m}^2)(12 \text{ m})$$
$$= 588\pi \text{ m}^3$$

7. Find the area of the circle shown below.

$$A = (6 \text{ in.})^2\pi$$
$$= 36\pi \text{ in}^2$$

8. Find the volume of the cone shown below.

$$V = \left(\frac{1}{3}\right)(36\pi \text{ in}^2)(10 \text{ in.})$$
$$= 120\pi \text{ in}^3$$

9. Find the area of the circle shown below.

$$A = (8 \text{ mm})^2\pi$$
$$= 64\pi \text{ mm}^2$$

10. Find the volume of the sphere shown below.

$$V = \left(\frac{4}{3}\right)\pi(64 \text{ mm}^2)(8 \text{ mm})$$
$$= \frac{2048}{3}\pi \text{ mm}^3$$

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Lesson 13: Comparing Irrational Numbers

Student Outcomes

- Students use finite decimal approximations of irrational numbers to compare the size of irrational numbers.
- Students place irrational numbers in their approximate locations on a number line.

Classwork

Exploratory Challenge/Exercises 1–11 (30 minutes)

Students work in pairs to complete Exercises 1–11. The first exercise may be used to highlight the process of answering and explaining the solution to each question. An emphasis should be placed on students' ability to explain their reasoning. Consider allowing students to use a calculator to check their work, but all decimal expansions should be done by hand. At the end of the Exploratory Challenge, consider asking students to state or write a description of their approach to solving each exercise.

Exploratory Challenge/Exercises 1–11

1. Rodney thinks that $\sqrt[3]{64}$ is greater than $\dfrac{17}{4}$. Sam thinks that $\dfrac{17}{4}$ is greater. Who is right and why?

$$\sqrt[3]{64} = \sqrt[3]{4^3} = 4$$

and

$$\frac{17}{4} = \frac{16}{4} + \frac{1}{4}$$
$$= 4 + \frac{1}{4}$$
$$= 4\frac{1}{4}$$

We see that $\sqrt[3]{64} < \dfrac{17}{4}$. *Sam is correct.*
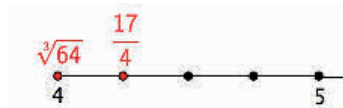
2. Which number is smaller, $\sqrt[3]{27}$ or 2.89? Explain.

$$\sqrt[3]{27} = \sqrt[3]{3^3} = 3$$

We see that 2.89 *is smaller than* $\sqrt[3]{27}$.

---

3. Which number is smaller, $\sqrt{121}$ or $\sqrt[3]{125}$? Explain.

$$\sqrt{121} = \sqrt{11^2} = 11$$
$$\sqrt[3]{125} = \sqrt[3]{5^3} = 5$$

We see that $\sqrt[3]{125}$ is smaller than $\sqrt{121}$.

4. Which number is smaller, $\sqrt{49}$ or $\sqrt[3]{216}$? Explain.

$$\sqrt{49} = \sqrt{7^2} = 7$$
$$\sqrt[3]{216} = \sqrt[3]{6^3} = 6$$

We see that $\sqrt[3]{216}$ is smaller than $\sqrt{49}$.

5. Which number is greater, $\sqrt{50}$ or $\dfrac{319}{45}$? Explain.

Students may use any method to determine the decimal expansion of the fraction.

The number $\dfrac{319}{45}$ is equal to $7.0\overline{8}$.

The number $\sqrt{50}$ is between 7 and 8 because $7^2 < 50 < 8^2$. The number $\sqrt{50}$ is between 7.0 and 7.1 because $7^2 < 50 < 7.1^2$. The number $\sqrt{50}$ is between 7.07 and 7.08 because $7.07^2 < 50 < 7.08^2$. The approximate decimal value of $\sqrt{50}$ is 7.07. Since $7.07 < 7.0\overline{8}$, then $\sqrt{50} < \dfrac{319}{45}$; therefore, the fraction $\dfrac{319}{45}$ is greater than $\sqrt{50}$.

Alternately: $\left(\sqrt{50}\right)^2 = 50$ and $\left(\dfrac{319}{45}\right)^2 = \dfrac{101761}{2025} > \dfrac{101250}{2025} = 50$. So, $\dfrac{319}{45}$ must be larger.

6. Which number is greater, $\dfrac{5}{11}$ or $0.\overline{4}$? Explain.

Students may use any method to determine the decimal expansion of the fraction.

The number $\dfrac{5}{11}$ is equal to $0.\overline{45}$. Since $0.44444\ldots < 0.454545\ldots$, then $0.\overline{4} < \dfrac{5}{11}$; therefore, the fraction $\dfrac{5}{11}$ is greater than $0.\overline{4}$.

Alternately: $0.444\ldots = \dfrac{4}{9}$, and we can compare the fractions $\dfrac{4}{9}$ and $\dfrac{5}{11}$ using their equivalents, $\dfrac{44}{99}$ and $\dfrac{45}{99}$ to see that $\dfrac{5}{11}$ is larger.

7. Which number is greater, $\sqrt{38}$ or $\dfrac{154}{25}$? Explain.

Students may use any method to determine the decimal expansion of the fraction.

$$\frac{154}{25} = \frac{154 \times 4}{25 \times 4} = \frac{616}{10^2} = 6.16$$

The number $\sqrt{38}$ is between 6 and 7 because $6^2 < 38 < 7^2$. The number $\sqrt{38}$ is between 6.1 and 6.2 because $6.1^2 < 38 < 6.2^2$. The number $\sqrt{38}$ is between 6.16 and 6.17 because $6.16^2 < 38 < 6.17^2$. Since $\sqrt{38}$ is greater than 6.16, then $\sqrt{38}$ is greater than $\dfrac{154}{25}$.

Alternately: $\left(\sqrt{38}\right)^2 = 38$ and $\left(\dfrac{154}{25}\right)^2 = \dfrac{23716}{625} < \dfrac{23750}{625} = 38$. So, $\sqrt{38}$ must be larger.

8. Which number is greater, $\sqrt{2}$ or $\dfrac{15}{9}$? Explain.

    Students may use any method to determine the decimal expansion of the fraction.

    The number $\dfrac{15}{9}$ is equal to $1.\overline{6}$.

    The number $\sqrt{2}$ is between 1 and 2 because $1^2 < 2 < 2^2$. The number $\sqrt{2}$ is between 1.4 and 1.5 because $1.4^2 < 2 < 1.5^2$. Therefore, $\sqrt{2} < \dfrac{15}{9}$; the fraction $\dfrac{15}{9}$ is greater.

    Alternately: $\left(\sqrt{2}\right)^2 = 2$ and $\left(\dfrac{15}{9}\right)^2 = \left(\dfrac{5}{3}\right)^2 = \dfrac{25}{9} > 2$. So, $\dfrac{15}{9}$ must be larger.

9. Place each of the following numbers at its approximate location on the number line: $\sqrt{25}, \sqrt{28}, \sqrt{30}, \sqrt{32}, \sqrt{35}$, and $\sqrt{36}$.

    The solutions are shown in red:
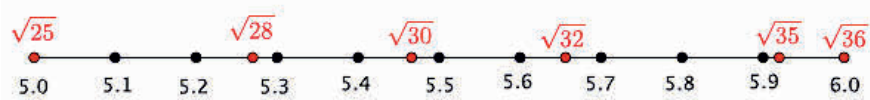
    The number $\sqrt{25} = \sqrt{5^2} = 5$.

    The numbers $\sqrt{28}, \sqrt{30}, \sqrt{32}$, and $\sqrt{35}$ are between 5 and 6. The number $\sqrt{28}$ is between 5.2 and 5.3 because $5.2^2 < 28 < 5.3^2$. The number $\sqrt{30}$ is between 5.4 and 5.5 because $5.4^2 < 30 < 5.5^2$. The number $\sqrt{32}$ is between 5.6 and 5.7 because $5.6^2 < 32 < 5.7^2$. The number $\sqrt{35}$ is between 5.9 and 6.0 because $5.9^2 < 35 < 6^2$.

    The number $\sqrt{36} = \sqrt{6^2} = 6$.

10. Challenge: Which number is larger, $\sqrt{5}$ or $\sqrt[3]{11}$?

    The number $\sqrt{5}$ is between 2 and 3 because $2^2 < 5 < 3^2$. The number $\sqrt{5}$ is between 2.2 and 2.3 because $2.2^2 < 5 < 2.3^2$. The number $\sqrt{5}$ is between 2.23 and 2.24 because $2.23^2 < 5 < 2.24^2$. The number $\sqrt{5}$ is between 2.236 and 2.237 because $2.236^2 < 5 < 2.237^2$. The decimal expansion of $\sqrt{5}$ is approximately $2.236\ldots$.

    The number $\sqrt[3]{11}$ is between 2 and 3 because $2^3 < 11 < 3^3$. The number $\sqrt[3]{11}$ is between 2.2 and 2.3 because $2.2^3 < 11 < 2.3^3$. The number $\sqrt[3]{11}$ is between 2.22 and 2.23 because $2.22^3 < 11 < 2.23^3$. The decimal expansion of $\sqrt[3]{11}$ is approximately $2.22\ldots$. Since $2.22\ldots < 2.236\ldots$, then $\sqrt[3]{11} < \sqrt{5}$; therefore, $\sqrt{5}$ is larger.

    Alternately:

    $$\left(\sqrt[3]{11}\right)^6 = 11^2 = 121$$

    $$\left(\sqrt{5}\right)^6 = 5^3 = 125$$

    We see that $\sqrt{5}$ must be larger.

11. A certain chessboard is being designed so that each square has an area of 3 in^2. What is the length of one edge of the board rounded to the tenths place? (A chessboard is composed of 64 squares as shown.)
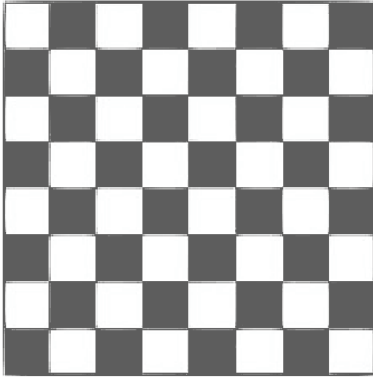
The area of one square is 3 in^2. So, if x is the length of one side of one square,

$$x^2 = 3$$
$$\sqrt{x^2} = \sqrt{3}$$
$$x = \sqrt{3}.$$

There are 8 squares along one edge of the board, so the length of one edge is $8 \times \sqrt{3}$. The number $\sqrt{3}$ is between 1 and 2 because $1^2 < 3 < 2^2$. The number $\sqrt{3}$ is between 1.7 and 1.8 because $1.7^2 < 3 < 1.8^2$. The number $\sqrt{3}$ is between 1.73 and 1.74 because $1.73^2 < 3 < 1.74^2$. The number $\sqrt{3}$ is approximately 1.73. So, the length of one edge of the chessboard is about 8×1.73 inches, which is approximately 13.8 in.

Note: Some students may determine the total area of the board, $64 \times 3 = 192$, and then determine the approximate value of $\sqrt{192}$ as 13.8 to answer the question.

Discussion (5 minutes)

▪ How do we know if a number is rational or irrational?

  □ *Numbers that can be expressed as a fraction $\dfrac{a}{b}$ for some integers a and b, where $b \neq 0$, are by definition, rational numbers. Any number that is not rational is irrational.*

▪ Is the number $1.\overline{6}$ rational or irrational? Explain.

  □ *The number $1.\overline{6}$ is rational because it is equal to $\dfrac{15}{9}$.*

▪ What strategy do you use to write the decimal expansion of a fraction? What strategy do you use to write the decimal expansion of square and cube roots?

  □ *Student responses will vary. Students will likely state that they use long division or equivalent fractions to write the decimal expansion of fractions. Students will say that they have to use the definition of square and cube roots to find in which tenth, in which hundredth, and so on, the root lies and so approximate its decimal expansion.*

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

▪ Finding the first few places of the decimal expansion of numbers allows us to compare the numbers.

Lesson Summary

Finding the first few places of the decimal expansion of numbers allows us to compare the numbers.

Exit Ticket (5 minutes)

Lesson 13: Comparing Irrational Numbers

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Name _____   Date _____

Lesson 13: Comparing Irrational Numbers

Exit Ticket

Place each of the following numbers at its approximate location on the number line: $\sqrt{12}$, $\sqrt{16}$, $\frac{20}{6}$, $3.\overline{53}$, and $\sqrt[3]{27}$.
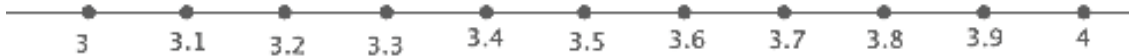
Exit Ticket Sample Solutions

Place each of the following numbers at its approximate location on the number line: $\sqrt{12}$, $\sqrt{16}$, $\frac{20}{6}$, $3.\overline{53}$, and $\sqrt[3]{27}$.

Students may use any method to compute the first few decimal places of a fraction.

The number $\sqrt{12}$ is between 3.4 and 3.5 since $3.4^2 < 12 < 3.5^2$.

The number $\sqrt{16} = \sqrt{4^2} = 4$.

The number $\frac{20}{6}$ is equal to $3.\overline{3}$.

The number $\sqrt[3]{27} = \sqrt[3]{3^3} = 3$.

The solutions are shown in red:
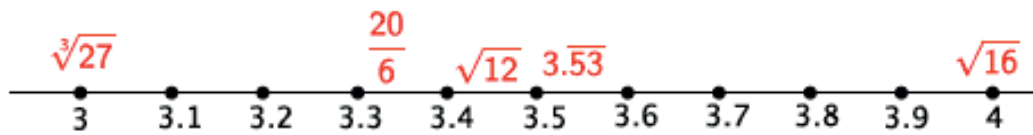
Problem Set Sample Solutions

1. Which number is smaller, $\sqrt[3]{343}$ or $\sqrt{48}$? Explain.

$$\sqrt[3]{343} = \sqrt[3]{7^3} = 7$$

The number $\sqrt{48}$ is between 6 and 7 but definitely less than 7. Therefore, $\sqrt{48} < \sqrt[3]{343}$, and $\sqrt{48}$ is smaller.

2. Which number is smaller, $\sqrt{100}$ or $\sqrt[3]{1000}$? Explain.

$$\sqrt{100} = \sqrt{10^2} = 10$$
$$\sqrt[3]{1000} = \sqrt[3]{10^3} = 10$$

The numbers $\sqrt{100}$ and $\sqrt[3]{1000}$ are equal because both are equal to 10.

3. Which number is larger, $\sqrt{87}$ or $\frac{929}{99}$? Explain.

Students may use any method to compute the first few decimal places of a fraction.

The number $\frac{929}{99}$ is equal to $9.\overline{38}$.

The number $\sqrt{87}$ is between 9 and 10 because $9^2 < 87 < 10^2$. The number $\sqrt{87}$ is between 9.3 and 9.4 because $9.3^2 < 87 < 9.4^2$. The number $\sqrt{87}$ is between 9.32 and 9.33 because $9.32^2 < 87 < 9.33^2$. Since $\sqrt{87} < 9.3$, then $\sqrt{87} < \frac{929}{99}$. The fraction $\frac{929}{99}$ is larger.

4. Which number is larger, $\frac{9}{13}$ or $0.\overline{692}$? Explain.

    Students may use any method to compute the first few decimal places of a fraction.

    The number $\frac{9}{13}$ is equal to $0.\overline{692307}$. Since $0.692307 \ldots < 0.692692 \ldots$, then we see that $\frac{9}{13} < 0.\overline{692}$. The decimal $0.\overline{692}$ is larger.

5. Which number is larger, 9.1 or $\sqrt{82}$? Explain.

    The number $\sqrt{82}$ is between 9 and 10 because $9^2 < 82 < 10^2$. The number $\sqrt{82}$ is between 9.0 and 9.1 because $9.0^2 < 82 < 9.1^2$. Since $\sqrt{82} < 9.1$, then the number 9.1 is larger than the number $\sqrt{82}$.

6. Place each of the following numbers at its approximate location on the number line: $\sqrt{144}$, $\sqrt[3]{1000}$, $\sqrt{130}$, $\sqrt{110}$, $\sqrt{120}$, $\sqrt{115}$, and $\sqrt{133}$. Explain how you knew where to place the numbers.
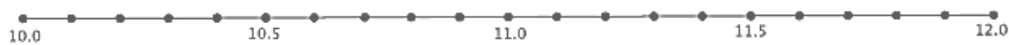
    The solutions are shown in red:
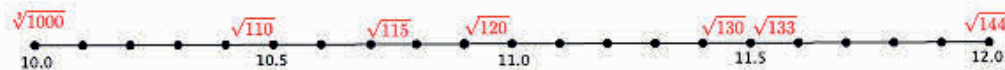
    The number $\sqrt{144} = \sqrt{12^2} = 12$.

    The number $\sqrt[3]{1000} = \sqrt[3]{10^3} = 10$.

    The numbers $\sqrt{110}$, $\sqrt{115}$, and $\sqrt{120}$ are all between 10 and 11 because when squared, their value falls between 10^2 and 11^2. The number $\sqrt{110}$ is between 10.4 and 10.5 because $10.4^2 < 110 < 10.5^2$. The number $\sqrt{115}$ is between 10.7 and 10.8 because $10.7^2 < 115 < 10.8^2$. The number $\sqrt{120}$ is between 10.9 and 11 because $10.9^2 < 120 < 11^2$.

    The numbers $\sqrt{130}$ and $\sqrt{133}$ are between 11 and 12 because when squared, their value falls between 11^2 and 12^2. The number $\sqrt{130}$ is between 11.4 and 11.5 because $11.4^2 < 130 < 11.5^2$. The number $\sqrt{133}$ is between 11.5 and 11.6 because $11.5^2 < 133 < 11.6^2$.

7. Which of the two right triangles shown below, measured in units, has the longer hypotenuse? Approximately how much longer is it?
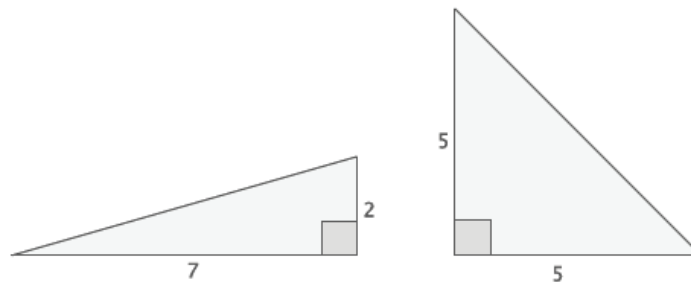
Let x represent the length of the hypotenuse of the triangle on the left.

$$7^2 + 2^2 = x^2$$
$$49 + 4 = x^2$$
$$53 = x^2$$
$$\sqrt{53} = x$$

The number $\sqrt{53}$ is between 7 and 8 because $7^2 < 53 < 8^2$. The number $\sqrt{53}$ is between 7.2 and 7.3 because $7.2^2 < 53 < 7.3^2$. The number $\sqrt{53}$ is between 7.28 and 7.29 because $7.28^2 < 53 < 7.29^2$. The approximate decimal value of $\sqrt{53}$ is 7.28.

Let y represent the length of the hypotenuse of the triangle on the right.

$$5^2 + 5^2 = y^2$$
$$25 + 25 = y^2$$
$$50 = y^2$$
$$\sqrt{50} = y$$

The number $\sqrt{50}$ is between 7 and 8 because $7^2 < 50 < 8^2$. The number $\sqrt{50}$ is between 7.0 and 7.1 because $7.0^2 < 50 < 7.1^2$. The number $\sqrt{50}$ is between 7.07 and 7.08 because $7.07^2 < 50 < 7.08^2$. The approximate decimal value of $\sqrt{50}$ is 7.07.

The triangle on the left has the longer hypotenuse. It is approximately 0.21 units longer than the hypotenuse of the triangle on the right.

Note: Based on their experience, some students may reason that $\sqrt{50} < \sqrt{53}$. To answer completely, students must determine the decimal expansion to approximate how much longer one hypotenuse is than the other.

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Lesson 14: Decimal Expansion of π

Student Outcomes

- Students calculate the first few places of the decimal expansion of π using basic properties of area.
- Students estimate the value of numbers such as π^2.

Lesson Notes

For this lesson, students need grid paper and a compass. Lead students through the activity that produces the decimal expansion of π. Quarter circles on grids of 10 by 10 and 20 by 20 are included at the end of the lesson if the teacher prefers to hand out the grids as opposed to students making their own with grid paper and a compass.

Classwork

Opening Exercise (5 minutes)

The purpose of the Opening Exercise is to remind students of what they know about the number π.

Opening Exercise

a. Write an equation for the area, A, of the circle shown.

6.3 cm
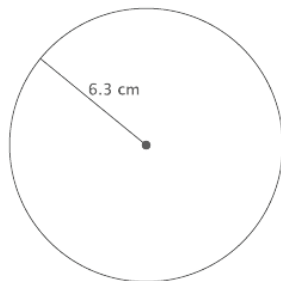
$$A = \pi(6.3)^2$$
$$= 39.69\pi$$

The area of the circle is 39.69π cm^2.

b. Write an equation for the circumference, C, of the circle shown.

9.7 mm
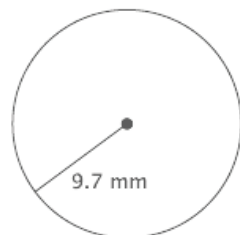
$$C = 2\pi(9.7)$$
$$= 19.4\pi$$

The circumference of the circle is 19.4π mm.

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c. Each of the squares in the grid below has an area of 1 unit^2.

i. Estimate the area of the circle shown by counting squares.

Estimates will vary. The approximate area of the circle is 78 units^2.

ii. Calculate the area of the circle using a radius of 5 units. Use 3.14 as an approximation for π.

$$A \approx 3.14(5)^2$$
$$\approx 78.5$$

The area of the circle is approximately 78.5 units^2.
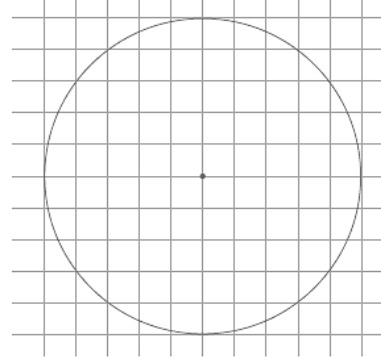
Discussion (25 minutes)

- The number pi, π, is defined as the area of a unit circle, that is, a circle with a radius of one unit. Our goal in this lesson is to determine the decimal expansion of π. What do you think that is?

  □ *Students will likely state that the decimal expansion of π is* 3.14 *because that is the number they have used in the past to approximate π.*

- The number 3.14 is often used to approximate π. How do we know if this is a good approximation for π? Does the decimal expansion of π begin 3.14? How could we check? Any thoughts?

Provide time for students to try to develop a plan for determining the decimal expansion of π. Have students share their ideas with the class.

- The Opening Exercise part (c) gives a strategy. Since the area of the unit circle is equal to π, we can count squares after placing the unit circle on a grid and estimate its area. Actually, let's decrease the amount of work doing this by focusing on the area of just a quarter circle. What is the area of a quarter of the unit circle?

1 unit
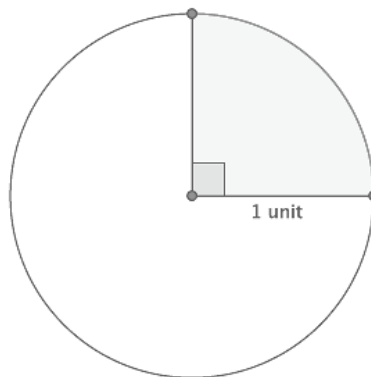
  □ *Since the unit circle has an area of π, then* $\frac{1}{4}\pi$ *will be the area of* $\frac{1}{4}$ *of the unit circle.*

- On a piece of graph paper, mark a center O near the bottom left corner of the paper. Use your ruler to draw two lines through O, one horizontal and one vertical. Let's call 10 widths of the grid squares on the graph paper *one unit*. (So one square width is a tenth of a unit.) Use your compass to measure 10 of the grid squares, and then make an arc to represent the outer edge of the quarter circle. Make sure your arc intersects the horizontal and vertical lines you drew.

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Verify that all students have a quarter circle on their graph paper.

- We have inner squares, squares that are fully inside the quarter circle, and partially inner squares, squares with just some portion sitting inside the quarter circle. Mark a border just inside the quarter circle enclosing all those fully inner squares (as shown in red below). Also mark a border just outside the quarter circle that encloses all the inner and partially inner squares (as shown in black below).
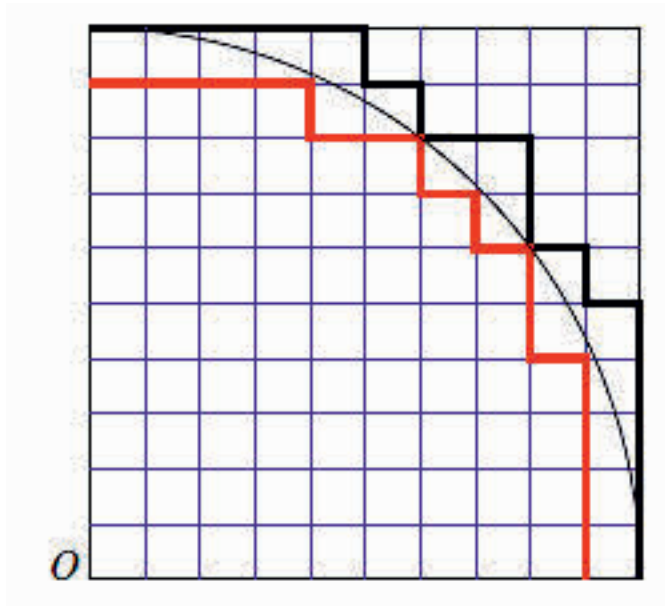
- What is the area of the quarter circle?

    □ $\dfrac{\pi}{4}$

- What is the area of one square?

    □ $\dfrac{1}{10} \times \dfrac{1}{10} = \dfrac{1}{100}$

- How many fully inner squares are there? What does that count say about the approximate value of $\dfrac{\pi}{4}$?

    □ *There are* 69 *fully inner squares. So the area of the quarter circle is approximately* $69 \times \dfrac{1}{100} = 0.69$.

- Is that estimate larger or smaller than the true value of $\dfrac{\pi}{4}$?

    □ *It is smaller. The area enclosed by the red border is smaller than the area of the quarter circle.*

- What is the smallest number of squares that cover the entire area of the quarter circle?

    □ *These are the squares within the second border we drew. There are* 86 *fully inner and partially inner squares covering the area of the quarter circle.*

- Use that count to obtain another estimate for $\dfrac{\pi}{4}$.

    □ *These* 86 *squares have a total area of* $86 \times \dfrac{1}{100} = 0.86$. *This also approximates the area of the quarter circle.*

- Is this estimate larger or smaller than the true value of $\frac{\pi}{4}$?

  - *Larger. We are computing an area larger than the area of the quarter circle.*

- So we then have

$$0.69 < \frac{\pi}{4} < 0.86.$$

Multiplying by 4 throughout gives

$$2.76 < \pi < 3.44.$$

- Does this bound on the number π seem reasonable?

  - *Yes, because we frequently use 3.14 to represent π, and $2.76 < 3.14 < 3.44$.*

- But this inequality trying to capture the value of π is somewhat broad. Can we get a better estimate than the number 69 for the count of squares inside the quarter circle? Shall we count partial squares?
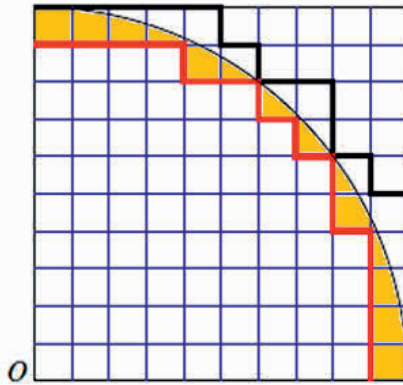
- Estimate the fraction of each square that is shaded here, add up the total amount, and round that total amount to some whole number of squares.

  - *The shaded area corresponds to something like $1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3} + \frac{1}{8} + \frac{7}{8} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{7}{8} + \frac{1}{8} + \frac{1}{3} + \frac{2}{3} + \frac{3}{4} + 1$, which is $8 + \frac{1}{2} + \frac{2}{3}$, or approximately 9. The shaded area is approximately 9 squares. (Accept any answer up to and including 9.)*

- So we can improve our lower estimate for the area of the quarter circle from 0.69 to 0.78 because $69 \times \frac{1}{100} = 0.69$ and $(69 + 9) \times \frac{1}{100} = 0.78$.

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- Can we improve on the number 86 too for the count of squares just covering the quarter circle? This time we want to estimate the fractions of squares shaded here and make the appropriate use of that approximation.
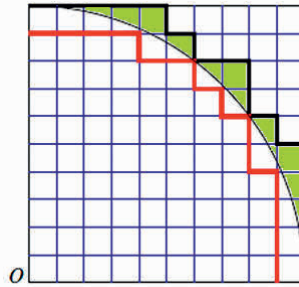
  □ *Students will likely struggle before they realize that they should subtract the fractional counts from the count of 86 squares. The improved estimate for the number of squares covering the unit circle will be something like* $86 - \frac{1}{4} - \frac{1}{3} - \frac{2}{3} - \frac{7}{8} - \frac{1}{8} - \frac{2}{3} - \frac{1}{2} - \frac{1}{2} - \frac{2}{3} - \frac{1}{8} - \frac{7}{8} - \frac{2}{3} - \frac{1}{3} - \frac{1}{4}$, *which is* $80 - \frac{1}{2} - \frac{1}{3}$, *or approximately* 79. *(Accept any answer larger than or equal to 79.)*

- So an improved upper estimate for the value of $\frac{\pi}{4}$ is $79 \times \frac{1}{100}$, or 0.79. We have

$$0.78 < \frac{\pi}{4} < 0.79.$$

This gives

$$3.12 < \pi < 3.16.$$

Does our estimate of 3.14 for π indeed seem reasonable.

  □ *Yes*

- What could we do to make our approximation of π significantly better?
  □ *We could decrease the size of the squares we are using to develop the area of the quarter circle.*

- As you have stated, one way to improve our approximation is by using smaller squares. Instead of having 100 squares, we have 400 squares, so that now each square has an area of $\frac{1}{20} \times \frac{1}{20} = \frac{1}{400}$.
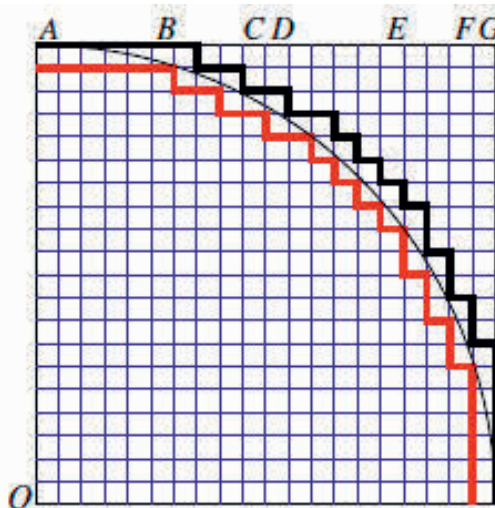
If time permits, allow students to repeat the first part of the process we just went through, counting only whole number squares. If time does not permit, then provide them with the information below.

- The inner region has 294 full squares, and the outer region just covering the quarter circle has 333 full squares. This means that

$$\frac{294}{400} < \frac{\pi}{4} < \frac{333}{400}.$$

- Multiplying by 4 throughout, we have

$$\frac{294}{100} < \pi < \frac{333}{100}$$

$$2.94 < \pi < 3.33.$$

- If we count fractional squares, then we can improve these counts to 310 and 321, respectively.

$$\frac{310}{400} < \frac{\pi}{4} < \frac{321}{400}$$

$$\frac{310}{100} < \pi < \frac{321}{100}$$

$$3.10 < \pi < 3.21$$

- We could continue the process of refining our estimate several more times to see that

$$3.14159 < \pi < 3.14160$$

and then continue on to get an even more precise estimate of π. But at this point, it should be clear that we have a fairly good one already.

- Imagine what would happen if we could repeat this process using even smaller squares. What would that do to our estimate? And how would that affect our decimal expansion?

  □ *If we used even smaller squares, we would get a better estimate, likely with more digits in the decimal expansion.*

Consider showing the first million digits of pi located at http://www.piday.org/million/. Share with students that as of 2013 over 12 trillion digits have been found in the decimal expansion of pi, and still no pattern of digits has emerged. That makes pi an infamous irrational number.

- We finish by making one more observation. If we have a bound on a number between two finite decimals, then we can find bounds on algebraic manipulations of that number. For example, from

$$3.14159 < \pi < 3.14160,$$

we see

$$6.28318 < 2\pi < 6.28320,$$

for example, and

$$3.14159^2 < \pi^2 < 3.14160^2$$

$$9.8695877281 < \pi^2 < 9.86965056.$$

Notice the repeat of the first 4 digits, 9.869, in this inequality. Therefore, we can say that $\pi^2 = 9.869$ is correct up to 3 decimal digits.

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Exercises 1–4 (5 minutes)

Students work on Exercises 1–4 independently or in pairs. If necessary, model for students how to use the given decimal digits of the irrational number to *trap* the number in the inequality for Exercises 2–4. An online calculator is used to determine the decimal values of the squared numbers in Exercises 2–4. If handheld calculators are used, then the decimal values will be truncated to 8 places. However, this does not affect the estimate of the irrational numbers.

Exercises 1–4

1. Gerald and Sarah are building a wheel with a radius of 6.5 cm and are trying to determine the circumference. Gerald says, "Because $6.5 \times 2 \times 3.14 = 40.82$, the circumference is 40.82 cm." Sarah says, "Because $6.5 \times 2 \times 3.10 = 40.3$ and $6.5 \times 2 \times 3.21 = 41.73$, the circumference is somewhere between 40.3 and 41.73." Explain the thinking of each student.

   Gerald is using a common approximation for the number π to determine the circumference of the wheel. That is why he used 3.14 in his calculation. Sarah is using an interval between which the value of π falls, based on the work we did in class. We know that $3.10 < \pi < 3.21$; therefore, her calculations of the circumference uses numbers we know π to be between.

2. Estimate the value of the number $(6.12486\ldots)^2$.

$$6.12486^2 < (6.12486\ldots)^2 < 6.12487^2$$
$$37.513\,910\,019\,6 < (6.12486\ldots)^2 < 37.514\,032\,516\,9$$

   $(6.12486\ldots)^2 = 37.51$ *is correct up to two decimal digits.*

3. Estimate the value of the number $(9.204107\ldots)^2$.

$$9.204\,107^2 < (9.204\,107\ldots)^2 < 9.204\,108^2$$
$$84.715\,585\,667\,449 < (9.204\,107\ldots)^2 < 84.715\,604\,075\,664$$

   $(9.204\,107\ldots)^2 = 84.715$ *is correct up to three decimal digits.*

4. Estimate the value of the number $(4.014325\ldots)^2$.

$$4.014\,325^2 < (4.014\,325\ldots)^2 < 4.014\,326^2$$
$$16.114\,805\,205\,625 < (4.014\,325\ldots)^2 < 16.114813234276$$

   $(4.014\,325\ldots)^2 = 16.1148$ *is correct up to four decimal digits.*

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- The area of a unit circle is π.
- We learned a method to estimate the value of π using graph paper, a unit circle, and areas.

> **Lesson Summary**
>
> Numbers, such as π, are frequently approximated in order to compute with them. Common approximations for π are 3.14 and $\frac{22}{7}$. It should be understood that using an approximate value of a number for computations produces an answer that is accurate to only the first few decimal digits.

Exit Ticket (5 minutes)

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Name _____ Date _____

Lesson 14: Decimal Expansion of π

Exit Ticket

Describe how we found a decimal approximation for π.

Exit Ticket Sample Solutions

Describe how we found a decimal approximation for π.

To make our work easier, we looked at the number of unit squares in a quarter circle that comprised its area. We started by counting just the whole number of unit squares. Then, we continued to revise our estimate of the area by looking at parts of squares specifically to see if parts could be combined to make a whole unit square. We looked at the inside and outside boundaries and said that the value of π would be between these two numbers. The inside boundary is a conservative estimate of the value of π, and the outside boundary is an overestimate of the value of π. We could continue this process with smaller squares in order to refine our estimate for the decimal approximation of π.

Problem Set Sample Solutions

Students estimate the values of numbers squared.

1. Caitlin estimated π to be $3.10 < \pi < 3.21$. If she uses this approximation of π to determine the area of a circle with a radius of 5 cm, what could the area be?

   The area of the circle with radius 5 cm will be between 77.5 cm^2 and 80.25 cm^2.

2. Myka estimated the circumference of a circle with a radius of 4.5 in. to be 28.44 in. What approximate value of π did she use? Is it an acceptable approximation of π? Explain.

$$C = 2\pi r$$
$$28.44 = 2\pi(4.5)$$
$$28.44 = 9\pi$$
$$\frac{28.44}{9} = \pi$$
$$3.16 = \pi$$

   Myka used 3.16 to approximate π. Student responses may vary with respect to whether or not 3.16 is an acceptable approximation for π. Accept any reasonable explanation.

3. A length of ribbon is being cut to decorate a cylindrical cookie jar. The ribbon must be cut to a length that stretches the length of the circumference of the jar. There is only enough ribbon to make one cut. When approximating π to calculate the circumference of the jar, which number in the interval $3.10 < \pi < 3.21$ should be used? Explain.

   In order to make sure the ribbon is long enough, we should use an estimate of π that is closer to 3.21. We know that 3.10 is a fair estimate of π but less than the actual value of π. Similarly, we know that 3.21 is a fair estimate of π but greater than the actual value of π. Since we can only make one cut, we should cut the ribbon so that there is a little more than we need, not less than. For that reason, an approximation of π closer to 3.21 should be used.

4. Estimate the value of the number $(1.86211\ldots)^2$.

$$1.86211^2 < (1.86211\ldots)^2 < 1.86212^2$$
$$3.4674536521 < (1.86211\ldots)^2 < 3.4674908944$$

   $(1.86211\ldots)^2 = 3.4674$ *is correct up to four decimal digits.*

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5. Estimate the value of the number $(5.9035687\ldots)^2$.

$$5.903\,568\,7^2 < (5.903\,568\,7\ldots)^2 < 5.903\,568\,8^2$$
$$34.852\,123\,395\,619\,69 < (5.903\,568\,7\ldots)^2 < 34.852\,124\,576\,333\,44$$

$(5.903\,568\,7\ldots)^2 = 34.85212$ *is correct up to five decimal digits.*

6. Estimate the value of the number $(12.30791\ldots)^2$.

$$12.30791^2 < (12.30791\ldots)^2 < 12.30792^2$$
$$151.484\,648\,568\,1 < (12.30791\ldots)^2 < 151.484\,894\,726\,4$$

$(12.30791\ldots)^2 = 151.484$ *is correct up to three decimal digits.*

7. Estimate the value of the number $(0.6289731\ldots)^2$.

$$0.628\,973\,1^2 < (0.628\,973\,1\ldots)^2 < 0.628\,973\,2^2$$
$$0.395\,607\,160\,523\,61 < (0.628\,973\,1\ldots)^2 < 0.395\,607\,286\,3182\,4$$

$(0.6289731\ldots)^2 = 0.395607$ *is correct up to six decimal digits.*

8. Estimate the value of the number $(1.112223333\ldots)^2$.

$$1.112\,223\,333^2 < (1.112\,223\,333\ldots)^2 < 1.112\,223\,334^2$$
$$1.237\,040\,742\,469\,628\,9 < (1.112\,223\,333\ldots)^2 < 1.237\,040\,744\,694\,075\,6$$

$(1.112\,223\,333\ldots)^2 = 1.237\,040\,74$ *is correct up to eight decimal digits.*

9. Which number is a better estimate for π, $\dfrac{22}{7}$ or 3.14? Explain.

    Allow for both answers to be correct as long as the student provides a reasonable explanation.

    A sample answer might be as follows.

    I think that $\dfrac{22}{7}$ is a better estimate because when I find the decimal expansion, $\dfrac{22}{7} \approx 3.142857\ldots$; compared to the number 3.14, $\dfrac{22}{7}$ is closer to the actual value of π.

10. To how many decimal digits can you correctly estimate the value of the number $(4.56789012\ldots)^2$?

$$4.567\,890\,12^2 < (4.567\,890\,12\ldots)^2 < 4.567\,890\,13^2$$
$$20.865\,620\,148\,393\,614\,4 < (1.112\,223\,333\ldots)^2 < 20.865\,620\,239\,751\,416\,9$$

$(4.567\,890\,12\ldots)^2 = 20.865\,620$ *is correct up to six decimal digits.*

10 by 10 Grid
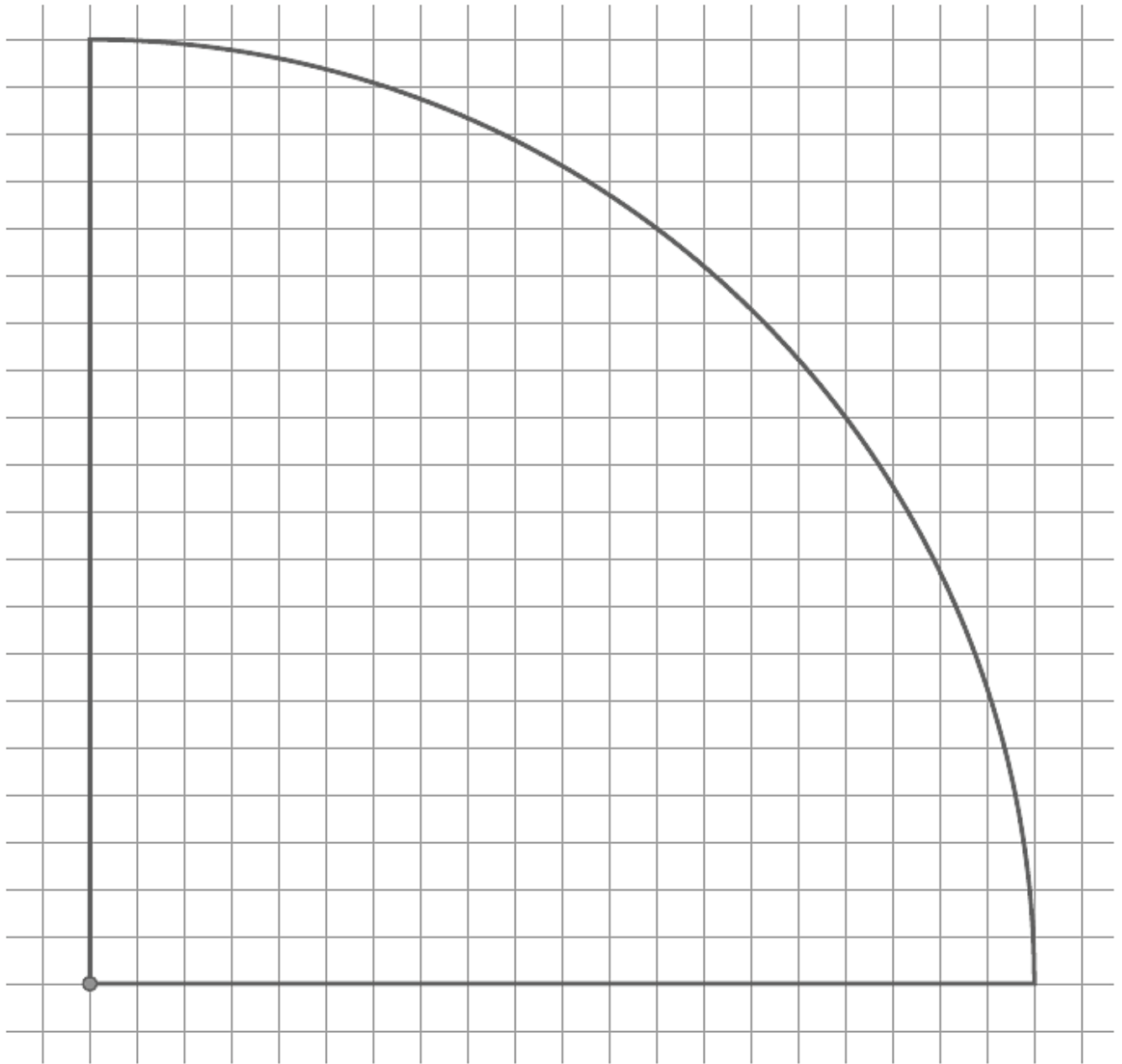
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20 by 20 Grid

Name _____   Date _____

1.

    a.   What is the decimal expansion of the number $\frac{35}{7}$? Is the number $\frac{35}{7}$ rational or irrational? Explain.

    b.   What is the decimal expansion of the number $\frac{4}{33}$? Is the number $\frac{4}{33}$ rational or irrational? Explain.

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2.

   a.   Write $0.\overline{345}$ as a fraction.

   b.   Write $2.8\overline{40}$ as a fraction.

   c.   Brandon stated that 0.66 and $\frac{2}{3}$ are equivalent. Do you agree? Explain why or why not.

d.   Between which two positive integers does $\sqrt{33}$ lie?

e.   For what integer x is $\sqrt{x}$ closest to 5.25? Explain.

3. Identify each of the following numbers as rational or irrational. If the number is irrational, explain how you know.

   a. $\sqrt{29}$

   b. $5.\overline{39}$

   c. $\dfrac{12}{4}$

   d. $\sqrt{36}$

   e. $\sqrt{5}$

   f. $\sqrt[3]{27}$

   g. $\pi = 3.141592\cdots$

   h. Order the numbers in parts (a)–(g) from least to greatest, and place them on a number line.

4. Circle the greater number in each of the pairs (a)–(e) below.

    a. Which is greater, 8 or $\sqrt{60}$?

    b. Which is greater, 4 or $\sqrt{26}$?

    c. Which is greater, $\sqrt[3]{64}$ or $\sqrt{16}$?

    d. Which is greater, $\sqrt[3]{125}$ or $\sqrt{30}$?

    e. Which is greater, -7 or $-\sqrt{42}$?

    f. Put the numbers 9, $\sqrt{52}$, and $\sqrt[3]{216}$ in order from least to greatest. Explain how you know which order to put them in.
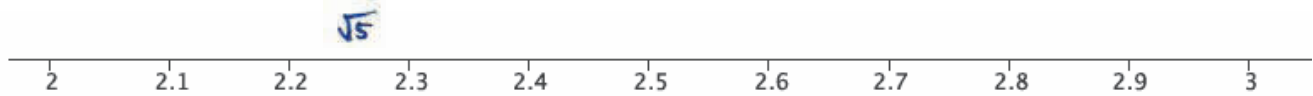
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5.
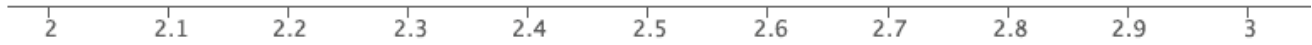
a. Between which two labeled points on the number line would $\sqrt{5}$ be located?

b. Explain how you know where to place $\sqrt{5}$ on the number line.

c. How could you improve the accuracy of your estimate?

6. Determine the positive solution for each of the following equations.

   a. $121 = x^2$

   b. $x^3 = 1000$

   c. $17 + x^2 = 42$

   d. $x^3 + 3x - 9 = x - 1 + 2x$

e.   The cube shown has a volume of 216 cm^3.

   i.   Write an equation that could be used to determine the length, l, of one side.
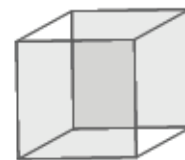
$V = 216$ cm^3

   ii. Solve the equation, and explain how you solved it.

A Progression Toward Mastery					
Assessment Task Item		STEP 1 **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	STEP 2 **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	STEP 3 **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	STEP 4 **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	a–b	Student makes little or no attempt to respond to either part of the problem. OR Student answers both parts incorrectly.	Student identifies one or both numbers as rational. Student may not write the decimal expansions of the numbers and does not reference the decimal expansions of the numbers in her explanation.	Student identifies both numbers as rational. Student correctly writes the decimal expansion of each number. Student may not reference the decimal expansion in his explanation but uses another explanation (e.g., the numbers are quotients of integers).	Student identifies both numbers as rational. Student correctly writes the decimal expansion of $\frac{35}{7}$ as 5.000…, or 5, and $\frac{4}{33}$ as 0.121212…, or $0.\overline{12}$. Student explains that the numbers are rational by stating that every rational number has a decimal expansion that repeats eventually. Student references the decimal expansion of $\frac{35}{7}$ with the repeating decimal of zero and the decimal expansion of $\frac{4}{33}$ with the repeating decimal of 12.

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2	a–b	Student does not attempt the problem or writes answers that are incorrect for both parts.	Student is able to write one of the parts (a)–(b) correctly as a fraction. OR Student answers both parts incorrectly but shows some evidence of understanding how to convert an infinite, repeating decimal to a fraction.	Student is able to write both parts (a)–(b) correctly as fractions. OR Student writes one part correctly but makes computational errors leading to an incorrect answer for the other part.	Student correctly writes both parts (a)–(b) as fractions. Part (a) is written as $0.\overline{345} = \frac{115}{333}$ (or equivalent), and part (b) is written as $2.8\overline{40} = \frac{2812}{990}$ (or equivalent).
	c	Student agrees with Brandon or writes an explanation unrelated to the problem.	Student does not agree with Brandon. Student writes a weak explanation defending his position.	Student does not agree with Brandon. Student writes an explanation that shows why the equivalence was incorrect, reasoning that 0.66 does not equal $\frac{2}{3}$ or that $\frac{2}{3}$ does not equal 0.66, but fails to include both explanations.	Student does not agree with Brandon. Student writes an explanation that shows 0.66 does not equal $\frac{2}{3}$ and that $\frac{2}{3}$ does not equal 0.66.
	d–e	Student does not attempt the problem or writes answers that are incorrect for both parts (d)–(e).	Student is able to answer at least one of the parts (d)–(e) correctly. Student may or may not provide a weak justification for answer selection.	Student is able to answer both parts (d)–(e) correctly. Student may or may not provide a justification for answer selection. The explanation includes some evidence of mathematical reasoning.	Student correctly answers both parts (d)–(e); for part (d), $\sqrt{33}$ is between positive integers 5 and 6; for part (e), $x \approx 28$. Student provides an explanation that includes solid reasoning related to rational approximation.
3	a–f	Student does not attempt the problem or writes correct answers for one or two parts of (a)–(g).	Student correctly identifies three or four parts of (a)–(g) as rational or irrational. Student may or may not provide a weak explanation for those numbers that are irrational.	Student correctly identifies five or six parts of (a)–(g) correctly as rational or irrational. Student may provide an explanation for those numbers that are irrational but does not refer to their decimal expansion or any other mathematical reason.	Student correctly identifies all seven parts of (a)–(g); (a) irrational, (b) rational, (c) rational, (d) rational, (e) irrational, (f) rational, and (g) irrational. Student explains parts (a), (e), and (g) as irrational by referring to their decimal expansion or the fact that the radicand was not a perfect square.

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Module 7: Introduction to Irrational Numbers Using Geometry

	h	Student correctly places zero to two numbers correctly on the number line.	Student correctly places three or four of the numbers on the number line.	Student correctly places five of the six numbers on the number line.	Student correctly places all six numbers on the number line. (Correct answers are noted in blue below.)
4	**a–e**	Student correctly identifies the larger number zero to one time in parts (a)–(e).	Student correctly identifies the larger number two to three times in parts (a)–(e).	Student correctly identifies the larger number four times in parts (a)–(e).	Student correctly identifies the larger number in all of parts (a)–(e); (a) 8, (b) $\sqrt{26}$, (c) numbers are equal, (d) $\sqrt{30}$, and (e) $-\sqrt{42}$.
	f	Student does not attempt the problem or responds incorrectly. Student does not provide an explanation.	Student may correctly order the numbers from greatest to least. Student may or may not provide a weak explanation for how he put the numbers in order.	Student correctly orders the numbers from least to greatest. Student provides a weak explanation for how she put the numbers in order.	Student correctly orders the numbers from least to greatest: $\sqrt[3]{216}$, $\sqrt{52}$, 9. Explanation includes correct mathematical vocabulary (e.g., square root, cube root, between perfect squares).
5	**a–c**	Student makes little or no attempt to do the problem. OR Student may or may not place $\sqrt{5}$ correctly on the number line for part (a). For parts (b)–(c), student does not provide an explanation.	Student may or may not have placed $\sqrt{5}$ correctly on the number line for part (a). For parts (b)–(c), student may or may not provide a weak explanation for how the number was placed. Student may or may not provide a weak explanation for how to improve accuracy of approximation.	Student places $\sqrt{5}$ correctly on the number line for part (a). For parts (b)–(c), student may provide a weak explanation for how the number was placed. Student may provide a weak explanation for how to improve accuracy of approximation. Student references the method of rational approximation.	Student correctly places $\sqrt{5}$ between 2.2 and 2.3 on the number line for part (a). For parts (b)–(c), student explains the method of rational approximation to locate the approximate position of $\sqrt{5}$ on the number line. Student explains how to continue the rational approximation to include increasing smaller intervals to improve the accuracy of the estimate.

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6	a–b	Student makes little or no attempt to solve either equation or writes incorrect answers for both.	Student writes the correct answer for one of the equations but does not write the answer in the appropriate form (i.e., 11 or 10 instead of $x = 11$ or $x = 10$).	Student solves at least one of the equations correctly or solves both correctly but does not write the answer in the appropriate form (i.e., 11 and 10 instead of $x = 11$ and $x = 10$).	Student solves both equations correctly for parts (a)–(b); (a) $x = 11$, and (b) $x = 10$.
	c–d	Student makes little or no attempt to solve either equation or writes incorrect answers for both.	Student may solve one equation correctly. OR Student uses properties of rational numbers to transform the equations but cannot determine the correct value of x or makes computational errors leading to incorrect solutions for x.	Student solves one of the equations correctly but makes computational errors leading to an incorrect answer for the other equation.	Student solves both equations correctly for parts (c)–(d); (c) $x = 5$, and (d) $x = 2$.
	e	Student makes little or no attempt to solve the problem.	Student may state that the length of one side of the cube is 6 cm but does not write an equation or solve it.	Student states that the length of one side of the cube is 6 cm. Student correctly writes and solves an equation but provides a weak explanation for solving it.	Student states that the length of one side of the cube is 6 cm. Student correctly writes and solves the equation. Student provides a clear and complete explanation for solving the equation that includes some reference to cube roots and why $\sqrt[3]{216} = 6$.

Name _____   Date _____

1.

    a.   What is the decimal expansion of the number $\frac{35}{7}$? Is the number $\frac{35}{7}$ rational or irrational? Explain.

$$\frac{35}{7} = 5.000\ldots$$

The number $\frac{35}{7}$ is a rational number. Rational numbers have decimal expansions that repeat. In this case, the decimal that repeats is a zero.

    b.   What is the decimal expansion of the number $\frac{4}{33}$? Is the number $\frac{4}{33}$ rational or irrational? Explain.

$$
\begin{array}{r}
0.1212 \\
33\overline{)4.00000} \\
33 \\
\hline
70 \\
66 \\
\hline
40 \\
33 \\
\hline
70 \\
66 \\
\hline
4
\end{array}
$$

$$\frac{4}{33} = 0.1212\ldots = 0.\overline{12}$$

The number $\frac{4}{33}$ is rational. Numbers that have decimal expansions that repeat. The digits 12 repeat in the decimal expansion of $\frac{4}{33}$. So, $\frac{4}{33}$ is rational.

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2.

a. Write $0.\overline{345}$ as a fraction.

Let x be $0.\overline{345}$.

$$10^3 x = 10^3 (0.\overline{345})$$
$$1000x = 1000(0.\overline{345})$$
$$1000x = 345.\overline{345}$$
$$1000x = 345 + x$$
$$1000x - x = 345 + x - x$$
$$999x = 345$$
$$x = \frac{345}{999} = \frac{115}{333}$$

b. Write $2.8\overline{40}$ as a fraction.

Let x be $2.8\overline{40}$.

$$10x = 28.\overline{40}$$
$$10x = 28 + 0.\overline{40}$$
$$10x = 28 + \frac{40}{99}$$
$$10x = \frac{(28)(99) + 40}{99}$$
$$x = \frac{2812}{99} \left(\frac{1}{10}\right)$$
$$x = \frac{2812}{990} = \frac{1406}{495}$$

Let y be $0.\overline{40}$.

$$10^2 y = 10^2 (0.\overline{40})$$
$$100y = 40.\overline{40}$$
$$100y = 40 + y$$
$$100y - y = 40 + y - y$$
$$99y = 40$$
$$y = \frac{40}{99}$$

$$2.8\overline{40} = \frac{2812}{990} = \frac{1406}{495}$$

c. Brandon stated that 0.66 and $\frac{2}{3}$ are equivalent. Do you agree? Explain why or why not.

No, I do not agree with Brandon. The decimal 0.66 is equal to the fraction $\frac{66}{100} = \frac{33}{50}$, not $\frac{2}{3}$. Also, the number $\frac{2}{3}$ is equal to the infinite decimal $0.\overline{6}$. The number 0.66 is a finite decimal. Therefore, 0.66 and $\frac{2}{3}$ are not equivalent.

d. Between which two positive integers does $\sqrt{33}$ lie?

The number $\sqrt{33}$ is between 5 and 6 because
$5^2 < (\sqrt{33})^2 < 6^2$.

e. For what integer x is $\sqrt{x}$ closest to 5.25? Explain.

$(5.25)^2 = 27.5625$

Since $\sqrt{x}$ is the square root of x, then x^2 will give me the integer that belongs in the square root.
$(5.25)^2 = 27.5625$, which is closest to the integer 28.

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3. Identify each of the following numbers as rational or irrational. If the number is irrational, explain how you know.

   a. $\sqrt{29}$ Irrational because 29 is not a perfect square and $\sqrt{29}$ has an infinite decimal expansion that does not repeat.

   b. $5.\overline{39}$ Rational

   c. $\dfrac{12}{4}$ Rational

   d. $\sqrt{36}$ Rational

   e. $\sqrt{5}$ Irrational because 5 is not a perfect square and $\sqrt{5}$ has an infinite decimal expansion that does not repeat.

   f. $\sqrt[3]{27}$ Rational

   g. $\pi = 3.141592\cdots$ Irrational because pi has a decimal expansion that does not repeat.

   h. Order the numbers in parts (a)–(g) from least to greatest, and place them on a number line.

   $\sqrt{29}$: $5^2 < (\sqrt{29})^2 < 6^2$, $5.3^2 < (\sqrt{29})^2 < 5.4^2$, $5.38^2 < (\sqrt{29})^2 < 5.39^2$
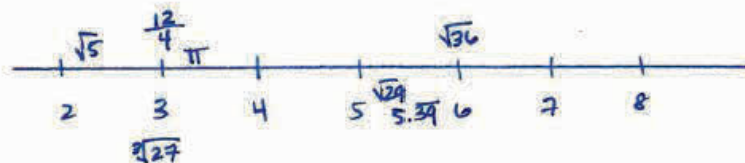
4. Circle the greater number in each of the pairs (a)–(e) below.

a. Which is greater? ⑧ or $\sqrt{60}$

b. Which is greater? 4 or $\boxed{\sqrt{26}}$

c. Which is greater? $\sqrt[3]{64}$ or $\sqrt{16}$

   The numbers are equal: $\sqrt[3]{64} = 4$, $\sqrt{16} = 4$.

d. Which is greater? $\sqrt[3]{125}$ or $\boxed{\sqrt{30}}$

e. Which is greater? -7 or $\boxed{-\sqrt{42}}$

f. Put the numbers 9, $\sqrt{52}$, and $\sqrt[3]{216}$ in order from least to greatest. Explain how you know which order to put them in.

   $\sqrt{52}$ is between 7 and 8.

   $\sqrt[3]{216} = 6$

   In order from least to greatest:

   $\sqrt[3]{216}$, $\sqrt{52}$, 9

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5.

a. Between which two labeled points on the number line would $\sqrt{5}$ be located?

The number $\sqrt{5}$ is between 2.2 and 2.3.

b. Explain how you know where to place $\sqrt{5}$ on the number line.

I knew that $\sqrt{5}$ was between 2 and 3 but closer to 2. So next, I checked intervals of tenths beginning with 2.0 to 2.1. The interval that $\sqrt{5}$ fit between was 2.2 and 2.3. because
$$2.2^2 < (\sqrt{5})^2 < 2.3^2, \quad 4.84 < 5 < 5.29.$$

c. How could you improve the accuracy of your estimate?

To improve the estimate, I would have to continue with the method of rational approximation to determine which interval of hundredths $\sqrt{5}$ fits between. Once I knew the interval of hundredths, I would check the interval of thousandths, and so on.

$$\sqrt{5}: \quad 2^2 < (\sqrt{5})^2 < 3^2, \quad 2.2^2 < (\sqrt{5})^2 < 2.3^2$$
$$4 < 5 \quad < 9 \qquad 4.84 < 5 < 5.29$$

6. Determine the positive solution for each of the following equations.

a. $121 = x^2$

$\sqrt{121} = \sqrt{x^2}$ $11^2 = 121$

$11 = x$ $121 = 121$

b. $x^3 = 1000$

$\sqrt[3]{x^3} = \sqrt[3]{1000}$ $10^3 = 1000$

$x = 10$ $1000 = 1000$

c. $17 + x^2 = 42$

$17 - 17 + x^2 = 42 - 17$ $17 + 5^2 = 42$

$x^2 = 25$ $17 + 25 = 42$

$\sqrt{x^2} = \sqrt{25}$ $42 = 42$

$x = 5$

d. $x^3 + 3x - 9 = x - 1 + 2x$

$x^3 + 3x - 3x - 9 = x - 1 + 2x - 3x$ $2^3 + (3)(2) - 9 = 2 - 1 + (2)(2)$

$x^3 - 9 = -1$ $8 + 6 - 9 = 2 - 1 + 4$

$x^3 - 9 + 9 = -1 + 9$ $14 - 9 = 1 + 4$

$x^3 = 8$ $5 = 5$

$\sqrt[3]{x^3} = \sqrt[3]{8}$

$x = 2$

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e.   The cube shown has a volume of 216 cm^3.

   i.   Write an equation that could be used to determine the length, l, of one side.

$$V = l^3$$
$$216 = l^3$$

$V = 216$ cm^3

   ii.   Solve the equation, and explain how you solved it.

$$216 = l^3$$
$$\sqrt[3]{216} = \sqrt[3]{l^3}$$
$$6 = l$$

The length of one side is 6cm.

To solve the equation, I had to take the cube root of both sides of the equation. The cube root of l^3, $\sqrt[3]{l^3}$, is l. The cube root of 216, $\sqrt[3]{216}$, is 6 because $6^3 = 216$. Therefore, the length of one side of the cube is 6cm.

8 GRADE

Mathematics Curriculum

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Topic C

The Pythagorean Theorem

Focus Standards:	■	Explain a proof of the Pythagorean Theorem and its converse.
	■	Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
	■	Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.
Instructional Days:	4	
	Lesson 15:	Pythagorean Theorem, Revisited (S)[1]
	Lesson 16:	Converse of the Pythagorean Theorem (S)
	Lesson 17:	Distance on the Coordinate Plane (P)
	Lesson 18:	Applications of the Pythagorean Theorem (E)

In Lesson 15, students engage with another proof of the Pythagorean theorem. This time, students compare the areas of squares that are constructed along each side of a right triangle in conjunction with what they know about similar triangles. Now that students know about square roots, students can determine the approximate length of an unknown side of a right triangle even when the length is not a whole number. Lesson 16 shows students another proof of the converse of the Pythagorean theorem based on the notion of congruence. Students practice explaining proofs in their own words in Lessons 15 and 16 and apply the converse of the theorem to make informal arguments about triangles as right triangles. Lesson 17 focuses on the application of the Pythagorean theorem to calculate the distance between two points on the coordinate plane. Lesson 18 gives students practice applying the Pythagorean theorem in a variety of mathematical and real-world scenarios. Students determine the height of isosceles triangles, determine the length of the diagonal of a rectangle, and compare lengths of paths around a right triangle.

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[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

![chat icon] **Lesson 15: Pythagorean Theorem, Revisited**

Student Outcomes

- Students use similar triangles to develop another proof of the Pythagorean theorem and explore a geometric consequence of this proof.
- Students explain a proof of the Pythagorean theorem.

Lesson Notes

The purpose of this lesson is for students to review and practice presenting the proof of the Pythagorean theorem using similar triangles. Then, students examine a geometric consequence of this proof.

Classwork

Discussion (20 minutes)

This discussion is an opportunity for students to practice explaining a proof of the Pythagorean theorem using similar triangles. Instead of leading the discussion, consider posing the questions, one at a time, to small groups of students and allowing time for discussions. Then, have select students share their reasoning while others critique.

- To prove the Pythagorean theorem, $a^2 + b^2 = c^2$, use a right triangle, shown below. Begin by drawing a segment from the right angle, perpendicular to side AB through point C. Label the intersection of the segments point D. $\overline{CD}$ is an altitude.

Proof of the Pythagorean Theorem
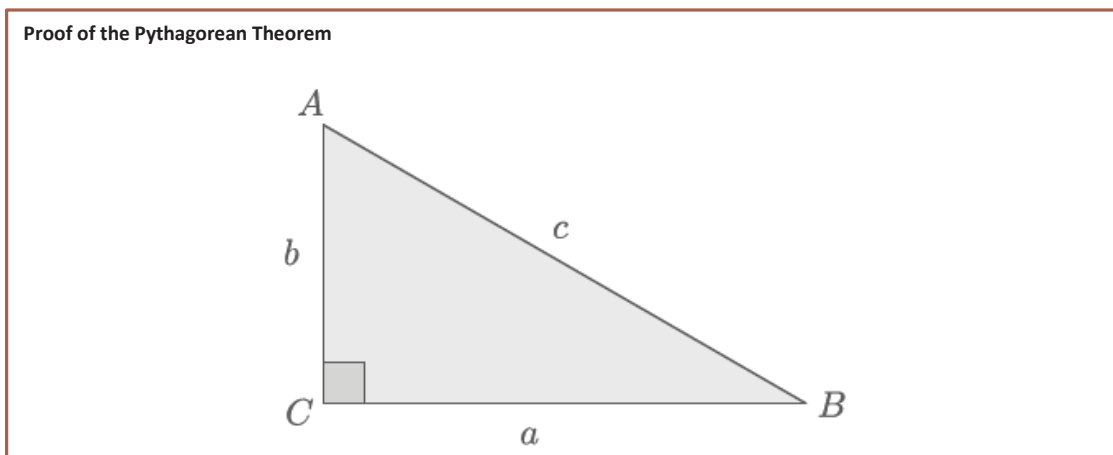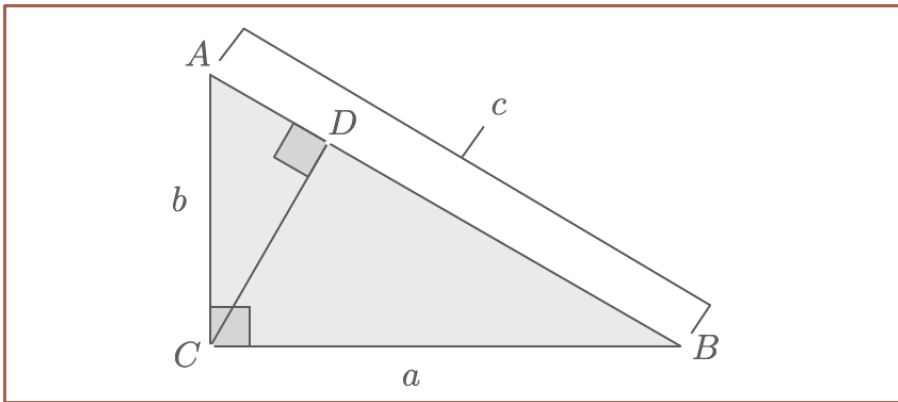
- Using one right triangle, we created 3 right triangles. Name those triangles.
  - *The three triangles are △ ABC, △ ACD, and △ BCD.*
- We can use our basic rigid motions to reorient the triangles so they are easier to compare, as shown below.
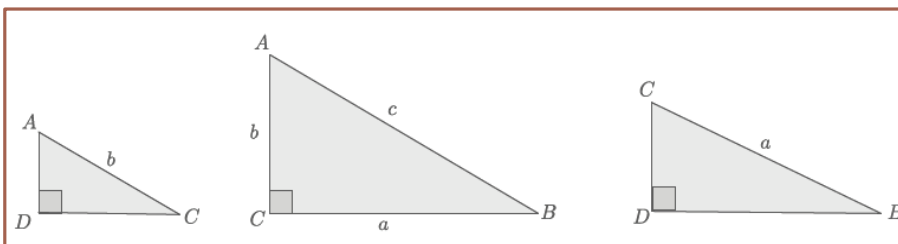
- The next step is to show that these triangles are similar. Begin by showing that △ ADC ~ △ ACB. Discuss in your group.
  - *△ ADC and △ ACB are similar because they each have a right angle, and they each share ∠A. Then, by the AA criterion for similarity, △ ADC ~ △ ACB.*
- Now, show that △ ACB ~ △ CDB. Discuss in your group.
  - *△ ACB ~ △ CDB because they each have a right angle, and they each share ∠B. Then, by the AA criterion for similarity, △ ACB ~ △ CDB.*
- Are △ ADC and △ CDB similar? Discuss in your group.
  - *We know that similarity has the property of transitivity; therefore, since △ ADC ~ △ ACB, and △ ACB ~ △ CDB, then △ ADC ~ △ CDB.*

Scaffolding:
- A good hands-on visual that can be used here requires a 3×5 notecard. Have students draw the diagonal and then draw the perpendicular line from C to side AB.
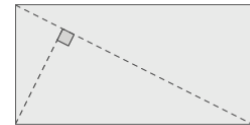
- Make sure students label all of the parts to match the triangle to the left. Next, have students cut out the three triangles. Students then have a notecard version of the three triangles shown and can use them to demonstrate the similarity among them.
- The next scaffolding box shows a similar diagram for the concrete case of a 6-8-10 triangle.

Scaffolding:
Also consider showing a concrete example, such as a 6-8-10 triangle, along with the general proof.
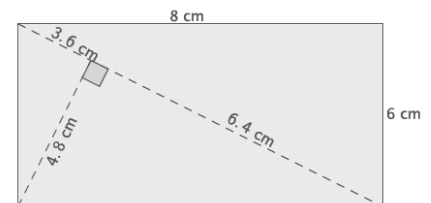
Have students verify similarity using a protractor to compare corresponding angle measures. There is a reproducible available at the end of the lesson.

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- Let's identify the segments that comprise side c as follows: $|AD| = x$ and $|BD| = y$. (Ensure that students note x and y in their student materials.) Using this notation, we see that side c is equal to the sum of the lengths x and y (i.e., $x + y = c$).
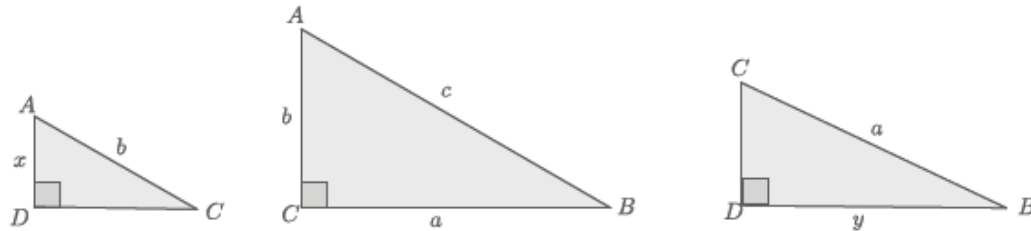
- If we consider $\triangle ADC$ and $\triangle ACB$, we can write a statement about corresponding sides being equal in a ratio that helps us reach our goal of showing $a^2 + b^2 = c^2$. Discuss in your group.

  □ *Using $\triangle ADC$ and $\triangle ACB$, we can write*

  $$\frac{x}{b} = \frac{b}{c}.$$

- Now solve the equation for x.

  $$x = \frac{b^2}{c}$$

- Using $\triangle ACB$ and $\triangle CDB$ gives us another piece that we need. Discuss in your group.

  □ *Using $\triangle ACB$ and $\triangle CDB$, we can write*

  $$\frac{a}{y} = \frac{c}{a}.$$

- Now solve the equation for y.

  $$\frac{a^2}{c} = y$$

- We know that $x + y = c$, and we just found expressions equal to x and y. Use this information to show that $a^2 + b^2 = c^2$. Discuss in your group.

  □ *By substituting $\dfrac{b^2}{c}$ for x and $\dfrac{a^2}{c}$ for y in c = x + y, we have*

  $$\frac{b^2}{c} + \frac{a^2}{c} = c.$$

  Multiplying through by c we have

  $$b^2 + a^2 = c^2.$$

  By the commutative property of addition we can rewrite the left side as

  $$a^2 + b^2 = c^2.$$

Discussion (15 minutes)

- Now, let's apply this knowledge to explore a geometric consequence of the proof we just completed. We begin with the right triangle with the altitude drawn as before.
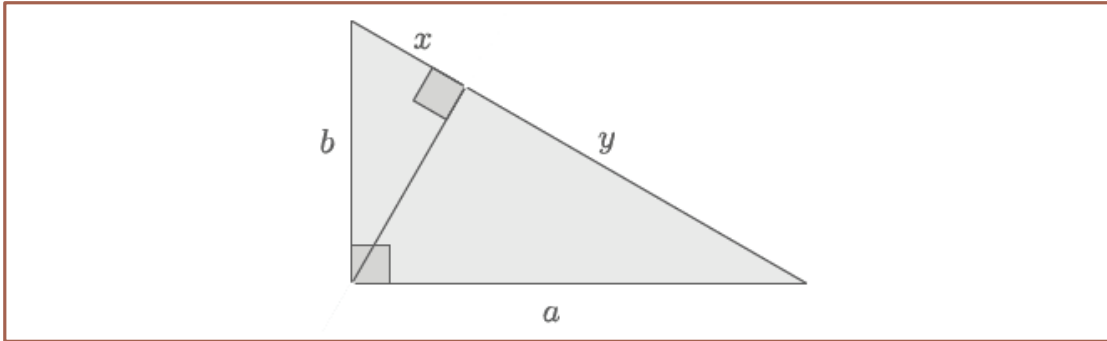
- Let's draw three squares on the right triangle. Notice that we can use the altitude to divide the large square, of area c^2, into two rectangles as shown. Call them rectangle I and rectangle II.
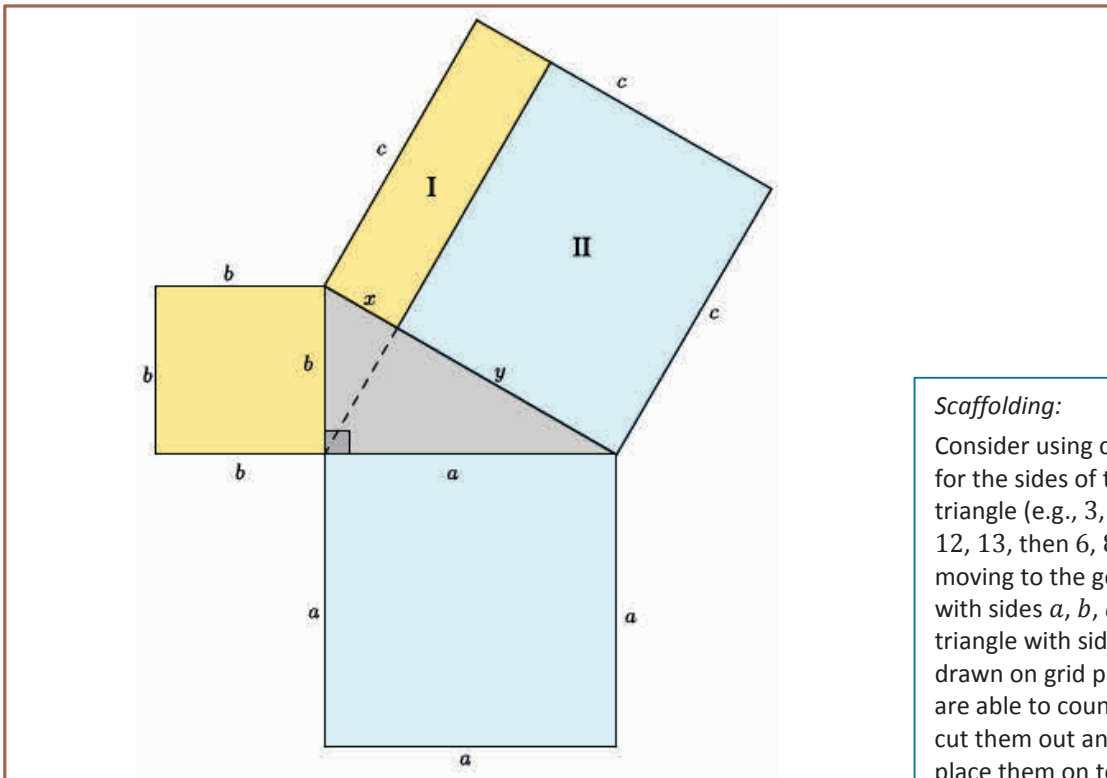
Scaffolding:

Consider using concrete values for the sides of the right triangle (e.g., 3, 4, 5, then 5, 12, 13, then 6, 8, 10) and then moving to the general triangle with sides a, b, c. Given a triangle with sides 3, 4, 5 drawn on grid paper, students are able to count squares or cut them out and physically place them on top of the larger square to compare the areas. An example of this is shown at the end of the lesson.

- What would it mean, geometrically, for $a^2 + b^2$ to equal c^2?
    - *It means that the sum of the areas of a^2 and b^2 is equal to the area c^2.*

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There are two possible ways to continue; one way is by examining special cases on grid paper, as mentioned in the scaffolding box on the previous page, and showing the relationship between the squares physically. The other way is by using the algebraic proof of the general case that continues below.

- What is the area of rectangle I?

  □   *The area of rectangle I is xc.*

- This is where the proof using similar triangles just completed is helpful. We said that $x = \dfrac{b^2}{c}$. Therefore, the area of rectangle I is

$$xc = \frac{b^2}{c} \cdot c = b^2.$$

- Now use similar reasoning to determine the area of rectangle II.

  □   *The area of rectangle II is yc. When we substitute $\dfrac{a^2}{c}$ for y we get $yc = \dfrac{a^2}{c} \cdot c = a^2$.*

- Explain how the work thus far shows that the Pythagorean theorem is true.

  □   *The Pythagorean theorem states that given a right triangle with lengths a, b, c, then $a^2 + b^2 = c^2$. The diagram shows that the area of the rectangles drawn off of side c have a sum of $a^2 + b^2$. The square constructed off of side c clearly has an area of c^2. Thus, the diagram shows that the areas $a^2 + b^2$ are equal to the area of c^2, which is exactly what the theorem states.*

To solidify student understanding of the proof, consider showing students the six-minute video located at http://www.youtube.com/watch?v=QCyvxYLFSfU. If multiple computers or tablets are available, have small groups of students watch the video together so they can pause and replay parts of the proof as necessary.
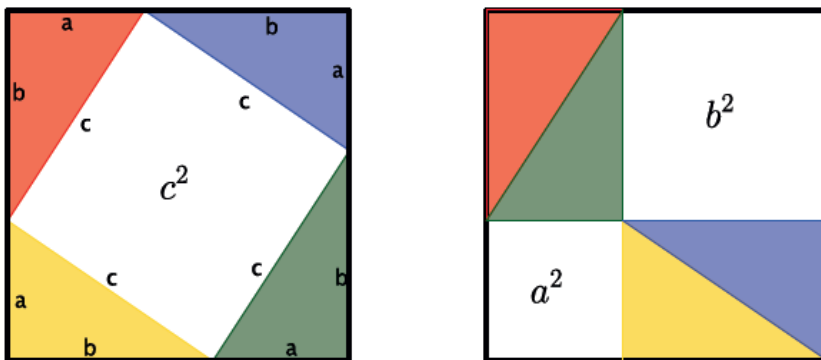
$$c^2 = a^2 + b^2$$

Scaffolding:

The geometric illustration of the proof, shown to the left, can be used as further support or as an extension to the claim that the sum of the areas of the smaller squares is equal to the area of the larger square.

Another short video that demonstrates $a^2 + b^2 = c^2$ using area is at the following link: http://9gag.com/gag/aOqPoMD/cool-demonstration-of-the-pythagorean-theorem.

It at least verifies the Pythagorean theorem for the squares drawn on the sides of one particular right triangle.

Closing (5 minutes)

The altitude of a right triangle drawn from the vertex with the right angle meeting the hypotenuse at some point divides the length of the hypotenuse into two sections. Consider having students explain what the lengths of those two segments would be for a right triangle with side lengths of $9, 40,$ and 41 units.

Summarize, or ask students to summarize, the main points from the lesson.

- We know a similarity proof of the Pythagorean theorem.
- The proof has a geometric consequence of showing how to divide the large square drawn on the side of a right triangle into two pieces with areas matching the areas of the two smaller squares drawn on the sides of the right triangle.

> **Lesson Summary**
>
> **The Pythagorean theorem can be proven by drawing an altitude in the given right triangle and identifying three similar triangles. We can see geometrically how the large square drawn on the hypotenuse of the triangle has an area summing to the areas of the two smaller squares drawn on the legs of the right triangle.**

Exit Ticket (5 minutes)

Name _____ Date _____

Lesson 15: Pythagorean Theorem, Revisited

Exit Ticket

Explain a proof of the Pythagorean theorem in your own words. Use diagrams and concrete examples, as necessary, to support your explanation.

Exit Ticket Sample Solutions

> **Explain a proof of the Pythagorean theorem in your own words. Use diagrams and concrete examples, as necessary, to support your explanation.**
>
> *Proofs will vary. The critical parts of the proof that demonstrate proficiency include an explanation of the similar triangles △ ADC, △ ACB, and △ CDB, including a statement about the ratio of their corresponding sides being equal, leading to the conclusion of the proof.*

Problem Set Sample Solutions

Students apply the concept of similar figures to show the Pythagorean theorem is true.

> 1. **For the right triangle shown below, identify and use similar triangles to illustrate the Pythagorean theorem.**
>
>
>
> *First, I must draw a segment that is perpendicular to side AB that goes through point C. The point of intersection of that segment and side AB will be marked as point D.*
>
>
>
> *Then, I have three similar triangles, △ ABC, △ CBD, and △ ACD, as shown below.*
>
>
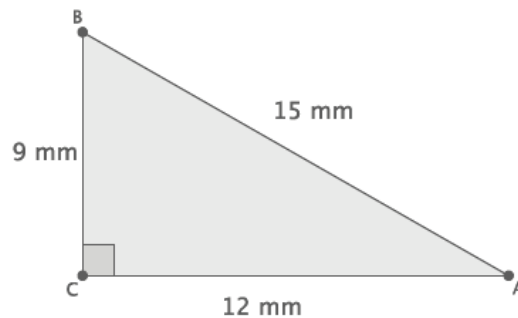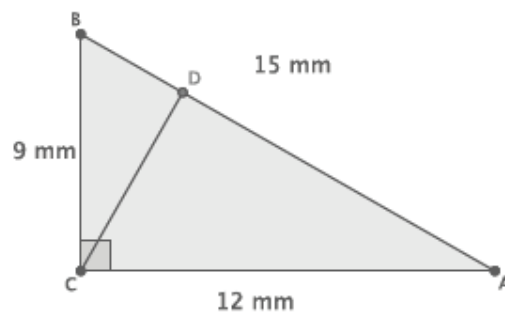
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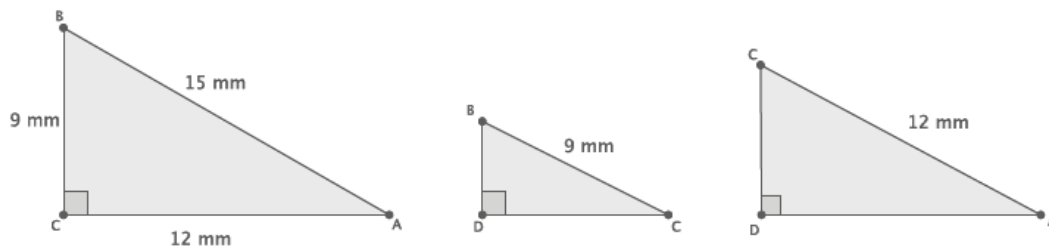
△ ABC and △ CBD are similar because each one has a right angle, and they both share ∠B. By AA criterion, △ ABC ~ △ CBD. △ ABC and △ ACD are similar because each one has a right angle, and they both share ∠A. By AA criterion, △ ABC ~ △ ACD. By the transitive property, we also know that △ ACD ~ △ CBD.

Since the triangles are similar, they have corresponding sides that are equal in ratio. For △ ABC and △ CBD,

$$\frac{9}{15} = \frac{|BD|}{9},$$

which is the same as $9^2 = 15(|BD|)$.

For △ ABC and △ ACD,

$$\frac{12}{15} = \frac{|AD|}{12},$$

which is the same as $12^2 = 15(|AD|)$.

Adding these two equations together I get

$$9^2 + 12^2 = 15(|BD|) + 15(|AD|).$$

By the distributive property,

$9^2 + 12^2 = 15(|BD| + |AD|)$;

however, $|BD| + |AD| = |AC| = 15$. Therefore,

$$9^2 + 12^2 = 15(15)$$
$$9^2 + 12^2 = 15^2.$$

2. **For the right triangle shown below, identify and use squares formed by the sides of the triangle to illustrate the Pythagorean theorem.**
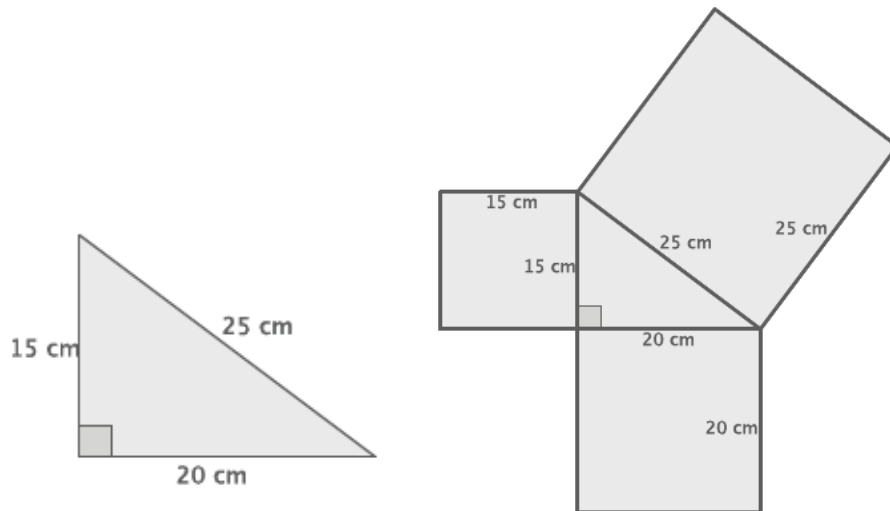
The sum of the areas of the smallest squares is $15^2\text{cm}^2 + 20^2\text{cm}^2 = 625 \text{ cm}^2$. The area of the largest square is $25^2\text{cm}^2 = 625 \text{ cm}^2$. The sum of the areas of the squares off of the legs is equal to the area of the square off of the hypotenuse; therefore, $a^2 + b^2 = c^2$.

3. Reese claimed that any figure can be drawn off the sides of a right triangle and that as long as they are similar figures, then the sum of the areas off of the legs will equal the area off of the hypotenuse. She drew the diagram by constructing rectangles off of each side of a known right triangle. Is Reese's claim correct for this example? In order to prove or disprove Reese's claim, you must first show that the rectangles are similar. If they are, then you can use computations to show that the sum of the areas of the figures off of the sides a and b equals the area of the figure off of side c.
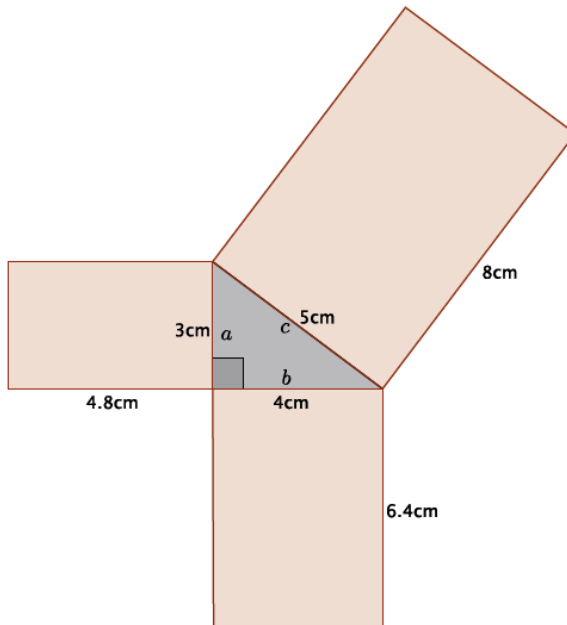
The rectangles are similar because their corresponding side lengths are equal in scale factor. That is, if we compare the longest side of the rectangle to the side with the same length as the right triangle sides, we get the ratios

$$\frac{4.8}{3} = \frac{6.4}{4} = \frac{8}{5} = 1.6.$$

Since the corresponding sides were all equal to the same constant, then we know we have similar rectangles. The areas of the smaller rectangles are 14.4 cm^2 and 25.6 cm^2, and the area of the largest rectangle is 40 cm^2. The sum of the smaller areas is equal to the larger area:

$$14.4 + 25.6 = 40$$
$$40 = 40.$$

Therefore, we have shown that the sum of the areas of the two smaller rectangles is equal to the area of the larger rectangle, and Reese's claim is correct.

4. After learning the proof of the Pythagorean theorem using areas of squares, Joseph got really excited and tried explaining it to his younger brother. He realized during his explanation that he had done something wrong. Help Joseph find his error. Explain what he did wrong.
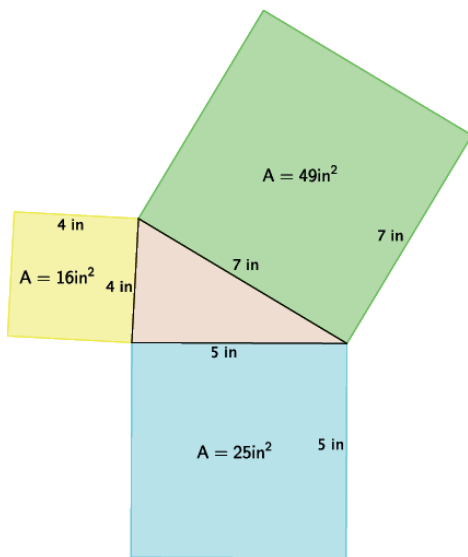
Based on the proof shown in class, we would expect the sum of the two smaller areas to be equal to the sum of the larger area (i.e., $16 + 25$ should equal 49). However, $16 + 25 = 41$. Joseph correctly calculated the areas of each square, so that was not his mistake. His mistake was claiming that a triangle with side lengths of 4, 5, and 7 was a right triangle. We know that the Pythagorean theorem only works for right triangles. Considering the converse of the Pythagorean theorem, when we use the given side lengths, we do not get a true statement.

$$4^2 + 5^2 = 7^2$$
$$16 + 25 = 49$$
$$41 \neq 49$$

Therefore, the triangle Joseph began with is not a right triangle, so it makes sense that the areas of the squares were not adding up like we expected.

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5. Draw a right triangle with squares constructed off of each side that Joseph can use the next time he wants to show his younger brother the proof of the Pythagorean theorem.

   Answers will vary. Verify that students begin, in fact, with a right triangle and do not make the same mistake that Joseph did. Consider having students share their drawings and explanations of the proof in future class meetings.

6. Explain the meaning of the Pythagorean theorem in your own words.

   If a triangle is a right triangle, then the sum of the squares of the legs will be equal to the square of the hypotenuse. Specifically, if the leg lengths are a and b, and the hypotenuse is length c, then for right triangles $a^2 + b^2 = c^2$.

7. Draw a diagram that shows an example illustrating the Pythagorean theorem.

   Diagrams will vary. Verify that students draw a right triangle with side lengths that satisfy the Pythagorean theorem.

Diagrams referenced in scaffolding boxes can be reproduced for student use.
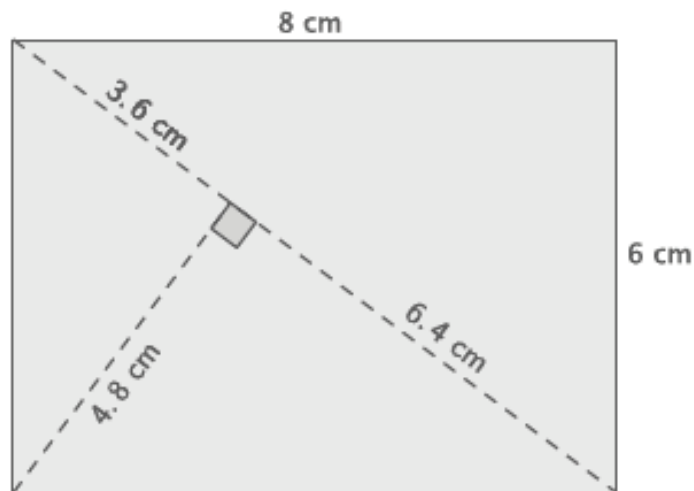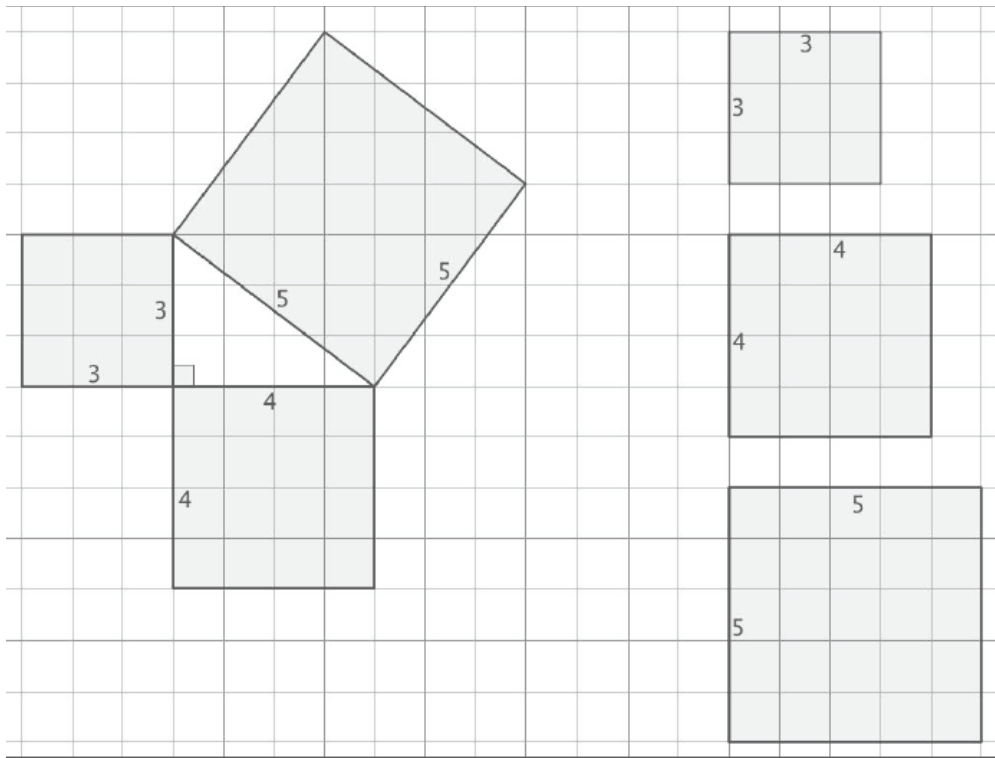
Lesson 16: Converse of the Pythagorean Theorem

Student Outcomes

- Students explain a proof of the converse of the Pythagorean theorem.
- Students apply the theorem and its converse to solve problems.

Lesson Notes

Students had their first experience with the converse of the Pythagorean theorem in Module 3 Lesson 14. In that lesson, students learned the proof of the converse by contradiction. That is, students were asked to draw a triangle with sides a, b, c, where the angle between sides a and b is different from $90°$. The proof using the Pythagorean theorem led students to an expression that was not possible; that is, two times a length was equal to zero. This contradiction meant that the angle between sides a and b was in fact $90°$. In this lesson, students are given two triangles with base and height dimensions of a and b. They are told that one of the triangles is a right triangle and has lengths that satisfy the Pythagorean theorem. Students must use computation and their understanding of the basic rigid motions to show that the triangle with an unmarked angle is also a right triangle. The proof is subtle, so it is important from the beginning that students understand the differences between the triangles used in the discussion of the proof of the converse.

Classwork

Discussion (20 minutes)

- So far you have seen three different proofs of the Pythagorean theorem:

  THEOREM: *If the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then* $a^2 + b^2 = c^2$.

Provide students time to explain to a partner a proof of the Pythagorean theorem. Allow them to choose any one of the three proofs they have seen. Remind them of the proof from Module 2 that was based on congruent triangles, knowledge about angle sum of a triangle, and angles on a line. Also remind them of the proof from Module 3 that was based on their knowledge of similar triangles and corresponding sides being equal in ratio. Select students to share their proofs with the class. Encourage other students to critique the reasoning of the student providing the proof.

- What do you recall about the meaning of the word converse?

Consider pointing out the hypothesis and conclusion of the Pythagorean theorem and then asking students to describe the converse in those terms.

  ▫ *The converse is when the hypothesis and conclusion of a theorem are reversed.*

> *Scaffolding:*
>
> Provide students samples of converses (and note that converses are not always true):
>
> - If a whole number has a final digit of zero, then it is divisible by 10. Converse: If it is divisible by 10, then its final digit is zero.
> - If it is raining, I will study inside the house. Converse: If I study inside the house, it is raining.

- You have also seen one proof of the converse:
  - *If the lengths of three sides of a triangle a, b, and c satisfy $c^2 = a^2 + b^2$, then the triangle is a right triangle, and furthermore, the side of length c is opposite the right angle.*

- The following is another proof of the converse. Assume we are given a triangle ABC so that the sides a, b, and c satisfy $c^2 = a^2 + b^2$. We want to show that $\angle ACB$ is a right angle. To do so, we construct a right triangle $A'B'C'$ with leg lengths of a and b and right angle $\angle A'C'B'$.

> **Proof of the Converse of the Pythagorean Theorem**
>
>
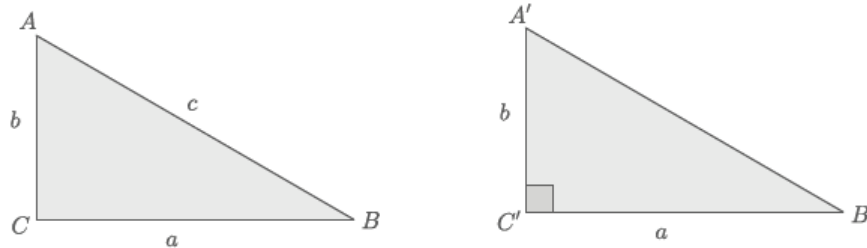
- What do we know or not know about each of these triangles?
  - *In the first triangle, ABC, we know that $a^2 + b^2 = c^2$. We do not know if angle C is a right angle. In the second triangle, $A'B'C'$, we know that it is a right triangle.*

- What conclusions can we draw from this?
  - *By applying the Pythagorean theorem to $\triangle A'B'C'$, we get $|A'B'|^2 = a^2 + b^2$. Since we are given $c^2 = a^2 + b^2$, then by substitution, $|A'B'|^2 = c^2$, and then $|A'B'| = c$. Since c is also $|AB|$, then $|A'B'| = |AB|$. That means that both triangles have sides a, b, and c that are the exact same lengths.*

- Recall that we would like to prove that $\angle ACB$ is a right angle, that it maps to $\angle A'C'B'$. If we can translate $\triangle ABC$ so that A goes to A', B goes to B', and C goes to C', it follows that all three angles in the triangle will match. In particular, that $\angle ACB$ maps to the right angle $\angle A'C'B'$, and so is a right angle, too.

- We can certainly perform a translation that takes B to B' and C to C' because segments BC and $B'C'$ are the same length. Must this translation take A to A'? What goes wrong mathematically if it misses and translates to a different point A'' as shown below?

In this picture, we've drawn A'' to the left of $\overline{A'C'}$. The reasoning that follows works just as well for a picture with A'' to the right of $\overline{A'C'}$ instead.

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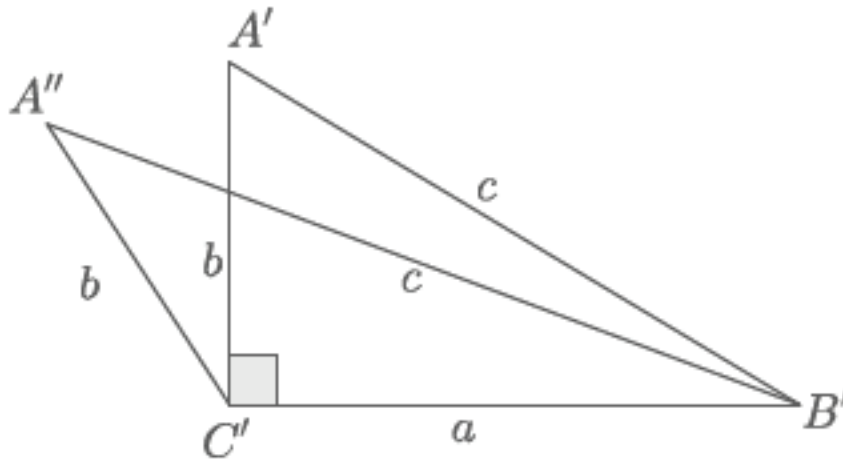
Provide time for students to think of what may go wrong mathematically. If needed, prompt them to notice the two isosceles triangles in the diagram, $\triangle A''C'A'$ and $\triangle A''B'A'$ and the four angles w_1, w_2, w_3, w_4 labeled as shown in the diagram below.
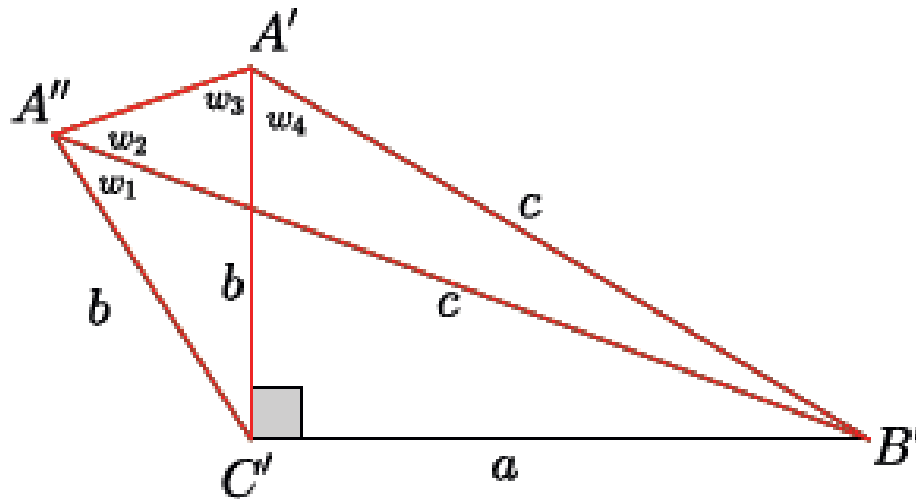
□ *$\triangle A''C'A'$ is isosceles and therefore has base angles that are equal in measure:*

$$w_1 + w_2 = w_3.$$

$\triangle A''B'A'$ is isosceles and therefore has base angles that are equal in measure:

$$w_2 = w_3 + w_4.$$

These two equations give $w_1 + w_3 + w_4 = w_3$, which is equal to $w_1 + w_4 = 0$, which is obviously not true.

▪ Therefore, the translation must map A to A', and since translations preserve the measures of angles, we can conclude that the measure of $\angle ACB$ is equal to the measure of $\angle A'C'B'$, and $\angle ACB$ is a right angle.

▪ Finally, if a triangle has side lengths of a, b and c, with c the longest length, that don't satisfy the equation $a^2 + b^2 = c^2$, then the triangle cannot be a right triangle.

Provide students time to explain to a partner a proof of the converse of the Pythagorean theorem. Allow them to choose either proof that they have seen. Remind them of the proof from Module 3 that was a proof by contradiction, where we assumed that the triangle was not a right triangle and then showed that the assumption was wrong. Select students to share their proofs with the class. Encourage other students to critique the reasoning of the student providing the proof.

Exercises 1–7 (15 minutes)

Students complete Exercises 1–7 independently. Remind students that since each of the exercises references the side length of a triangle, we need only consider the positive square root of each number because we cannot have a negative length.

Exercises 1–7

1. Is the triangle with leg lengths of 3 mi. and 8 mi. and hypotenuse of length $\sqrt{73}$ mi. a right triangle? Show your work, and answer in a complete sentence.

$$3^2 + 8^2 = \left(\sqrt{73}\right)^2$$
$$9 + 64 = 73$$
$$73 = 73$$

Yes, the triangle with leg lengths of 3 mi. and 8 mi. and hypotenuse of length $\sqrt{73}$ mi. is a right triangle because it satisfies the Pythagorean theorem.

2. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.

4 in

1 in
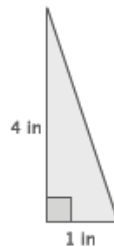
Let c in. represent the length of the hypotenuse of the triangle.

$$1^2 + 4^2 = c^2$$
$$1 + 16 = c^2$$
$$17 = c^2$$
$$\sqrt{17} = c$$
$$4.1 \approx c$$

The length of the hypotenuse of the right triangle is exactly $\sqrt{17}$ inches and approximately 4.1 inches.

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3. **What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.**
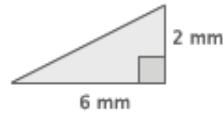
Let c mm represent the length of the hypotenuse of the triangle.

$$2^2 + 6^2 = c^2$$
$$4 + 36 = c^2$$
$$40 = c^2$$
$$\sqrt{40} = c$$
$$\sqrt{2^3} \times \sqrt{5} = c$$
$$\sqrt{2^2} \times \sqrt{2} \times \sqrt{5} = c$$
$$2\sqrt{10} = c$$

The length of the hypotenuse of the right triangle is exactly $2\sqrt{10}$ mm and approximately 6.3 mm.

4. **Is the triangle with leg lengths of 9 in. and 9 in. and hypotenuse of length $\sqrt{175}$ in. a right triangle? Show your work, and answer in a complete sentence.**

$$9^2 + 9^2 = \left(\sqrt{175}\right)^2$$
$$81 + 81 = 175$$
$$162 \neq 175$$

No, the triangle with leg lengths of 9 in. and 9 in. and hypotenuse of length $\sqrt{175}$ in. is not a right triangle because the lengths do not satisfy the Pythagorean theorem.

5. **Is the triangle with leg lengths of $\sqrt{28}$ cm and 6 cm and hypotenuse of length 8 cm a right triangle? Show your work, and answer in a complete sentence.**

$$\left(\sqrt{28}\right)^2 + 6^2 = 8^2$$
$$28 + 36 = 64$$
$$64 = 64$$

Yes, the triangle with leg lengths of $\sqrt{28}$ cm and 6 cm and hypotenuse of length 8 cm is a right triangle because the lengths satisfy the Pythagorean theorem.

6. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence.
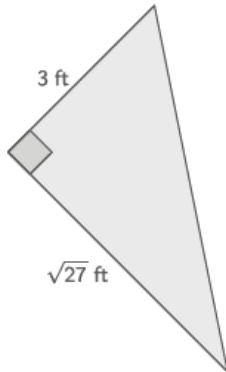
Let c ft. *represent the length of the hypotenuse of the triangle.*

$$3^2 + \left(\sqrt{27}\right)^2 = c^2$$
$$9 + 27 = c^2$$
$$36 = c^2$$
$$\sqrt{36} = \sqrt{c^2}$$
$$6 = c$$

The length of the hypotenuse of the right triangle is 6 ft.

7. The triangle shown below is an isosceles right triangle. Determine the length of the legs of the triangle. Show your work, and answer in a complete sentence.
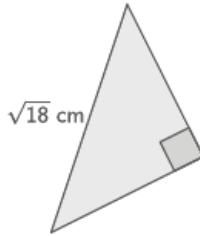
Let x cm *represent the length of each of the legs of the isosceles triangle.*

$$x^2 + x^2 = \left(\sqrt{18}\right)^2$$
$$2x^2 = 18$$
$$\frac{2x^2}{2} = \frac{18}{2}$$
$$x^2 = 9$$
$$\sqrt{x^2} = \sqrt{9}$$
$$x = 3$$

The leg lengths of the isosceles triangle are 3 cm.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- The converse of the Pythagorean theorem states that if side lengths of a triangle a, b, c satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle.
- If the side lengths of a triangle a, b, c do not satisfy $a^2 + b^2 = c^2$, then the triangle is not a right triangle.
- We know how to explain a proof of the Pythagorean theorem and its converse.

Lesson Summary

The converse of the Pythagorean theorem states that if a triangle with side lengths a, b, and c satisfies $a^2 + b^2 = c^2$, then the triangle is a right triangle.

The converse can be proven using concepts related to congruence.

Exit Ticket (5 minutes)

Name _____   Date _____

Lesson 16: Converse of the Pythagorean Theorem

Exit Ticket

1. Is the triangle with leg lengths of 7 mm and 7 mm and a hypotenuse of length 10 mm a right triangle? Show your work, and answer in a complete sentence.

2. What would the length of the hypotenuse need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.

3. If one of the leg lengths is 7 mm, what would the other leg length need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.

Exit Ticket Sample Solutions

1. Is the triangle with leg lengths of 7 mm and 7 mm and a hypotenuse of length 10 mm a right triangle? Show your work, and answer in a complete sentence.

$$7^2 + 7^2 = 10^2$$
$$49 + 49 = 100$$
$$98 \neq 100$$

No, the triangle with leg lengths of 7 mm and 7 mm and hypotenuse of length 10 mm is not a right triangle because the lengths do not satisfy the Pythagorean theorem.

2. What would the length of the hypotenuse need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.

Let c mm represent the length of the hypotenuse.

Then,

$$7^2 + 7^2 = c^2$$
$$49 + 49 = c^2$$
$$98 = c^2$$
$$\sqrt{98} = c$$

The hypotenuse would need to be $\sqrt{98}$ mm for the triangle with sides of 7 mm and 7 mm to be a right triangle.

3. If one of the leg lengths is 7 mm, what would the other leg length need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.

Let a mm represent the length of one leg.

Then,

$$a^2 + 7^2 = 10^2$$
$$a^2 + 49 = 100$$
$$a^2 + 49 - 49 = 100 - 49$$
$$a^2 = 51$$
$$a = \sqrt{51}$$

The leg length would need to be $\sqrt{51}$ mm so that the triangle with one leg length of 7 mm and the hypotenuse of 10 mm is a right triangle.

Problem Set Sample Solutions

1. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.
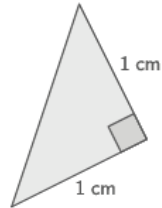
    Let c cm represent the length of the hypotenuse of the triangle.

    $$1^2 + 1^2 = c^2$$
    $$1 + 1 = c^2$$
    $$2 = c^2$$
    $$\sqrt{2} = \sqrt{c^2}$$
    $$1.4 \approx c$$

    The length of the hypotenuse is exactly $\sqrt{2}$ cm and approximately 1.4 cm.

2. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.
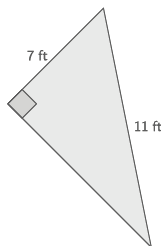
    Let x ft. represent the unknown length of the triangle.

    $$7^2 + x^2 = 11^2$$
    $$49 + x^2 = 121$$
    $$49 - 49 + x^2 = 121 - 49$$
    $$x^2 = 72$$
    $$\sqrt{x^2} = \sqrt{72}$$
    $$x = \sqrt{2^2} \cdot \sqrt{2} \cdot \sqrt{3^2}$$
    $$x = 6\sqrt{2}$$
    $$x \approx 8.5$$

    The length of the unknown side of the triangle is exactly $6\sqrt{2}$ ft. and approximately 8.5 ft.

3. Is the triangle with leg lengths of $\sqrt{3}$ cm and 9 cm and hypotenuse of length $\sqrt{84}$ cm a right triangle? Show your work, and answer in a complete sentence.

    $$\left(\sqrt{3}\right)^2 + 9^2 = \left(\sqrt{84}\right)^2$$
    $$3 + 81 = 84$$
    $$84 = 84$$

    Yes, the triangle with leg lengths of $\sqrt{3}$ cm and 9 cm and hypotenuse of length $\sqrt{84}$ cm is a right triangle because the lengths satisfy the Pythagorean theorem.

4. Is the triangle with leg lengths of $\sqrt{7}$ km and 5 km and hypotenuse of length $\sqrt{48}$ km a right triangle? Show your work, and answer in a complete sentence.

    $$\left(\sqrt{7}\right)^2 + 5^2 = \left(\sqrt{48}\right)^2$$
    $$7 + 25 = 48$$
    $$32 \neq 48$$

    No, the triangle with leg lengths of $\sqrt{7}$ km and 5 km and hypotenuse of length $\sqrt{48}$ km is not a right triangle because the lengths do not satisfy the Pythagorean theorem.

5. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.
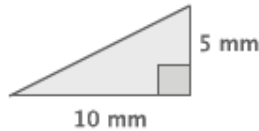
Let c mm represent the length of the hypotenuse of the triangle.

$$5^2 + 10^2 = c^2$$
$$25 + 100 = c^2$$
$$125 = c^2$$
$$\sqrt{125} = \sqrt{c^2}$$
$$\sqrt{5^3} = c$$
$$\sqrt{5^2} \times \sqrt{5} = c$$
$$5\sqrt{5} = c$$
$$11.2 \approx c$$

The length of the hypotenuse is exactly $5\sqrt{5}$ mm and approximately 11.2 mm.

6. Is the triangle with leg lengths of 3 and 6 and hypotenuse of length $\sqrt{45}$ a right triangle? Show your work, and answer in a complete sentence.

$$3^2 + 6^2 = \left(\sqrt{45}\right)^2$$
$$9 + 36 = 45$$
$$45 = 45$$

Yes, the triangle with leg lengths of 3 and 6 and hypotenuse of length $\sqrt{45}$ is a right triangle because the lengths satisfy the Pythagorean theorem.

7. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.
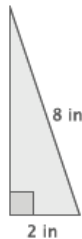
Let x in. represent the unknown side length of the triangle.

$$2^2 + x^2 = 8^2$$
$$4 + x^2 = 64$$
$$4 - 4 + x^2 = 64 - 4$$
$$x^2 = 60$$
$$\sqrt{x^2} = \sqrt{60}$$
$$x = \sqrt{2^2} \cdot \sqrt{3} \cdot \sqrt{5}$$
$$x = 2\sqrt{15}$$
$$x \approx 7.7$$

The length of the unknown side of the triangle is exactly $2\sqrt{15}$ inches and approximately 7.7 inches.

8. Is the triangle with leg lengths of 1 and $\sqrt{3}$ and hypotenuse of length 2 a right triangle? Show your work, and answer in a complete sentence.

$$1^2 + \left(\sqrt{3}\right)^2 = 2^2$$
$$1 + 3 = 4$$
$$4 = 4$$

Yes, the triangle with leg lengths of 1 and $\sqrt{3}$ and hypotenuse of length 2 is a right triangle because the lengths satisfy the Pythagorean theorem.

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9. Corey found the hypotenuse of a right triangle with leg lengths of 2 and 3 to be $\sqrt{13}$. Corey claims that since $\sqrt{13} = 3.61$ when estimating to two decimal digits, that a triangle with leg lengths of 2 and 3 and a hypotenuse of 3.61 is a right triangle. Is he correct? Explain.

    No, Corey is not correct.

    $$2^2 + 3^2 = (3.61)^2$$
    $$4 + 9 = 13.0321$$
    $$13 \neq 13.0321$$

    No, the triangle with leg lengths of 2 and 3 and hypotenuse of length 3.61 is not a right triangle because the lengths do not satisfy the Pythagorean theorem.

10. Explain a proof of the Pythagorean theorem.

    Consider having students share their proof with a partner while their partner critiques their reasoning. Accept any of the three proofs that students have seen.

11. Explain a proof of the converse of the Pythagorean theorem.

    Consider having students share their proof with a partner while their partner critiques their reasoning. Accept either of the proofs that students have seen.

Lesson 17: Distance on the Coordinate Plane

Student Outcomes

▪ Students determine the distance between two points on a coordinate plane using the Pythagorean theorem.

Lesson Notes

Calculators are helpful in this lesson for determining values of radical expressions.

Classwork

Example 1 (6 minutes)

Example 1

What is the distance between the two points A and B on the coordinate plane?
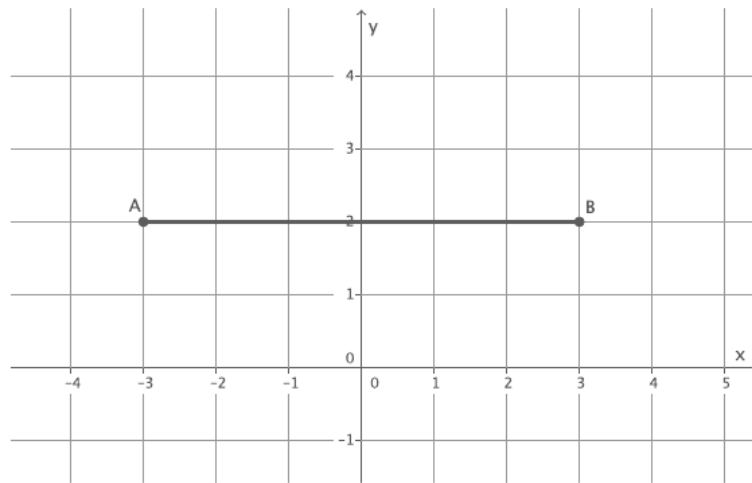
▪ What is the distance between the two points A and B on the coordinate plane?

  □ *The distance between points A and B is 6 units.*

Scaffolding:

Students may benefit from physically measuring lengths to understand finding distance.

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> **What is the distance between the two points *A* and *B* on the coordinate plane?**
>
>
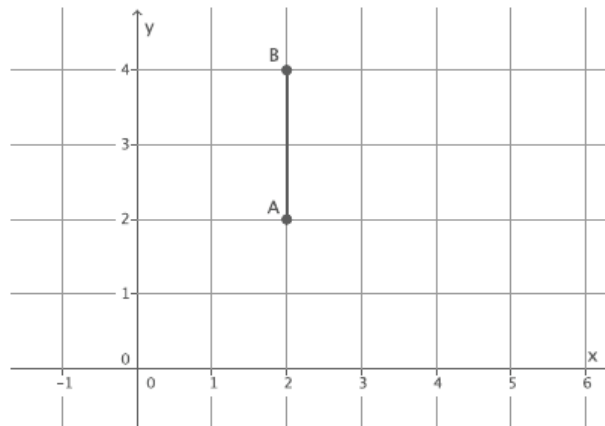
▪ What is the distance between the two points *A* and *B* on the coordinate plane?

  ▫ *The distance between points A and B is 2 units.*

> **What is the distance between the two points *A* and *B* on the coordinate plane? Round your answer to the tenths place.**
>
>
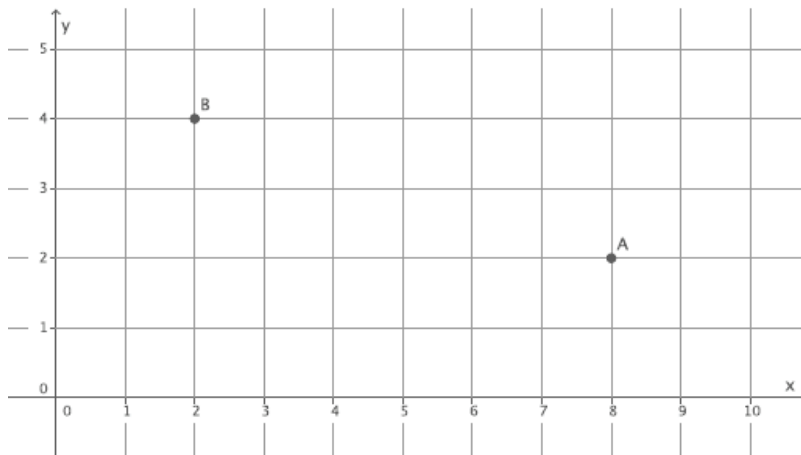
▪ What is the distance between the two points *A* and *B* on the coordinate plane? Round your answer to the tenths place.

Provide students time to solve the problem. Have students share their work and estimations of the distance between the points. The questions that follow can be used to guide students' thinking.

- We cannot simply count units between the points because the line that connects A to B is not horizontal or vertical. What have we done recently that allowed us to find the length of an unknown segment?
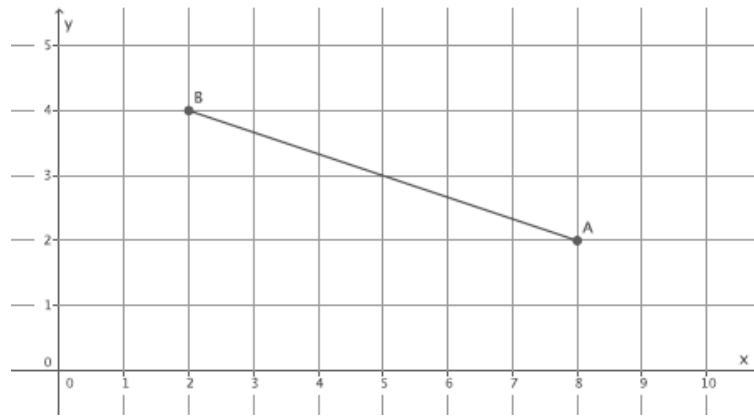
  □ *The Pythagorean theorem allows us to determine the length of an unknown side of a right triangle.*

- Use what you know about the Pythagorean theorem to determine the distance between points A and B.

Provide students time to solve the problem now that they know that the Pythagorean theorem can help them. If necessary, the questions below can guide students' thinking.

- We must draw a right triangle so that $|AB|$ is the hypotenuse. How can we construct the right triangle that we need?

  □ *Draw a vertical line through B and a horizontal line through A. Or, draw a vertical line through A and a horizontal line through B.*
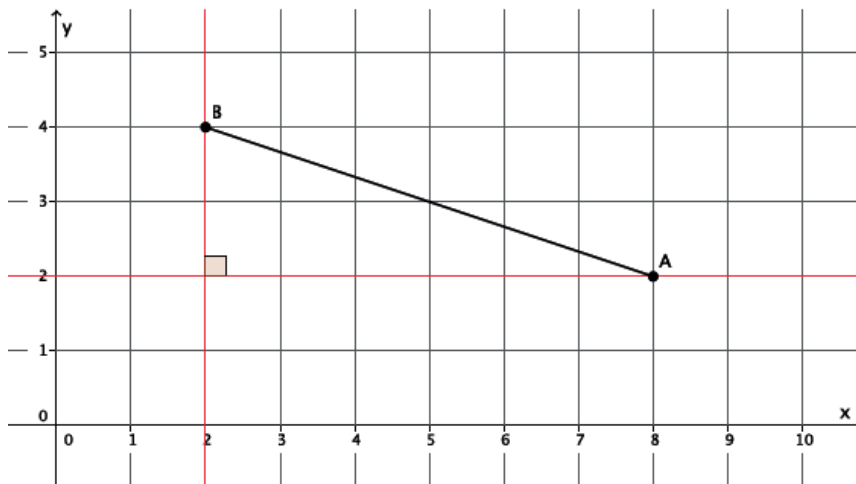
- Let's mark the point of intersection of the horizontal and vertical lines we drew as point C. What is the length of $\overline{AC}$? $\overline{BC}$?
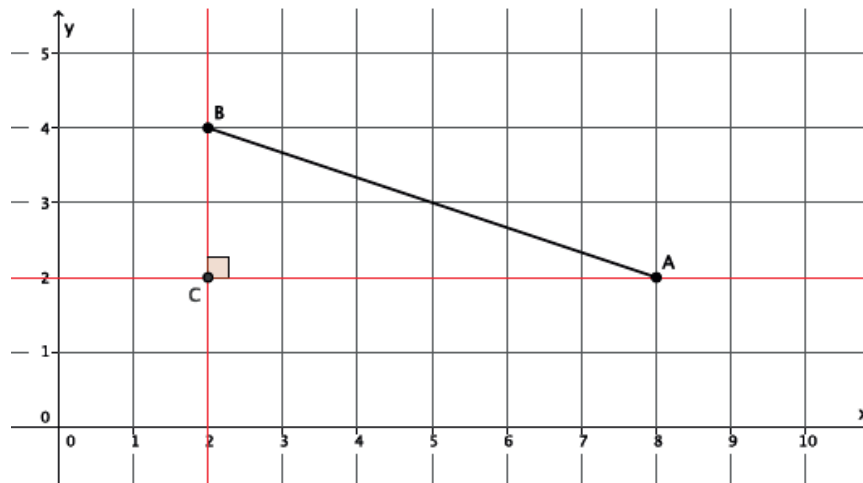
  - $|AC| = 6$ units, and $|BC| = 2$ units

- Now that we know the lengths of the legs of the right triangle, we can determine the length of $\overline{AB}$.

Remind students that because we are finding a length, we need only consider the positive value of the square root because a negative length does not make sense. If necessary, remind students of this fact throughout their work in this lesson.

  - Let c be the length of $\overline{AB}$.

$$2^2 + 6^2 = c^2$$
$$4 + 36 = c^2$$
$$40 = c^2$$
$$\sqrt{40} = c$$
$$6.3 \approx c$$

The distance between points A and B is approximately 6.3 units.

Example 2 (6 minutes)

Example 2

Given two points A and B on the coordinate plane, determine the distance between them. First, make an estimate; then, try to find a more precise answer. Round your answer to the tenths place.
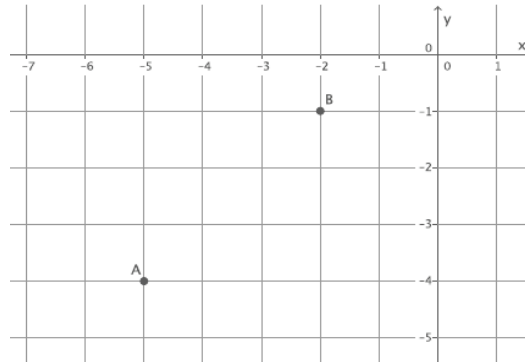
Provide students time to solve the problem. Have students share their work and estimations of the distance between the points. The questions below can be used to guide students' thinking.

▪ We know that we need a right triangle. How can we draw one?
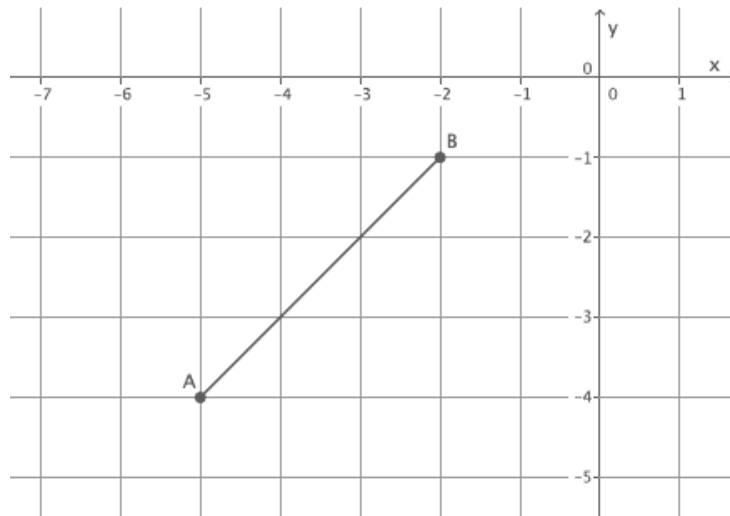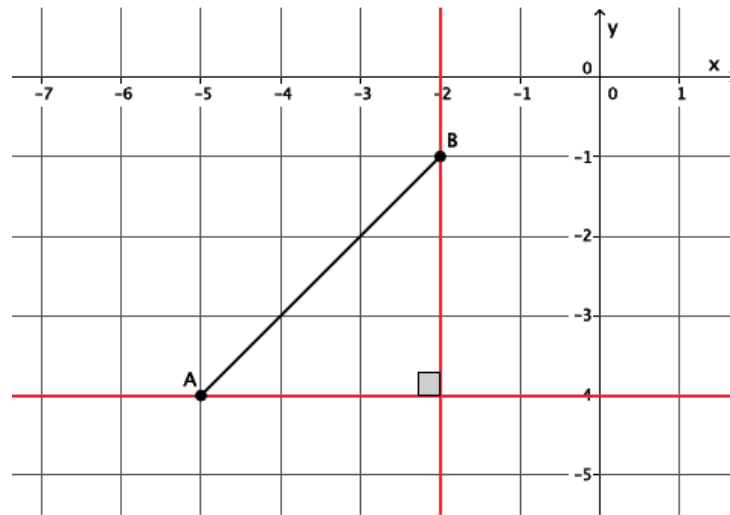
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- ▫ *Draw a vertical line through B and a horizontal line through A. Or draw a vertical line through A and a horizontal line through B.*
- ▪ Mark the point C at the intersection of the horizontal and vertical lines. What do we do next?
  - ▫ *Count units to determine the lengths of the legs of the right triangle, and then use the Pythagorean theorem to find $|AB|$.*

Show the last diagram, and ask a student to explain the answer.
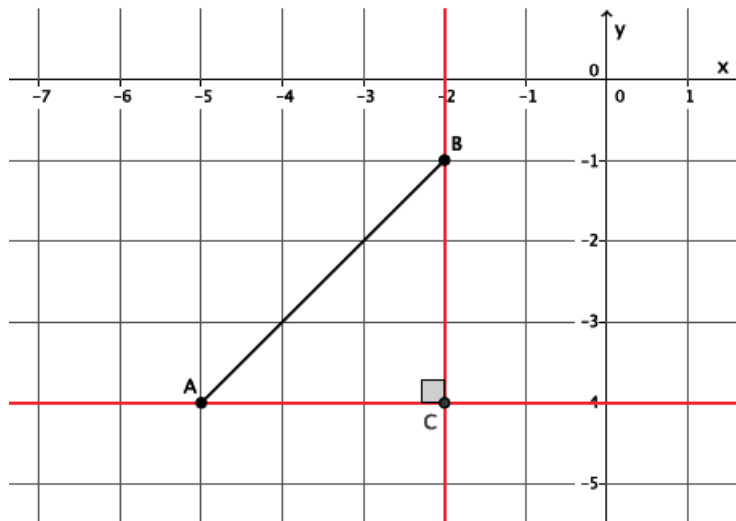
- ▫ *$|AC| = 3$ units, and $|BC| = 3$ units. Let c be $|AB|$.*

$$3^2 + 3^2 = c^2$$
$$9 + 9 = c^2$$
$$18 = c^2$$
$$\sqrt{18} = c$$
$$4.2 \approx c$$

The distance between points A and B is approximately 4.2 units.

Exercises 1–4 (12 minutes)

Students complete Exercises 1–4 independently.

Exercises 1–4

For each of the Exercises 1–4, determine the distance between points A and B on the coordinate plane. Round your answer to the tenths place.

1.
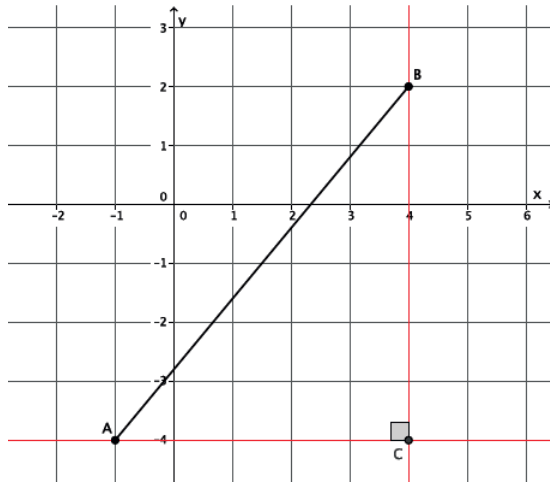
Let c represent $|AB|$.

$$5^2 + 6^2 = c^2$$
$$25 + 36 = c^2$$
$$61 = c^2$$
$$\sqrt{61} = c$$
$$7.8 \approx c$$

The distance between points A and B is about 7.8 units.

2.
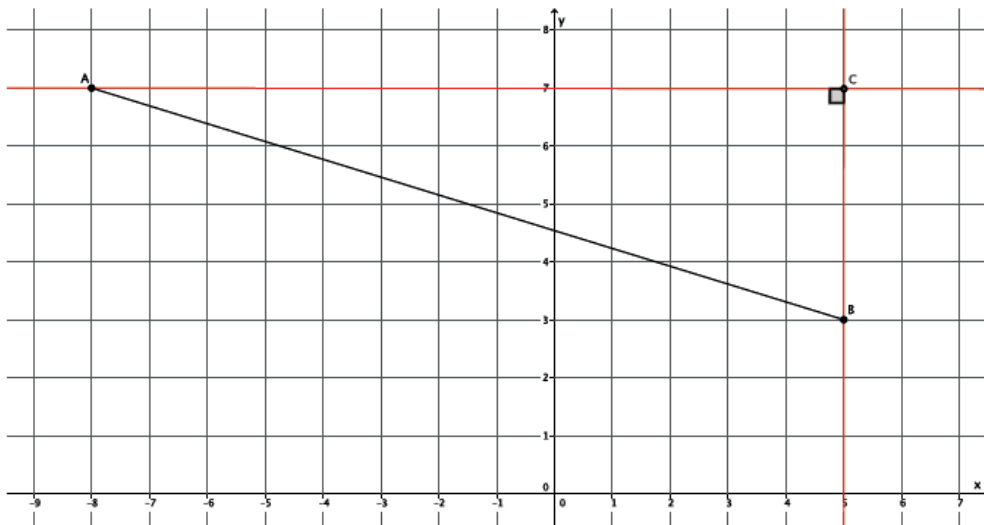
Let c represent $|AB|$.

$$13^2 + 4^2 = c^2$$
$$169 + 16 = c^2$$
$$185 = c^2$$
$$\sqrt{185} = c$$
$$13.6 \approx c$$

The distance between points A and B is about 13.6 units.

Lesson 17:   Distance on the Coordinate Plane

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3.
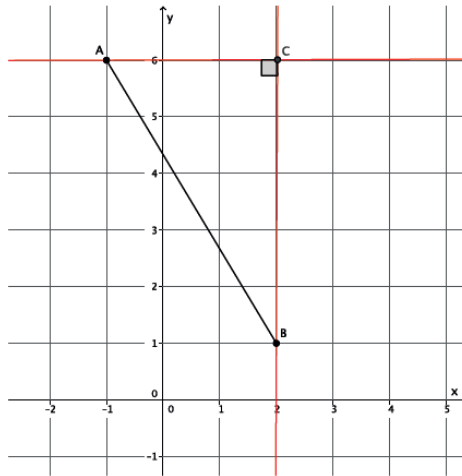
Let c represent $|AB|$.

$$3^2 + 5^2 = c^2$$
$$9 + 25 = c^2$$
$$34 = c^2$$
$$\sqrt{34} = c$$
$$5.8 \approx c$$

The distance between points A and B is about 5.8 *units.*

4.
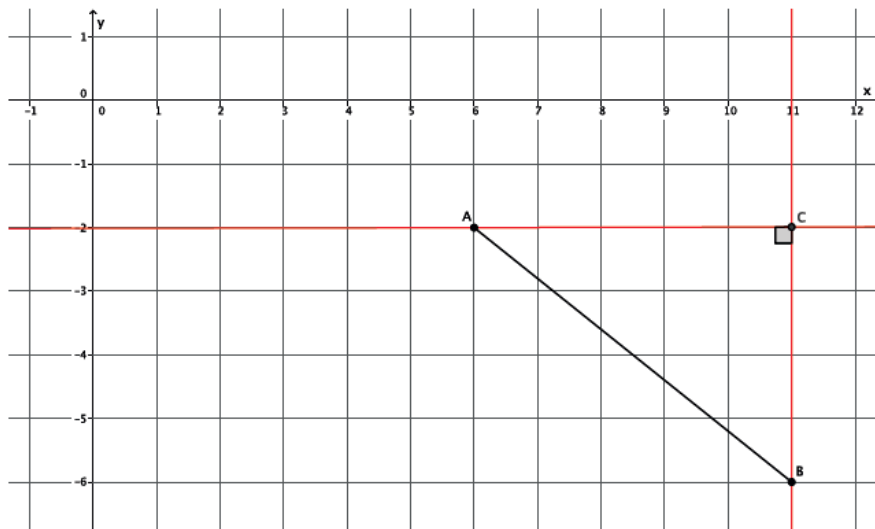
Let c represent $|AB|$.

$$5^2 + 4^2 = c^2$$
$$25 + 16 = c^2$$
$$41 = c^2$$
$$\sqrt{41} = c$$
$$6.4 \approx c$$

The distance between points A and B is about 6.4 *units.*

Example 3 (14 minutes)

- ▪ Is the triangle formed by the points A, B, C a right triangle?

Provide time for small groups of students to discuss and determine if the triangle formed is a right triangle. Have students share their reasoning with the class. If necessary, use the questions below to guide their thinking.

> **Example 3**
>
> **Is the triangle formed by the points A, B, C a right triangle?**
>
>
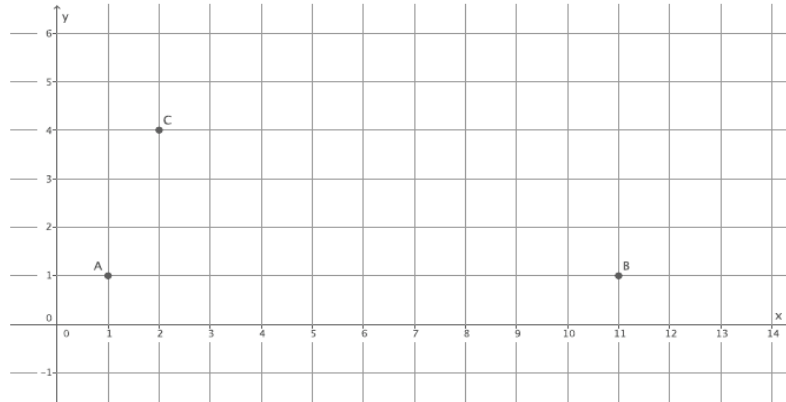
- ▪ How can we verify if a triangle is a right triangle?
    - ▫ *Use the converse of the Pythagorean theorem.*
- ▪ What information do we need about the triangle in order to use the converse of the Pythagorean theorem, and how would we use it?
    - ▫ *We need to know the lengths of all three sides; then, we can check to see if the side lengths satisfy the Pythagorean theorem.*

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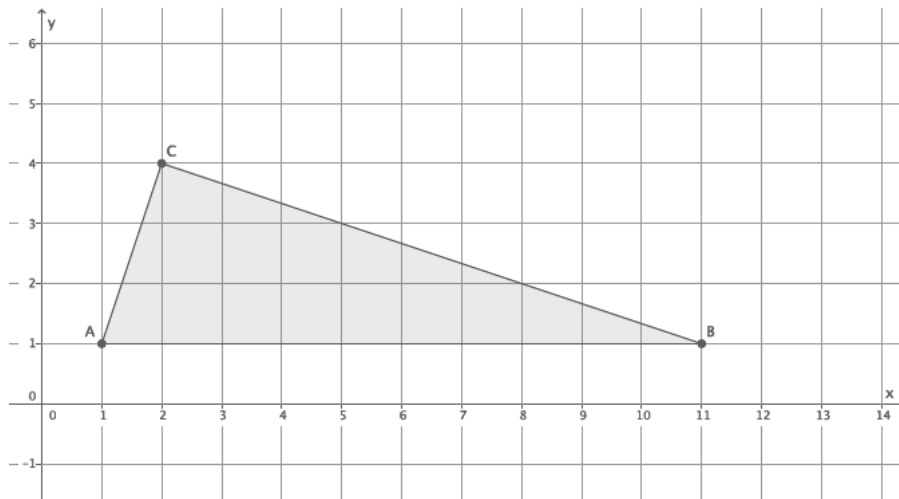
- Clearly, $|AB| = 10$ units. How can we determine $|AC|$?

  □ *To find $|AC|$, follow the same steps used in the previous problem. Draw horizontal and vertical lines to form a right triangle, and use the Pythagorean theorem to determine the length.*

- Determine $|AC|$. Leave your answer in square root form unless it is a perfect square.
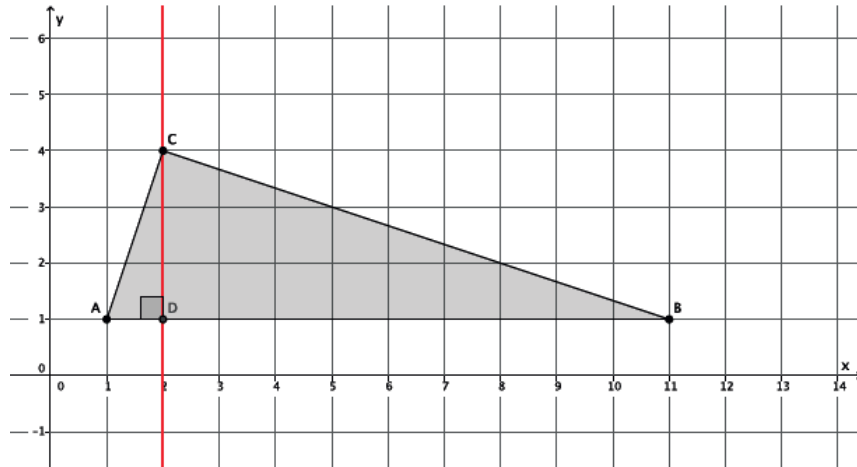
  □ *Let c represent $|AC|$.*

$$1^2 + 3^2 = c^2$$
$$1 + 9 = c^2$$
$$10 = c^2$$
$$\sqrt{10} = c$$

- Now, determine $|BC|$. Again, leave your answer in square root form unless it is a perfect square.
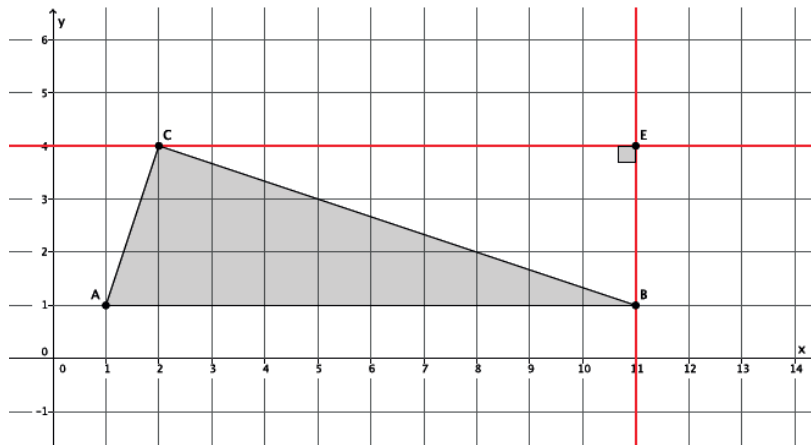
  □ *Let c represent $|BC|$.*

$$9^2 + 3^2 = c^2$$
$$81 + 9 = c^2$$
$$90 = c^2$$
$$\sqrt{90} = c$$

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- The lengths of the three sides of the triangle are 10 units, $\sqrt{10}$ units, and $\sqrt{90}$ units. Which number represents the hypotenuse of the triangle? Explain.
  - *The side $\overline{AB}$ must be the hypotenuse because it is the longest side. When estimating the lengths of the other two sides, I know that $\sqrt{10}$ is between 3 and 4, and $\sqrt{90}$ is between 9 and 10. Therefore, the side that is 10 units in length is the hypotenuse.*
- Use the lengths 10, $\sqrt{10}$, and $\sqrt{90}$ to determine if the triangle is a right triangle.
  - *Sample response:*

$$\left(\sqrt{10}\right)^2 + \left(\sqrt{90}\right)^2 = 10^2$$
$$10 + 90 = 100$$
$$100 = 100$$

- Therefore, the points A, B, C form a right triangle.

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- To find the distance between two points on the coordinate plane, draw a right triangle, and use the Pythagorean theorem.
- To verify if a triangle in the plane is a right triangle, use both the Pythagorean theorem and its converse.

Lesson Summary

To determine the distance between two points on the coordinate plane, begin by connecting the two points. Then, draw a vertical line through one of the points and a horizontal line through the other point. The intersection of the vertical and horizontal lines forms a right triangle to which the Pythagorean theorem can be applied.

To verify if a triangle is a right triangle, use the converse of the Pythagorean theorem.

Exit Ticket (4 minutes)

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Name _____   Date _____

Lesson 17: Distance on the Coordinate Plane

Exit Ticket

Use the following diagram to answer the questions below.
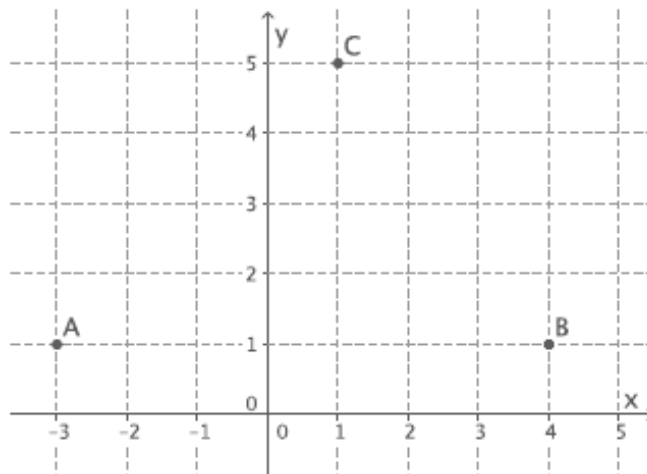
1. Determine $|AC|$. Leave your answer in square root form unless it is a perfect square.

2. Determine $|CB|$. Leave your answer in square root form unless it is a perfect square.

3. Is the triangle formed by the points A, B, C a right triangle? Explain why or why not.

Exit Ticket Sample Solutions

Use the following diagram to answer the questions below.
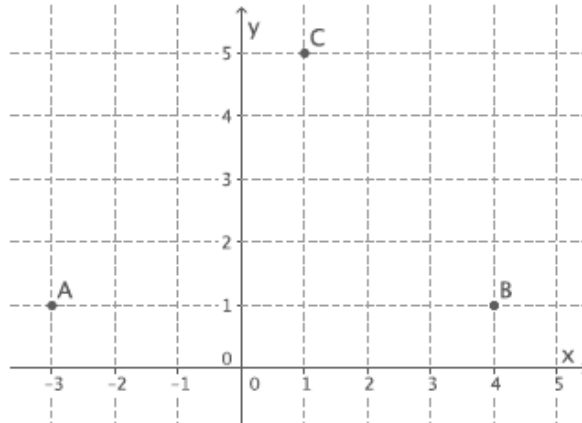
1. Determine $|AC|$. Leave your answer in square root form unless it is a perfect square.

    Let c represent $|AC|$.

    $$4^2 + 4^2 = c^2$$
    $$16 + 16 = c^2$$
    $$32 = c^2$$
    $$\sqrt{32} = c$$

2. Determine $|CB|$. Leave your answer in square root form unless it is a perfect square.

    Let d represent $|CB|$.

    $$3^2 + 4^2 = d^2$$
    $$9 + 16 = d^2$$
    $$25 = d^2$$
    $$\sqrt{25} = d$$
    $$5 = d$$

3. Is the triangle formed by the points A, B, C a right triangle? Explain why or why not.

    Using the lengths $5, \sqrt{32}$, and $|AB| = 7$ to determine if the triangle is a right triangle, I have to check to see if

    $$5^2 + \left(\sqrt{32}\right)^2 = 7^2$$
    $$25 + 32 \neq 49$$

    Therefore, the triangle formed by the points A, B, C is not a right triangle because the lengths of the triangle do not satisfy the Pythagorean theorem.

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Problem Set Sample Solutions

For each of the Problems 1–4, determine the distance between points A and B on the coordinate plane. Round your answer to the tenths place.

1.
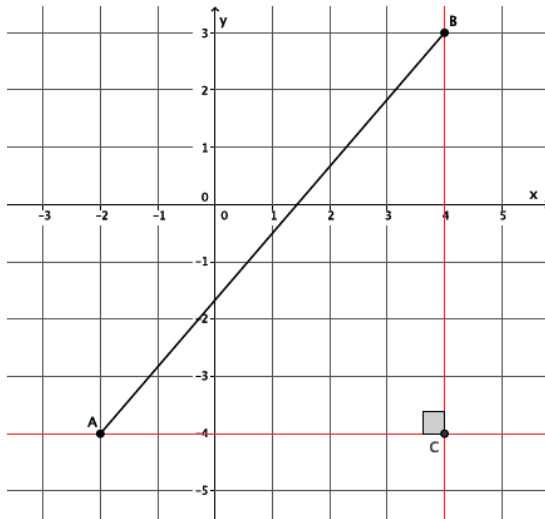
Let c represent $|AB|$.

$$6^2 + 7^2 = c^2$$
$$36 + 49 = c^2$$
$$85 = c^2$$
$$\sqrt{85} = c$$
$$9.2 \approx c$$

The distance between points A and B is about 9.2 units.

2.
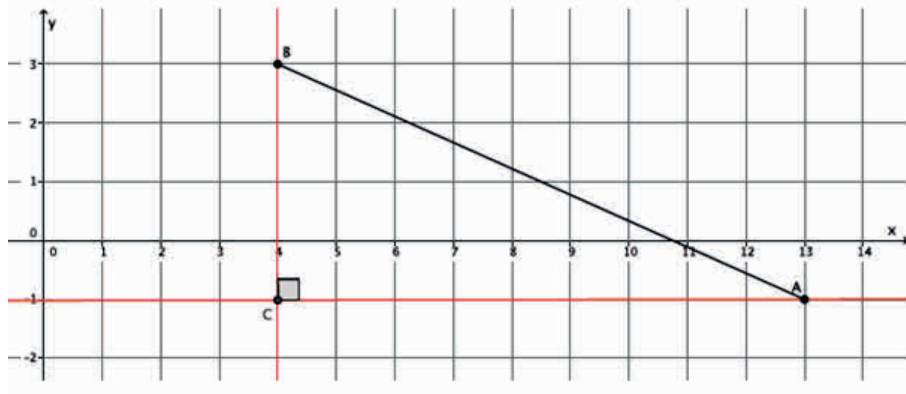
Let c represent $|AB|$.

$$9^2 + 4^2 = c^2$$
$$81 + 16 = c^2$$
$$97 = c^2$$
$$\sqrt{97} = c$$
$$9.8 \approx c$$

The distance between points A and B is about 9.8 units.

3.
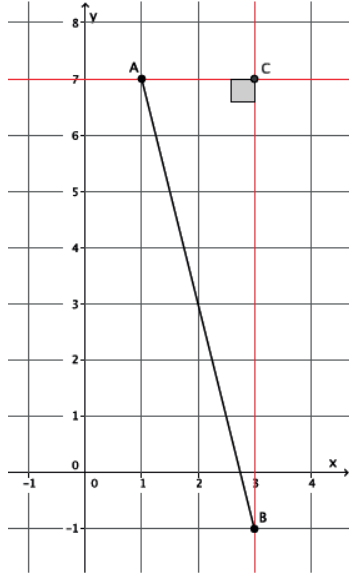
Let c represent |AB|.

$$2^2 + 8^2 = c^2$$
$$4 + 64 = c^2$$
$$68 = c^2$$
$$\sqrt{68} = c$$
$$8.2 \approx c$$

The distance between points A and B is about 8.2 units.

4.
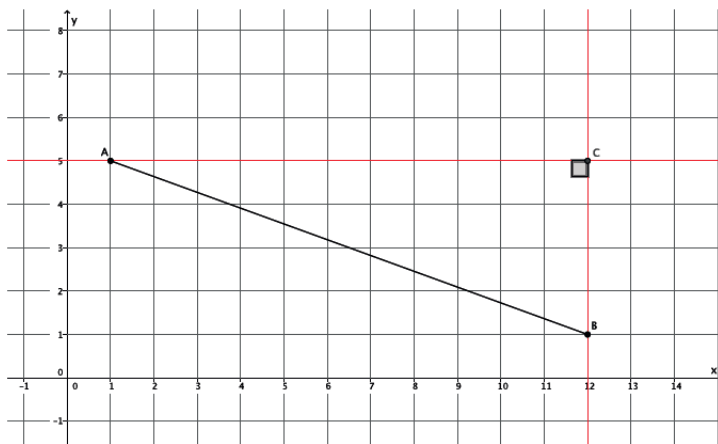
Let c represent |AB|.

$$11^2 + 4^2 = c^2$$
$$121 + 16 = c^2$$
$$137 = c^2$$
$$\sqrt{137} = c$$
$$11.7 \approx c$$

The distance between points A and B is about 11.7 units.

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5. Is the triangle formed by points A, B, C a right triangle?
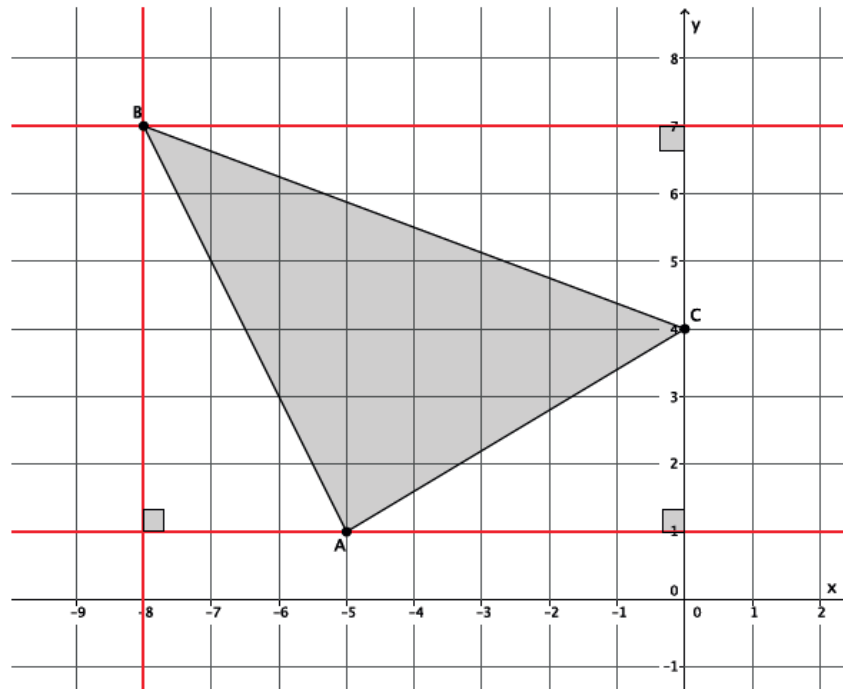
Let c represent $|AB|$.

$$3^2 + 6^2 = c^2$$
$$9 + 36 = c^2$$
$$45 = c^2$$
$$\sqrt{45} = c$$

Let c represent $|AC|$.

$$3^2 + 5^2 = c^2$$
$$9 + 25 = c^2$$
$$34 = c^2$$
$$\sqrt{34} = c$$

Let c represent $|BC|$.

$$3^2 + 8^2 = c^2$$
$$9 + 64 = c^2$$
$$73 = c^2$$
$$\sqrt{73} = c$$

$$\left(\sqrt{45}\right)^2 + \left(\sqrt{34}\right)^2 = \left(\sqrt{73}\right)^2$$
$$45 + 34 = 73$$
$$79 \neq 73$$

No, the points do not form a right triangle.

Lesson 18: Applications of the Pythagorean Theorem

Student Outcomes

- Students apply the Pythagorean theorem to real-world and mathematical problems in two dimensions.

Lesson Notes

It is recommended that students have access to a calculator as they work through the exercises. However, it is not recommended that students use calculators to answer the questions but only to check their work or estimate the value of an irrational number using rational approximation. Make clear to students that they can use calculators but that all mathematical work should be shown. This lesson includes a Fluency Exercise that takes approximately 10 minutes to complete. The Fluency Exercise is a white board exchange with problems on volume that can be found at the end of this lesson. It is recommended that the Fluency Exercise take place at the beginning of the lesson or after the discussion that concludes the lesson.

Classwork

Exploratory Challenge/Exercises 1–5 (20 minutes)

Students complete Exercises 1–5 in pairs or small groups. These problems are applications of the Pythagorean theorem and are an opportunity to remind students to make sense of problems and persevere in solving them. Students should compare their solutions and solution methods in their pairs, small groups, and as a class. If necessary, remind students that we are finding lengths, which means we need only consider the positive square root of a number.

Exercises

1. The area of the right triangle shown below is 26.46 in^2. What is the perimeter of the right triangle? Round your answer to the tenths place.
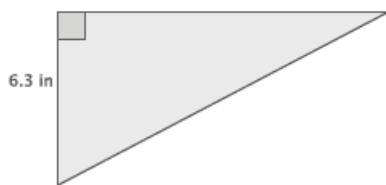
Let b in. represent the length of the base of the triangle where $h = 6.3$.

$$A = \frac{bh}{2}$$
$$26.46 = \frac{6.3b}{2}$$
$$52.92 = 6.3b$$
$$\frac{52.92}{6.3} = \frac{6.3b}{6.3}$$
$$8.4 = b$$

Let c in. represent the length of the hypotenuse.

$$6.3^2 + 8.4^2 = c^2$$
$$39.69 + 70.56 = c^2$$
$$110.25 = c^2$$
$$\sqrt{110.25} = \sqrt{c^2}$$
$$\sqrt{110.25} = c$$

The number $\sqrt{110.25}$ is between 10 and 11. When comparing with tenths, the number is actually equal to 10.5 because $10.5^2 = 110.25$. Therefore, the length of the hypotenuse is 10.5 in. The perimeter of the triangle is $6.3 \text{ in.} + 8.4 \text{ in.} + 10.5 \text{ in.} = 25.2 \text{ in.}$

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2.   The diagram below is a representation of a soccer goal.

10 ft

8 ft c 8 ft c

3 ft   3 ft
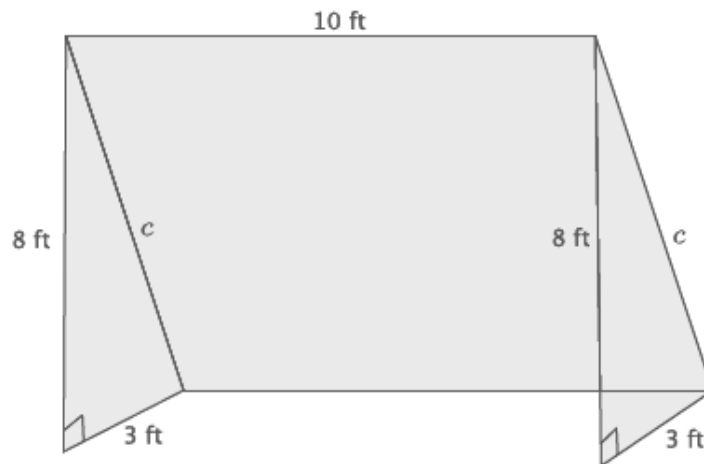
a.   Determine the length of the bar, c, that would be needed to provide structure to the goal. Round your answer to the tenths place.

Let c ft. represent the hypotenuse of the right triangle.

$$8^2 + 3^2 = c^2$$
$$64 + 9 = c^2$$
$$73 = c^2$$
$$\sqrt{73} = \sqrt{c^2}$$

The number $\sqrt{73}$ is between 8 and 9. In the sequence of tenths, it is between 8.5 and 8.6 because $8.5^2 < \left(\sqrt{73}\right)^2 < 8.6^2$. In the sequence of hundredths, the number is between 8.54 and 8.55 because $8.54^2 < \left(\sqrt{73}\right)^2 < 8.55^2$. Since the number $\sqrt{73}$ is between 8.54 and 8.55, it would round to 8.5. The length of the bar that provides structure for the goal is approximately 8.5 ft.

b.   How much netting (in square feet) is needed to cover the entire goal?

The areas of the triangles are each 12 ft^2. The area of the rectangle in the back is approximately 85 ft^2. The total area of netting required to cover the goal is approximately 109 ft^2.

3. The typical ratio of length to width that is used to produce televisions is $4:3$.
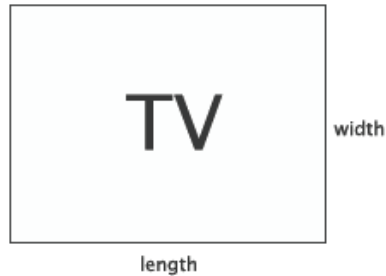
A TV with length 20 inches and width 15 inches, for example, has sides in a $4:3$ ratio; as does any TV with length $4x$ inches and width $3x$ inches for any number x.

a. What is the advertised size of a TV with length 20 inches and width 15 inches?

Let c in. be the length of the diagonal.

$$20^2 + 15^2 = c^2$$
$$400 + 225 = c^2$$
$$625 = c^2$$
$$\sqrt{625} = \sqrt{c^2}$$
$$25 = c$$

Since the TV has a diagonal length of 25 inches, then it is a $25"$ TV.

Note to Teacher:

Check in with students to make sure they understand how TVs are measured in terms of their diagonal length and what is meant by the ratio of $4:3$. To complete the problem, students must be clear that the size of a TV is not denoted by its length or width but by the length of the diagonal (hypotenuse). Also, students must have some sense that the ratio of length to width must be some multiple of the ratio $4:3$; otherwise, the TV would not give a good perspective. Consider showing students what a TV would look like with a ratio of $9:12$. They should notice that such dimensions yield a TV screen that is unfamiliar to them.

b. A $42"$ TV was just given to your family. What are the length and width measurements of the TV?

Let x be the factor applied to the ratio $4:3$.

$$(3x)^2 + (4x)^2 = 42^2$$
$$9x^2 + 16x^2 = 1764$$
$$(9 + 16)x^2 = 1764$$
$$25x^2 = 1764$$
$$\frac{25x^2}{25} = \frac{1764}{25}$$
$$x^2 = 70.56$$
$$\sqrt{x^2} = \sqrt{70.56}$$
$$x = \sqrt{70.56}$$

The number $\sqrt{70.56}$ is between 8 and 9. In working with the sequence of tenths, I realized the number $\sqrt{70.56}$ is actually equal to 8.4 because $8.4^2 = 70.56$. Therefore, $x = 8.4$, and the dimensions of the TV are (4×8.4) inches , which is 33.6 inches, and (3×8.4) inches, which is 25.2 inches.

c. Check that the dimensions you got in part (b) are correct using the Pythagorean theorem.

$$33.6^2 + 25.2^2 = 42^2$$
$$1128.96 + 635.04 = 1764$$
$$1764 = 1764$$

d. The table that your TV currently rests on is $30"$ in length. Will the new TV fit on the table? Explain.

The dimension for the length of the TV is 33.6 inches. It will not fit on a table that is 30 inches in length.

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4. Determine the distance between the following pairs of points. Round your answer to the tenths place. Use graph paper if necessary.

a. $(7, 4)$ and $(-3, -2)$

Let c represent the distance between the two points.

$$10^2 + 6^2 = c^2$$
$$100 + 36 = c^2$$
$$136 = c^2$$
$$\sqrt{136} = \sqrt{c^2}$$
$$\sqrt{136} = c$$

The number $\sqrt{136}$ is between 11 and 12. In the sequence of tenths, it is between 11.6 and 11.7 because $11.6^2 < \left(\sqrt{136}\right)^2 < 11.7^2$. In the sequence of hundredths, it is between 11.66 and 11.67, which means the number will round in tenths to 11.7. The distance between the two points is approximately 11.7 units.

b. $(-5, 2)$ and $(3, 6)$

Let c represent the distance between the two points.

$$8^2 + 4^2 = c^2$$
$$64 + 16 = c^2$$
$$80 = c^2$$
$$\sqrt{80} = \sqrt{c^2}$$
$$\sqrt{80} = c$$

The number $\sqrt{80}$ is between 8 and 9. In the sequence of tenths, it is between 8.9 and 9 because $8.9^2 < \left(\sqrt{80}\right)^2 < 9^2$. In the sequence of hundredths, it is between 8.94 and 8.95, which means it will round to 8.9. The distance between the two points is approximately 8.9 units.

c. Challenge: (x_1, y_1) and (x_2, y_2). Explain your answer.

Note: Deriving the distance formula using the Pythagorean theorem is not part of the Grade 8 content but does present an interesting challenge to students. Assign it only to students who need a challenge.

Let c represent the distance between the two points.

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = c^2$$
$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{c^2}$$
$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = c$$

I noticed that the dimensions of the right triangle were equal to the difference in x-values and the difference in y-values. Using those expressions and what I knew about solving radical equations, I was able to determine the length of c.

5. What length of ladder is needed to reach a height of 7 feet along the wall when the base of the ladder is 4 feet from the wall? Round your answer to the tenths place.

Let c feet represent the length of the ladder.

$$7^2 + 4^2 = c^2$$
$$49 + 16 = c^2$$
$$65 = c^2$$
$$\sqrt{65} = \sqrt{c^2}$$
$$\sqrt{65} = c$$

The number $\sqrt{65}$ is between 8 and 9. In the sequence of tenths, it is between 8 and 8.1 because $8^2 < \left(\sqrt{65}\right)^2 < 8.1^2$. In the sequence of hundredths, it is between 8.06 and 8.07, which means the number will round to 8.1. The ladder must be approximately 8.1 feet long to reach 7 feet up a wall when placed 4 feet from the wall.

7 feet

4 feet
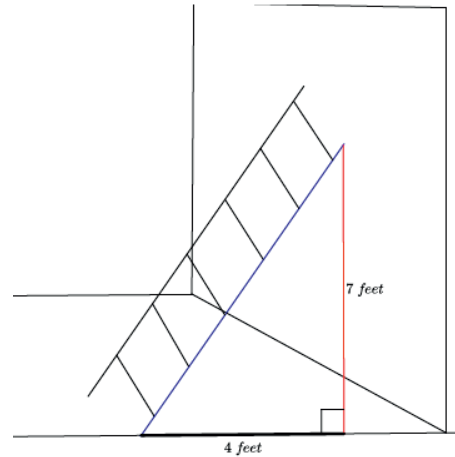
Discussion (5 minutes)

This discussion provides a challenge question to students about how the Pythagorean theorem might be applied to a three-dimensional situation. The next lesson focuses on using the Pythagorean theorem to answer questions about cones and spheres.

▪ The majority of our work with the Pythagorean theorem has been in two dimensions. Can you think of any applications we have seen so far that are in three dimensions?

□ *The soccer goal is three-dimensional. A ladder propped up against a wall is three-dimensional.*

▪ What new applications of the Pythagorean theorem in three dimensions do you think we will work on next?

Provide students time to think about this in pairs or small groups.

□ *We have worked with solids this year, so there may be an application involving cones and spheres.*

Fluency Exercise (10 minutes): Area and Volume II

RWBE: Refer to the Rapid White Board Exchanges section in the Module 1 Module Overview for directions to administer a Rapid White Board Exchange.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

▪ We know some basic applications of the Pythagorean theorem in terms of measures of a television, length of a ladder, and area and perimeter of right triangles.

▪ We know that there will be some three-dimensional applications of the theorem beyond what we have already seen.

Exit Ticket (5 minutes)

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Name _____   Date _____

Lesson 18: Applications of the Pythagorean Theorem

Use the diagram of the equilateral triangle shown below to answer the following questions. Show the work that leads to your answers.
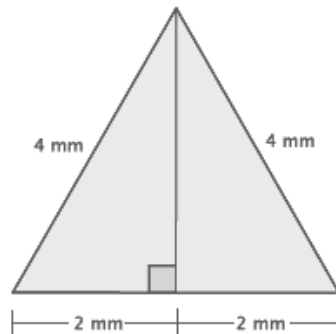
a.   What is the perimeter of the triangle?

b.   What is the height, h mm, of the equilateral triangle? Write an exact answer using a square root and an approximate answer rounded to the tenths place.

c.   Using the approximate height found in part (b), estimate the area of the equilateral triangle.

Exit Ticket Sample Solutions
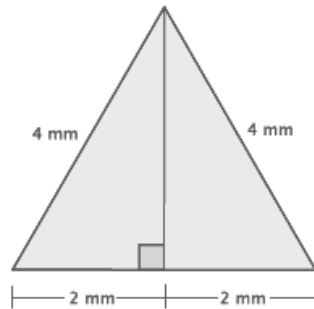
> **Use the diagram of the equilateral triangle shown below to answer the following questions. Show the work that leads to your answers.**
>
>
>
> a.   **What is the perimeter of the triangle?**
>
> $4 + 4 + 4 = 12$
>
>   *The perimeter is 12 mm.*
>
> b.   **What is the height, h mm, of the equilateral triangle? Write an exact answer using a square root and an approximate answer rounded to the tenths place.**
>
>   *Using the fact that the height is one leg length of a right triangle, and I know the hypotenuse is 4 mm and the other leg length is 2 mm, I can use the Pythagorean theorem to find h.*
>
> $$2^2 + h^2 = 4^2$$
> $$4 + h^2 = 16$$
> $$4 - 4 + h^2 = 16 - 4$$
> $$h^2 = 12$$
> $$h = \sqrt{12}$$
> $$h = \sqrt{4 \times 3}$$
> $$h = \sqrt{4} \times \sqrt{3}$$
> $$h = 2\sqrt{3}$$
>
>   *The number $\sqrt{3}$ is between 1 and 2. In the sequence of tenths, it is between 1.7 and 1.8 because $1.7^2 < \left(\sqrt{3}\right)^2 < 1.8^2$. In the sequence of hundredths, it is between 1.73 and 1.74. This would put h between 2×1.73 mm, which is 3.46 mm, and 2×1.74 mm, which is 3.48 mm. In terms of tenths, the approximate height is 3.5 mm. The exact height is $\sqrt{12}$ mm, or $2\sqrt{3}$ mm.*
>
> c.   **Using the approximate height found in part (b), estimate the area of the equilateral triangle.**
>
> $$A = \frac{bh}{2}$$
> $$A = \frac{4(3.5)}{2}$$
> $$A = \frac{14}{2}$$
> $$A = 7$$
>
>   *The approximate area of the equilateral triangle is 7 mm^2.*

Problem Set Sample Solutions

Students continue applying the Pythagorean theorem to solve real-world and mathematical problems.

1. A 70" TV is advertised on sale at a local store. What are the length and width of the television?

    The TV is in the ratio of $4:3$ and has measurements of $4x:3x$, where x is the scale factor of enlargement.

    $$(3x)^2 + (4x)^2 = 70^2$$
    $$9x^2 + 16x^2 = 4,900$$
    $$25x^2 = 4,900$$
    $$\frac{25x^2}{25} = \frac{4,900}{25}$$
    $$x^2 = 196$$
    $$\sqrt{x^2} = \sqrt{196}$$
    $$x = 14$$

    The length of the TV is (4×14) inches, which is 56 inches, and the width is (3×14) inches, which is 42 inches.

2. There are two paths that one can use to go from Sarah's house to James' house. One way is to take C Street, and the other way requires you to use A Street and B Street. How much shorter is the direct path along C Street?
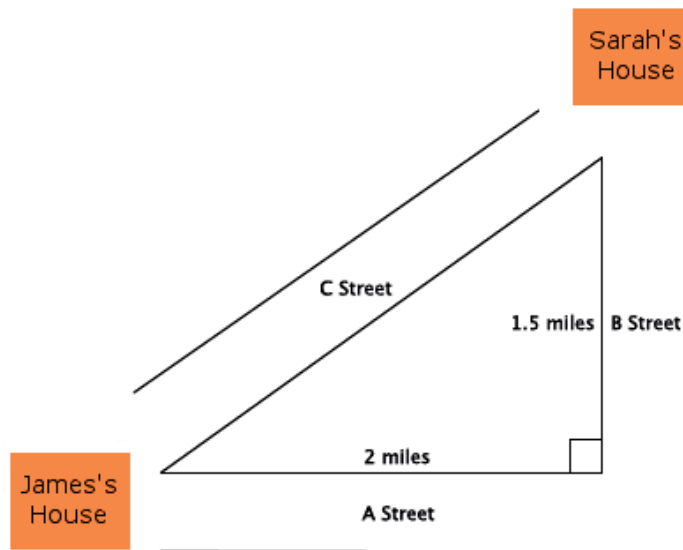
    Let c miles represent the length of the hypotenuse of the right triangle.

    $$2^2 + 1.5^2 = c^2$$
    $$4 + 2.25 = c^2$$
    $$6.25 = c^2$$
    $$\sqrt{6.25} = \sqrt{c^2}$$
    $$2.5 = c$$

    The path using A Street and B Street is 3.5 miles. The path along C Street is 2.5 miles. The path along C Street is exactly 1 mile shorter than the path along A Street and B Street.

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3. An isosceles right triangle refers to a right triangle with equal leg lengths, s, as shown below.
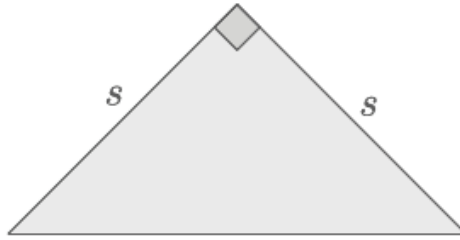
What is the length of the hypotenuse of an isosceles right triangle with a leg length of 9 cm? Write an exact answer using a square root and an approximate answer rounded to the tenths place.

Let c be the length of the hypotenuse of the isosceles triangle in centimeters.

$$9^2 + 9^2 = c^2$$
$$81 + 81 = c^2$$
$$162 = c^2$$
$$\sqrt{162} = \sqrt{c^2}$$
$$\sqrt{81 \times 2} = c$$
$$\sqrt{81} \times \sqrt{2} = c$$
$$9\sqrt{2} = c$$

The number $\sqrt{2}$ is between 1 and 2. In the sequence of tenths, it is between 1.4 and 1.5 because $1.4^2 < \left(\sqrt{2}\right)^2 < 1.5^2$. Since the number 2 is closer to 1.4^2 than 1.5^2, it would round to 1.4. $9 \times 1.4 = 12.6$. So, 12.6 cm is the approximate length of the hypotenuse, and $9\sqrt{2}$ cm is the exact length.

4. The area of the right triangle shown to the right is 66.5 cm^2.

   a. What is the height of the triangle?

   $$A = \frac{bh}{2}$$
   $$66.5 = \frac{9.5h}{2}$$
   $$133 = 9.5h$$
   $$\frac{133}{9.5} = \frac{9.5h}{9.5}$$
   $$14 = h$$

   The height of the triangle is 14 cm.

9.5 cm
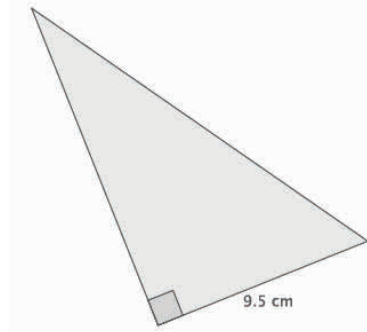
b. What is the perimeter of the right triangle? Round your answer to the tenths place.

Let c represent the length of the hypotenuse in centimeters.

$$9.5^2 + 14^2 = c^2$$
$$90.25 + 196 = c^2$$
$$286.25 = c^2$$
$$\sqrt{286.25} = \sqrt{c^2}$$
$$\sqrt{286.25} = c$$

The number $\sqrt{286.25}$ is between 16 and 17. In the sequence of tenths, the number is between 16.9 and 17 because $16.9^2 < \left(\sqrt{286.25}\right)^2 < 17^2$. Since 286.25 is closer to 16.9^2 than 17^2, then the approximate length of the hypotenuse is 16.9 cm.

The perimeter of the triangle is 9.5 cm $+ 14$ cm $+ 16.9$ cm $= 40.4$ cm.

5. What is the distance between points $(1, 9)$ and $(-4, -1)$? Round your answer to the tenths place.

Let c represent the distance between the points.

$$10^2 + 5^2 = c^2$$
$$100 + 25 = c^2$$
$$125 = c^2$$
$$\sqrt{125} = \sqrt{c^2}$$
$$\sqrt{125} = c$$
$$11.2 \approx c$$

The distance between the points is approximately 11.2 units.

6. An equilateral triangle is shown below. Determine the area of the triangle. Round your answer to the tenths place.
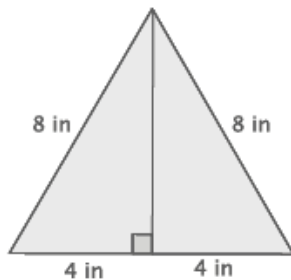
Let h in. represent the height of the triangle.

$$4^2 + h^2 = 8^2$$
$$16 + h^2 = 64$$
$$h^2 = 48$$
$$\sqrt{h^2} = \sqrt{48}$$
$$h = \sqrt{48}$$
$$h \approx 6.9$$

$$A = \frac{(8)(6.9)}{2} = (4)(6.9) = 27.6$$

The area of the triangle is approximately 27.6 in^2.

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Area and Volume II

1. Find the area of the square shown below.

$$A = (6 \text{ cm})^2$$
$$= 36 \text{ cm}^2$$

6 cm

2. Find the volume of the cube shown below.

$$V = (6 \text{ cm})^3$$
$$= 216 \text{ cm}^3$$

6 cm

3. Find the area of the rectangle shown below.

$$A = (9 \text{ m})(3 \text{ m})$$
$$= 27 \text{ m}^2$$

3 m

9 m

4. Find the volume of the rectangular prism shown below.

$$V = (27 \text{ m}^2)(5 \text{ m})$$
$$= 135 \text{ m}^3$$

3 m

5 m

9 m

5. Find the area of the circle shown below.

$$A = (5 \text{ m})^2 \pi$$
$$= 25\pi \text{ m}^2$$

10 m

6. Find the volume of the cylinder shown below.

$$V = (25\pi \text{ m}^2)(11 \text{ m})$$
$$= 275\pi \text{ m}^3$$

5 m
11 m

7. Find the area of the circle shown below.

$$A = (8 \text{ in.})^2\pi$$
$$= 64\pi \text{ in}^2$$

8 in

8. Find the volume of the cone shown below.

$$V = \left(\frac{1}{3}\right)(64\pi \text{ in}^2)(12 \text{ in.})$$
$$= 256\pi \text{ in}^3$$

8 in
12 in

9. Find the area of the circle shown below.

$$A = (6 \text{ mm})^2\pi$$
$$= 36\pi \text{ mm}^2$$

6 mm

10. Find the volume of the sphere shown below.

$$V = \left(\frac{4}{3}\right)\pi(6 \text{ mm})^3$$
$$= \frac{864 \text{ mm}^3}{3}\pi$$
$$= 288\pi \text{ mm}^3$$

6 mm
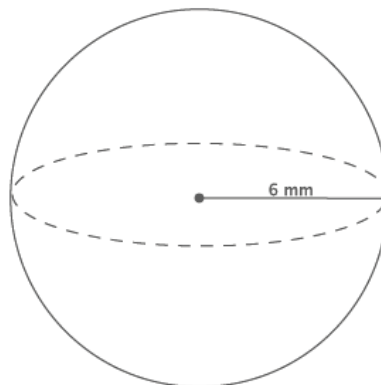
Topic D

Applications of Radicals and Roots

Focus Standards	■	Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
	■	Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.
Instructional Days: 5		
Lesson 19:	Cones and Spheres (P)[1]	
Lesson 20:	Truncated Cones (P)	
Lesson 21:	Volume of Composite Solids (E)	
Lesson 22:	Average Rate of Change (S)	
Lesson 23:	Nonlinear Motion (M)	

In Lesson 19, students use the Pythagorean theorem to determine the height, lateral length (slant height), or radius of the base of a cone. Students also use the Pythagorean theorem to determine the radius of a sphere given the length of a cord. Many problems in Lesson 19 also require students to use the height, length, or radius they determined using the Pythagorean theorem to then find the volume of a figure. In Lesson 20, students learn that the volume of a truncated cone can be determined using facts about similar triangles. Specifically, the fact that corresponding parts of similar triangles are equal in ratio is used to determine the height of the part of the cone that has been removed to make the truncated cone. Then, students calculate the volume of the whole cone (i.e., removed part and truncated part) and subtract the volume of the removed portion to determine the volume of the truncated cone. In this lesson, students learn that the formula to determine the volume of a pyramid is analogous to that of a cone. That is, the volume of a pyramid is exactly one-third the volume of a rectangular prism with the same base area and height. In Lesson 21, students determine the volume of solids comprised of cylinders, cones, spheres, and combinations of those figures as composite solids. Students consistently link their understanding of expressions (numerical and algebraic) to the volumes they represent. In Lesson 22, students apply their knowledge of volume to

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[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

compute the average rate of change in the height of the water level when water drains into a conical container. Students bring together much of what they have learned in Grade 8, such as Pythagorean theorem, volume of solids, similarity, constant rate, and rate of change, to work on challenging problems in Lessons 22 and 23. The optional modeling lesson, Lesson 23, challenges students with a problem about nonlinear motion. In describing the motion of a ladder sliding down a wall, students bring together concepts of exponents, roots, average speed, constant rate, functions, and the Pythagorean theorem. Throughout the lesson, students are challenged to reason abstractly and quantitatively while making sense of problems, applying their knowledge of concepts learned throughout the year to persevere in solving them.

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Lesson 19: Cones and Spheres

Student Outcomes

▪ Students use the Pythagorean theorem to determine an unknown dimension of a cone or a sphere.

▪ Students know that a pyramid is a special type of cone with triangular faces and a polygonal base.

▪ Students know how to use the lateral length of a cone and the length of a chord of a sphere to solve problems related to volume.

Classwork

Exercises 1–2 (5 minutes)

Students complete Exercises 1–2 individually. The purpose of these exercises is for students to perform computations that may help them relate the volume formula for a pyramid to the volume formula for a right circular cone, introduced in Module 5. Their response to part (b) of Exercise 2 is the starting point of the discussion that follows.

Exercises 1–2

Note: Figures not drawn to scale.

1. **Determine the volume for each figure below.**

   a. **Write an expression that shows volume in terms of the area of the base, B, and the height of the figure. Explain the meaning of the expression, and then use it to determine the volume of the figure.**

Scaffolding:

Printable nets, located at the end of the lesson, can be used to create 3-D models for Exercises 1–2.

$$V = Bh$$
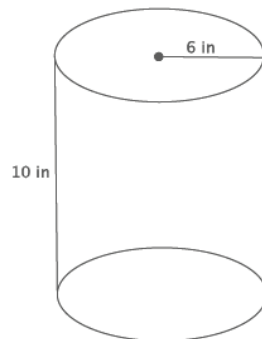
The expression $V = Bh$ means that the volume of the cylinder is found by multiplying the area of the base by the height. The base is a circle whose area can be found by squaring the radius, 6 in., and then multiplying by π. The volume is found by multiplying that area by the height of 10.

$$V = \pi(6)^2(10)$$
$$= 360\pi$$

The volume of the cylinder is 360π in^3.

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b.   Write an expression that shows volume in terms of the area of the base, B, and the height of the figure. Explain the meaning of the expression, and then use it to determine the volume of the figure.
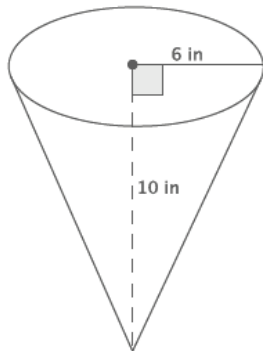
$$V = \frac{1}{3}Bh$$

The expression $V = \frac{1}{3}Bh$ means that the volume of the cone is found by multiplying the area of the base by the height and then taking one-third of that product. The base is a circle whose area can be found by squaring the radius, 6 in., and then multiplying by π. The volume is found by multiplying that area by the height of 10 in. and then taking one-third of that product.

$$V = \frac{1}{3}\pi(6)^2(10)$$
$$= \frac{360}{3}\pi$$
$$= 120\,\pi$$

The volume of the cone is 120π in³.

2.

a.   Write an expression that shows volume in terms of the area of the base, B, and the height of the figure. Explain the meaning of the expression, and then use it to determine the volume of the figure.

$$V = Bh$$
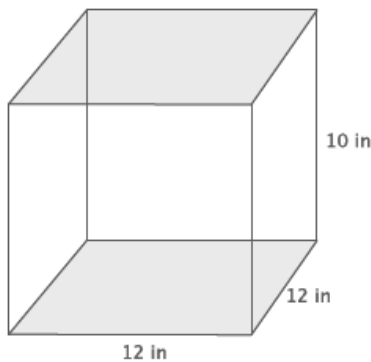
The expression $V = Bh$ means that the volume of the prism is found by multiplying the area of the base by the height. The base is a square whose area can be found by multiplying 12×12. The volume is found by multiplying that area, 144, by the height of 10.

$$V = 12(12)(10)$$
$$= 1,440$$

The volume of the prism is $1,440$ in³.

b.   The volume of the square pyramid shown below is 480 in³. What might be a reasonable guess for the formula for the volume of a pyramid? What makes you suggest your particular guess?
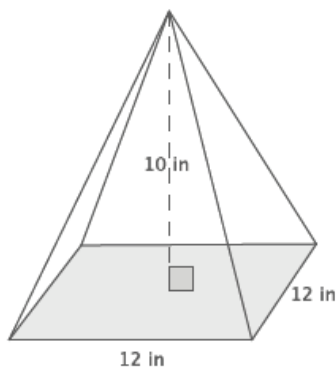
Since $480 = \frac{1440}{3}$, the formula to find the volume of a pyramid is likely $\frac{1}{3}Bh$, where B is the area of the base. This is similar to the volume of a cone compared to the volume of a cylinder with the same base and height. The volume of a square pyramid is $\frac{1}{3}$ of the volume of the rectangular prism with the same base and height.

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Discussion (5 minutes)

▪   What do you think the formula to find the volume of a pyramid is? Explain.

Ask students to share their response to part (b) of Exercise 2. If students do not see the connection between cones and cylinders to pyramids and prisms, then use the discussion points below.

▪   A pyramid is similar to a cone, but a pyramid has a polygonal base and faces that are shaped like triangles.
      For now we focus on right square pyramids only, that is, pyramids that have a base that is a square.
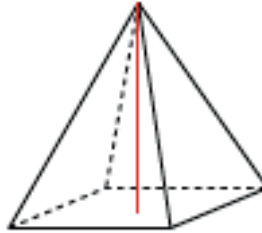
▪   The relationship between a cone and cylinder is similar for pyramids and prisms. How are the volumes of
      cones and cylinders related?

      ▫   *A cone is one-third the volume of a cylinder with the same base and height.*

▪   In general, we say that the volume of a cylinder is $V = Bh$, where B is the area of the base. Then the volume
      of a cone is $V = \frac{1}{3}Bh$, again where B is the area of the base.

▪   How do you think the volumes of rectangular pyramids and rectangular prisms are related?

      ▫   *The volume of a rectangular pyramid is one-third the volume of a rectangular prism with the same base*
            and height.

▪   In general, the volume of a rectangular prism is $V = Bh$, where B is the area of the base. Then the volume of a
      pyramid is $V = \frac{1}{3}Bh$, again where B is the area of the base.

Example 1

Example 1

State as many facts as you can about a cone.
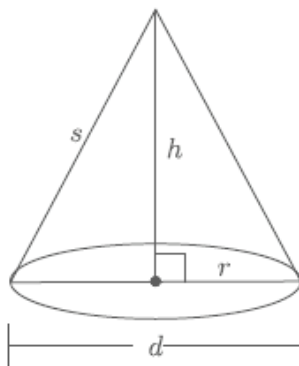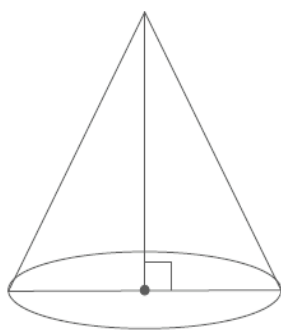
Area of the base:

$$A = \pi(r^2)$$

Circumference of the base:

$$C = 2\pi(r) = \pi(d)$$

Volume of the cone:

$$V = \frac{1}{3}\pi(r^2)h$$

Provide students with a minute or two to discuss as many facts as they can about a right circular cone, and then have them share their facts with the class. As they identify parts of the cone and facts about the cone, label the drawing above. Students should be able to state or identify the following: radius, diameter, height, base, area of a circle is $A = \pi r^2$, circumference of a circle is $C = 2\pi r = \pi d$, and the volume of a cone is $V = \frac{1}{3}Bh$, where B is the area of the base.

- ▪ What part of the cone have we not identified?
  - ▫ *The slanted part of the cone*
- ▪ The slanted part of the cone is known as the lateral length, which is also referred to as the slant height. We denote the lateral length of a cone by s.

Label the lateral length of the cone with s on the drawing on the previous page.

- ▪ Now that we know about the lateral length of a cone, we can begin using it in our work.

Exercises 3–6 (9 minutes)

Students work in pairs to complete Exercises 3–6. Students may need assistance determining the dimensions of the various parts of a cone. Let students reason through it first, offering guidance if necessary. Consider allowing students to use a calculator to approximate their answers or allow students to leave their answers as square roots, so as not to distract from the goal of the lesson. As needed, continue to remind students that we need only consider the positive square root of a number when our context involves length.

Exercises 3–10

3.   **What is the lateral length (slant height) of the cone shown below?**
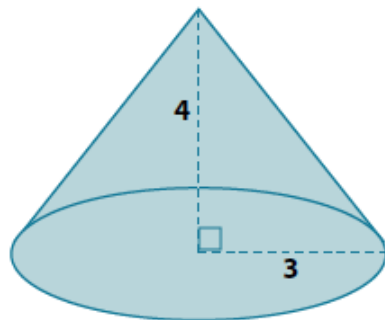
Let c be the lateral length.

$$3^2 + 4^2 = c^2$$
$$9 + 16 = c^2$$
$$25 = c^2$$
$$\sqrt{25} = \sqrt{c^2}$$
$$5 = c$$

The lateral length of the cone is 5 units.

4.   **Determine the exact volume of the cone shown below.**
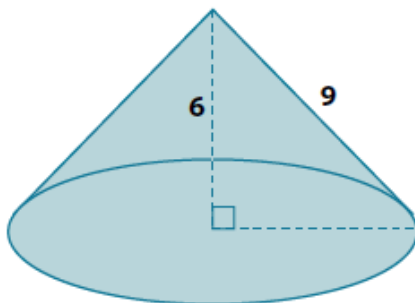
Let r be the radius of the base.

$$6^2 + r^2 = 9^2$$
$$36 + r^2 = 81$$
$$r^2 = 45$$

The area of the base is 45π units^2.

$$V = \frac{1}{3}Bh$$
$$V = \frac{1}{3}(45)\pi(6)$$
$$V = 90\pi$$

The volume of the cone is 90π units^3.

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5. What is the lateral length (slant height) of the pyramid shown below? Give an exact square root answer and an approximate answer rounded to the tenths place.
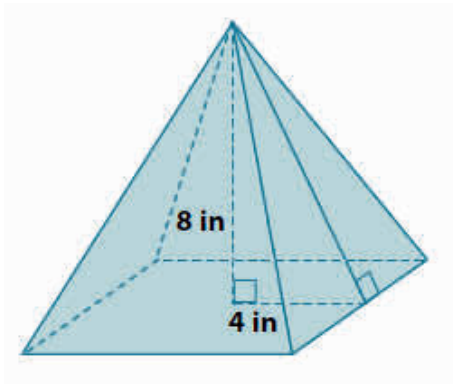
Let c in. represent the lateral length of the pyramid.

$$4^2 + 8^2 = c^2$$
$$16 + 64 = c^2$$
$$80 = c^2$$
$$\sqrt{80} = \sqrt{c^2}$$
$$\sqrt{80} = c$$

The number $\sqrt{80}$ is between 8 and 9. In the sequence of tenths, it is between 8.9 and 9.0. Since 80 is closer to 8.9^2 than 9^2, the approximate lateral length is 8.9 inches.

8 in

4 in

6. Determine the volume of the square pyramid shown below. Give an exact answer using a square root.
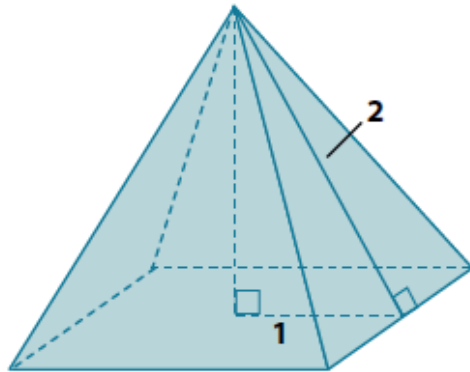
Let h be the height of the pyramid.

$$1^2 + h^2 = 2^2$$
$$1 + h^2 = 4$$
$$h^2 = 3$$
$$\sqrt{h^2} = \sqrt{3}$$
$$h = \sqrt{3}$$

The area of the base is 4 units^2.

$$V = \frac{1}{3}(4)(\sqrt{3})$$
$$= \frac{4\sqrt{3}}{3}$$

2

1

The volume of the pyramid is $\dfrac{4\sqrt{3}}{3}$ units^3.

Discussion (7 minutes)

- Let O be the center of a circle, and let P and Q be two points on the circle as shown. Then $\overline{PQ}$ is called a chord of the circle.
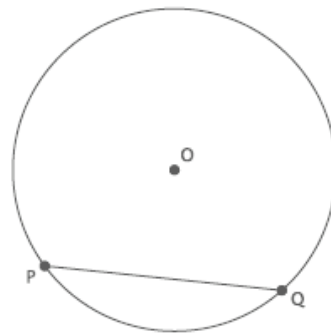
- What do you notice about the lengths $|OP|$ and $|OQ|$?
    - *Both lengths are equal to the radius, r, of the circle, which means they are equal in length to each other.*
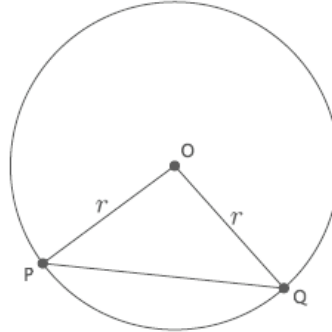
- Will lengths $|OP|$ and $|OQ|$ always be equal to r, no matter where the chord is drawn?

Provide students time to place points P and Q around the circle to get an idea that no matter where the endpoints of the chord are placed, the length from the center of the circle to each of those points is always equal to r. The reason is based on the definition of a chord. Points P and Q must lie on the circle in order for $\overline{PQ}$ to be identified as a chord.

- When the angle $\angle POQ$ is a right angle, we can use the Pythagorean theorem to determine the length of the chord given the length of the radius; or, if we know the length of the chord, we can determine the length of the radius.

- Similarly, when points P and Q are on the surface of a sphere, the segment that connects them is called a chord.
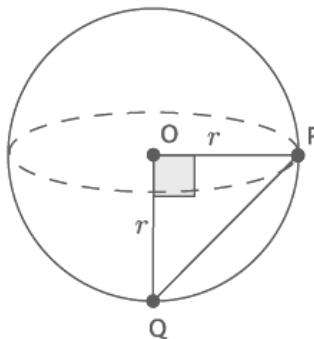
- Just like with circles, if the angle formed by POQ is a right angle, then we can use the Pythagorean theorem to find the length of the chord if we are given the length of the radius; or, given the length of the chord, we can determine the radius of the sphere.

Exercises 7–10 (9 minutes)

Students work in pairs to complete Exercises 7–10. Consider allowing students to use their calculators or to leave their answers as square roots (simplified square roots if that lesson was used with students), but not approximated, so as not to distract from the goal of the lesson.

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7. **What is the length of the chord of the sphere shown below? Give an exact answer using a square root.**

11 cm
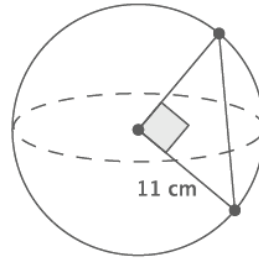
Let c cm *represent the length of the chord.*

$$11^2 + 11^2 = c^2$$
$$121 + 121 = c^2$$
$$242 = c^2$$
$$\sqrt{242} = \sqrt{c^2}$$
$$\sqrt{11^2 \times 2} = c$$
$$11\sqrt{2} = c$$

The length of the chord is $\sqrt{242}$ cm, or $11\sqrt{2}$ cm.

8. **What is the length of the chord of the sphere shown below? Give an exact answer using a square root.**

Let c in. *represent the length of the chord.*

4 in
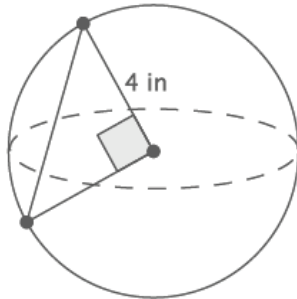
$$4^2 + 4^2 = c^2$$
$$16 + 16 = c^2$$
$$32 = c^2$$
$$\sqrt{32} = \sqrt{c^2}$$
$$\sqrt{4^2 \times 2} = c$$
$$4\sqrt{2} = c$$

The length of the chord is $\sqrt{32}$ in., or $4\sqrt{2}$ in.

9. **What is the volume of the sphere shown below? Give an exact answer using a square root.**
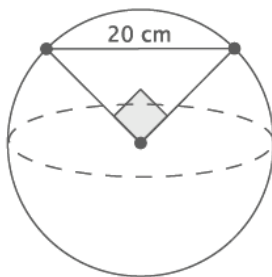
Let r cm *represent the radius of the sphere.*

20 cm

$$r^2 + r^2 = 20^2$$
$$2r^2 = 400$$
$$r^2 = 200$$
$$\sqrt{r^2} = \sqrt{200}$$
$$r = \sqrt{200}$$
$$r = \sqrt{10^2 \times 2}$$
$$r = 10\sqrt{2}$$

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(10\sqrt{2})^3$$
$$= \frac{4}{3}\pi(10^3)(\sqrt{2})^3$$
$$= \frac{4}{3}\pi(1000)(\sqrt{8})$$
$$= \frac{4}{3}\pi(1000)\left(\sqrt{2^2 \times 2}\right)$$
$$= \frac{4}{3}\pi(1000)(2)(\sqrt{2})$$
$$= \frac{8000\sqrt{2}}{3}\pi$$

The volume of the sphere is $\frac{4000\sqrt{8}}{3}\pi$ cm^3, or $\frac{8000\sqrt{2}}{3}\pi$ cm^3.

10. **What is the volume of the sphere shown below? Give an exact answer using a square root.**

12 mm
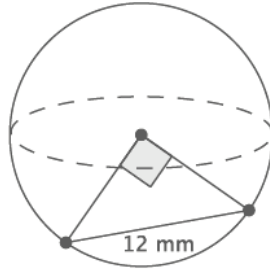
Let r mm represent the radius of the sphere.

$$r^2 + r^2 = 12^2$$
$$2r^2 = 144$$
$$r^2 = 72$$
$$\sqrt{r^2} = \sqrt{72}$$
$$r = \sqrt{6^2 \times 2}$$
$$r = 6\sqrt{2}$$

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi\left(6\sqrt{2}\right)^3$$
$$= \frac{4}{3}\pi(6^3)\left(\sqrt{2}\right)^3$$
$$= \frac{4}{3}\pi(216)(\sqrt{8})$$
$$= \frac{4}{3}\pi(216)\left(\sqrt{2^2 \times 2}\right)$$
$$= \frac{4}{3}\pi(216)(2)(\sqrt{2})$$
$$= \frac{1728\sqrt{2}}{3}\pi$$
$$= 576\sqrt{2}\pi$$

The volume of the sphere is $576\sqrt{2}\pi$ mm^3.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- The volume formulas for rectangular pyramids and rectangular prisms are similar to those of cones and cylinders.

- The formula to determine the volume of a pyramid is $\frac{1}{3}Bh$, where B is the area of the base. This is similar to the formula to determine the volume of a cone.

- The segment formed by two points on a circle is called a chord.

- We know how to apply the Pythagorean theorem to cones and spheres to determine volume.

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Lesson Summary

The volume formula for a right square pyramid is $V = \frac{1}{3}Bh$, where B is the area of the square base.

The lateral length of a cone, sometimes referred to as the slant height, is the side s, shown in the diagram below.
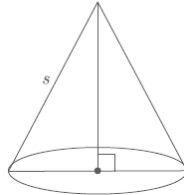
Given the lateral length and the length of the radius, the Pythagorean theorem can be used to determine the height of the cone.

Let O be the center of a circle, and let P and Q be two points on the circle. Then $\overline{PQ}$ is called a chord of the circle.
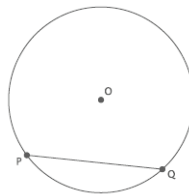
The segments OP and OQ are equal in length because both represent the radius of the circle. If the angle formed by POQ is a right angle, then the Pythagorean theorem can be used to determine the length of the radius when given the length of the chord, or the length of the chord can be determined if given the length of the radius.

Exit Ticket (5 minutes)

Name _____   Date _____

Lesson 19: Cones and Spheres

Exit Ticket

Which has the larger volume? Give an approximate answer rounded to the tenths place.

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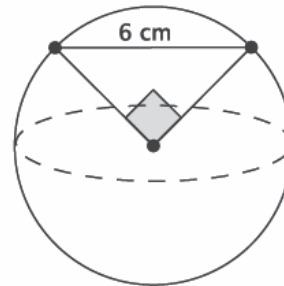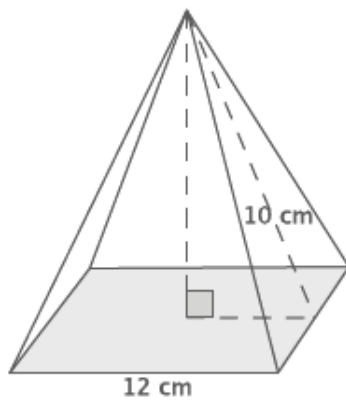
Exit Ticket Sample Solutions

Which has the larger volume? Give an approximate answer rounded to the tenths place.
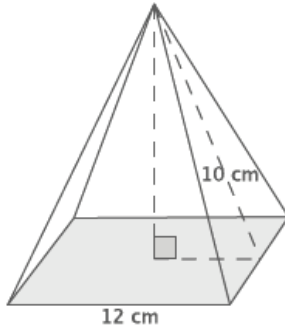
Let h cm represent the height of the square pyramid.

$$h^2 + 6^2 = 10^2$$
$$h^2 + 36 = 100$$
$$h^2 = 64$$
$$h = 8$$

$$V = \frac{1}{3}(12^2)(8)$$

$$V = \frac{1}{3}(144)(8)$$

$$V = 384$$

The volume of the square pyramid is 384 cm^3.
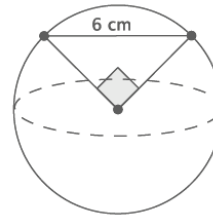
Let r represent the radius of the sphere in centimeters.

$$r^2 + r^2 = 6^2$$
$$2r^2 = 36$$
$$r^2 = 18$$
$$\sqrt{r^2} = \sqrt{18}$$
$$r = \sqrt{3^2 \times 2}$$
$$r = 3\sqrt{2}$$

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(3\sqrt{2})^3$$
$$= \frac{4}{3}\pi(3)^3(\sqrt{2})^3$$
$$= \frac{4}{3}\pi(27)(\sqrt{8})$$
$$= \frac{4}{3}\pi(27)\left(\sqrt{2^2 \times 2}\right)$$
$$= \frac{4}{3}\pi(27)(2)(\sqrt{2})$$
$$= 72\pi\sqrt{2}$$

The volume of the sphere is $72\pi\sqrt{2}$ cm^3.

The number $\sqrt{2}$ is between 1 and 2. In the sequence of tenths, it is between 1.4 and 1.5. Since 2 is closer to 1.4^2 than 1.5^2, the number is approximately 1.4.

We know from previous lessons we can estimate $\pi = 3.14$.

Then, we can calculate the approximate volume of the sphere:

$$V \approx (72)(1.4)(3.14)$$
$$V \approx 316.512$$

The approximate volume of the sphere is 316.512 cm^3. Therefore, the volume of the square pyramid is greater.

Problem Set Sample Solutions

Students use the Pythagorean theorem to solve mathematical problems in three dimensions.

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1. What is the lateral length (slant height) of the cone shown below? Give an approximate answer rounded to the tenths place.
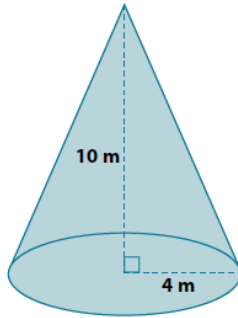
    Let c m be the lateral length.

    $$10^2 + 4^2 = c^2$$
    $$100 + 16 = c^2$$
    $$116 = c^2$$
    $$\sqrt{116} = \sqrt{c^2}$$
    $$\sqrt{116} = c$$

    The number $\sqrt{116}$ is between 10 and 11. In the sequence of tenths, it is between 10.7 and 10.8. Since 116 is closer to 10.8^2 than 10.7^2, the approximate value of the number is 10.8.

    The lateral length of the cone is approximately 10.8 m.

    10 m

    4 m

2. What is the volume of the cone shown below? Give an exact answer.
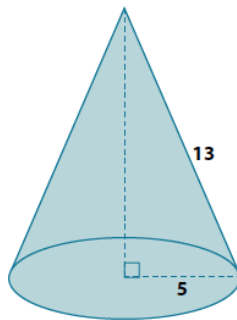
    Let h represent the height of a cone.

    $$5^2 + h^2 = 13^2$$
    $$25 + h^2 = 169$$
    $$h^2 = 144$$
    $$\sqrt{h^2} = \sqrt{144}$$
    $$h = 12$$

    The height of the cone is 12 units.

    $$V = \frac{1}{3}\pi(25)(12)$$
    $$= 100\pi$$

    The volume of the cone is 100π units^3.

    13

    5

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3. Determine the volume and surface area of the square pyramid shown below. Give exact answers.
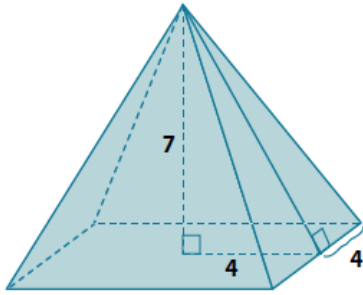
$$V = \frac{1}{3}(64)(7)$$
$$= \frac{448}{3}$$

The volume of the pyramid is $\frac{448}{3}$ units^3.

Let c represent the lateral length.

$$7^2 + 4^2 = c^2$$
$$49 + 16 = c^2$$
$$65 = c^2$$
$$\sqrt{65} = \sqrt{c^2}$$
$$\sqrt{65} = c$$

The area of each face of the pyramid is $4\sqrt{65}$ units^3 (since $\frac{1}{2} \times 8 \times \sqrt{65} = 4\sqrt{65}$), so the area of all four faces is $16\sqrt{65}$ units^2. Since the base area is 64 units^2, the total surface area of the pyramid is $(64 + 16\sqrt{65}$ units$^2)$.

4. Alejandra computed the volume of the cone shown below as 64π cm^3. Her work is shown below. Is she correct? If not, explain what she did wrong, and calculate the correct volume of the cone. Give an exact answer.
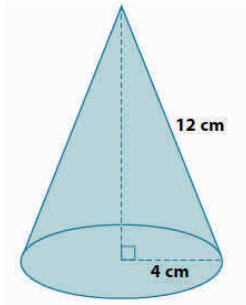
$$V = \frac{1}{3}\pi(4^2)(12)$$
$$= \frac{(16)(12)\pi}{3}$$
$$= 64\pi$$

The volume of the cone is 64π cm^3.

Alejandra's work is incorrect. She used the lateral length instead of the height of the cone to compute volume.

Let h cm represent the height.

$$4^2 + h^2 = 12^2$$
$$16 + h^2 = 144$$
$$h^2 = 128$$
$$\sqrt{h^2} = \sqrt{128}$$
$$h = \sqrt{128}$$
$$h = \sqrt{8^2 \times 2}$$
$$h = 8\sqrt{2}$$

$$V = \frac{1}{3}\pi(4)^2(8\sqrt{2})$$
$$V = \frac{1}{3}\pi(128\sqrt{2})$$
$$V = \frac{128\sqrt{2}}{3}\pi$$

The volume of the cone is $\frac{128\sqrt{2}}{3}\pi$ cm^3.

5.   What is the length of the chord of the sphere shown below? Give an exact answer using a square root.

Let c m represent the length of the chord.

$$9^2 + 9^2 = c^2$$
$$81 + 81 = c^2$$
$$162 = c^2$$
$$\sqrt{162} = \sqrt{c^2}$$
$$\sqrt{162} = c$$
$$\sqrt{9^2 \times 2} = c$$
$$9\sqrt{2} = c$$
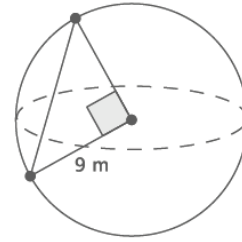
The length of the chord is $\sqrt{162}$ m, or $9\sqrt{2}$ m.

6.   What is the volume of the sphere shown below? Give an exact answer using a square root.

Let r in. represent the radius.

$$r^2 + r^2 = 14^2$$
$$2r^2 = 196$$
$$r^2 = 98$$
$$\sqrt{r^2} = \sqrt{98}$$
$$r = \sqrt{7^2 \times 2}$$
$$r = 7\sqrt{2}$$

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi\left(7\sqrt{2}\right)^3$$
$$= \frac{4}{3}\pi(343)\left(\sqrt{8}\right)$$
$$= \frac{4}{3}\pi(343)\left(2\sqrt{2}\right)$$
$$= \frac{2744\sqrt{2}}{3}\pi$$
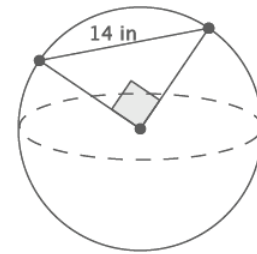
The volume of the sphere is $\frac{4}{3}\left(\sqrt{98}\right)^3 \pi$ in^3, or $\frac{2744\sqrt{2}}{3}\pi$ in^3.

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Lesson 19:   Cones and Spheres

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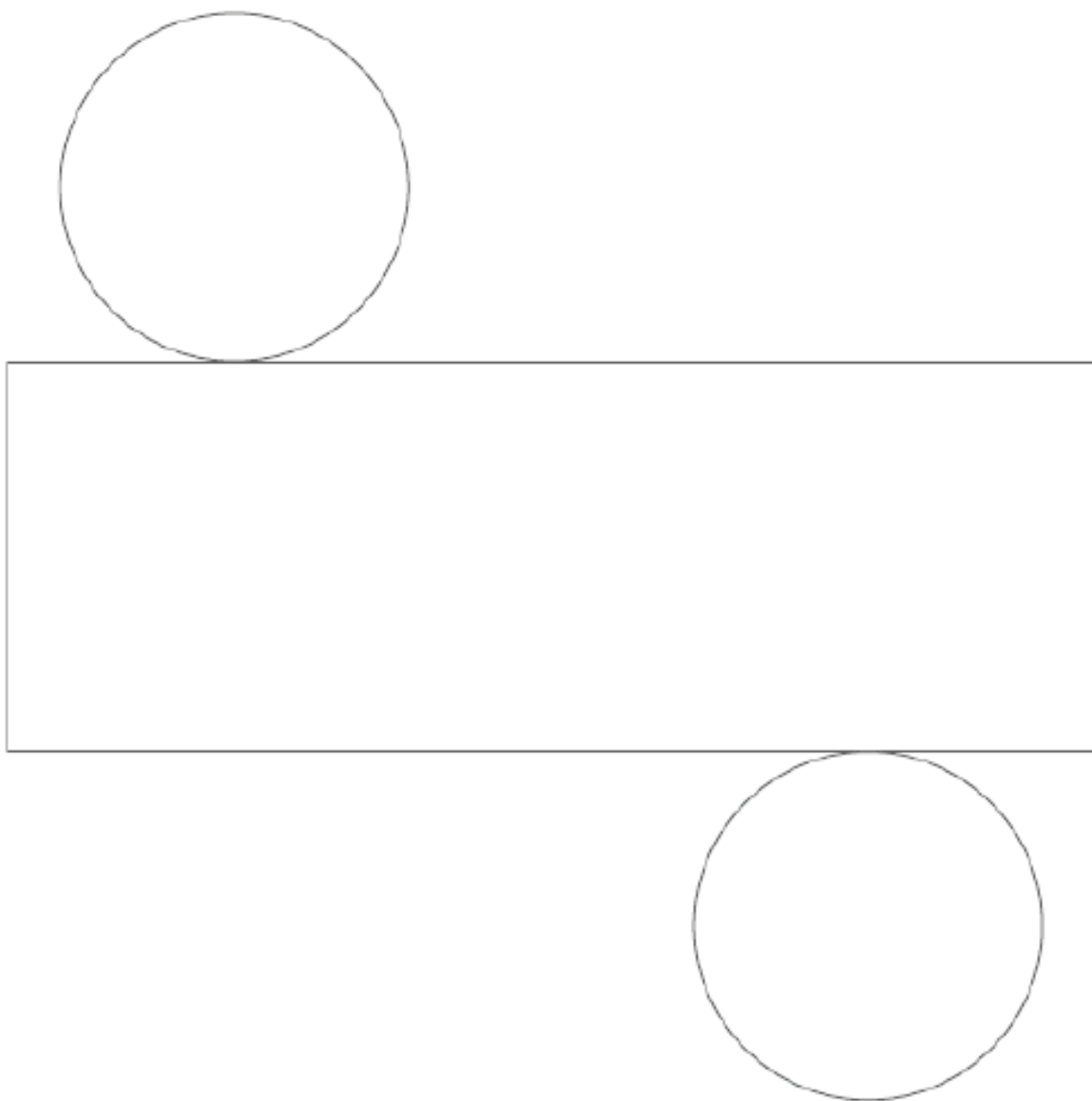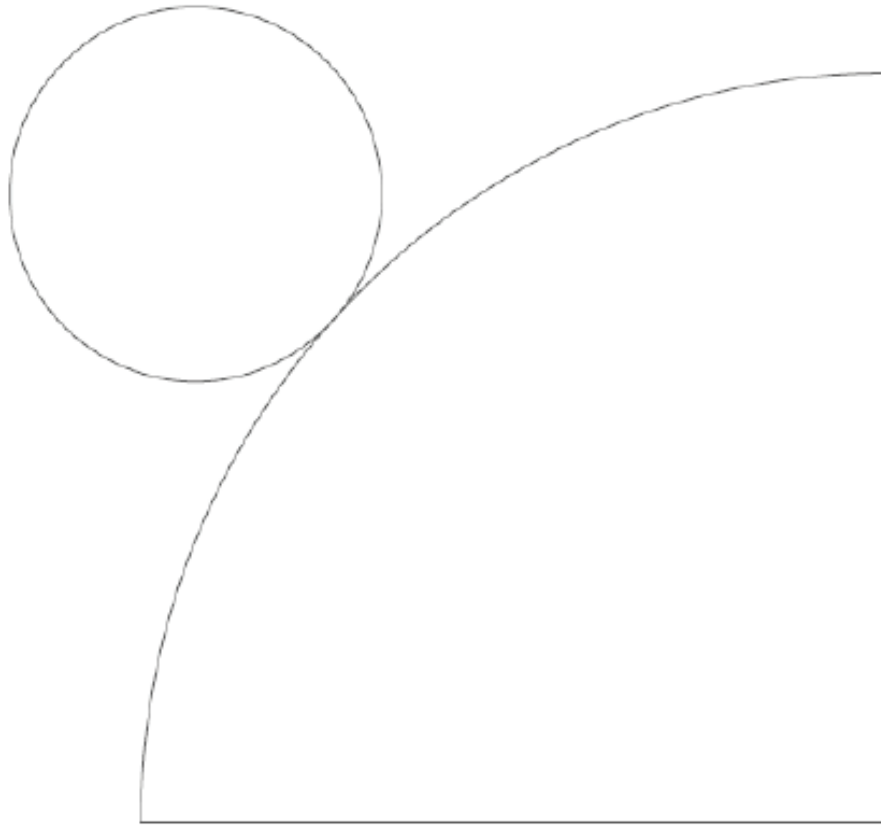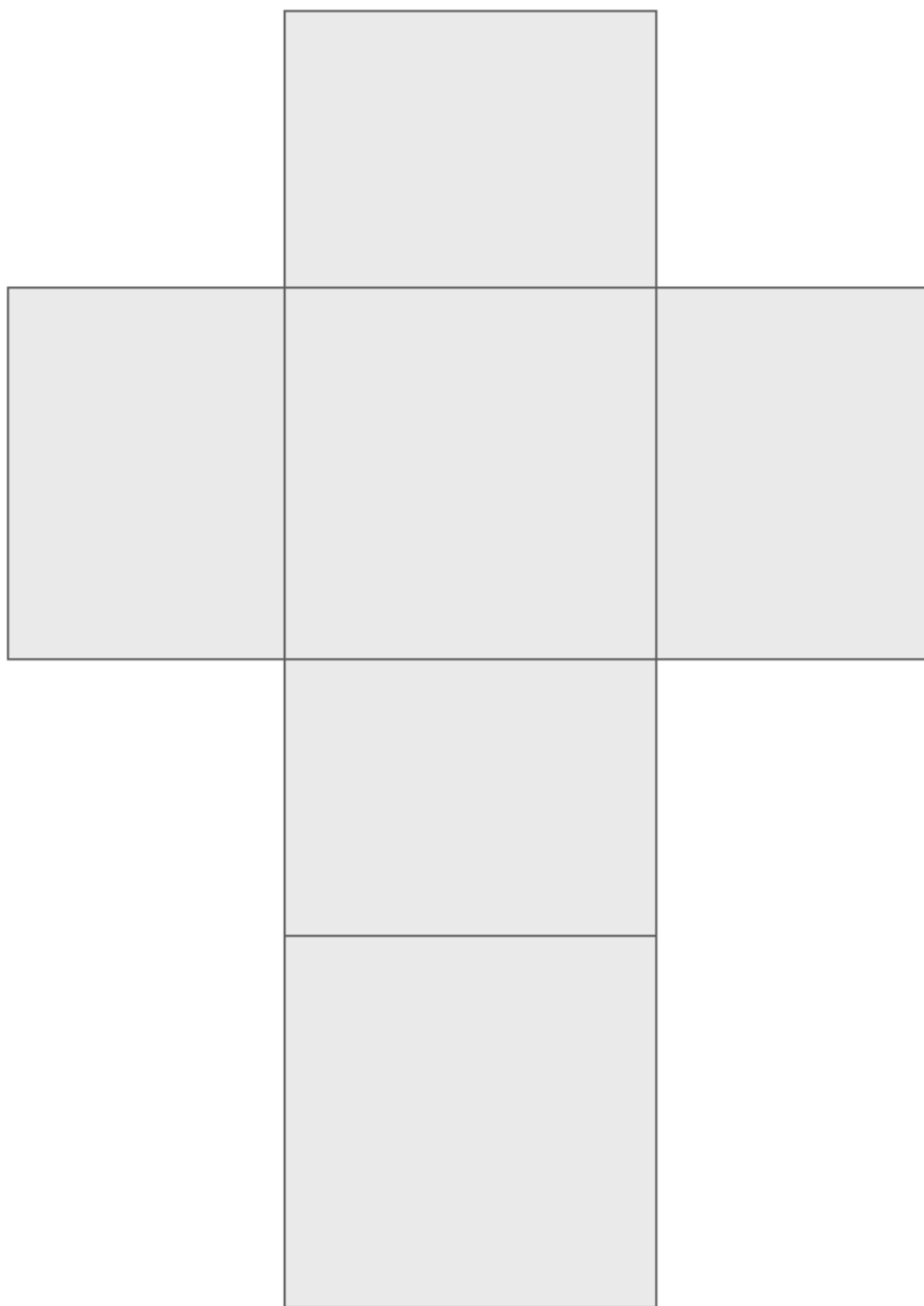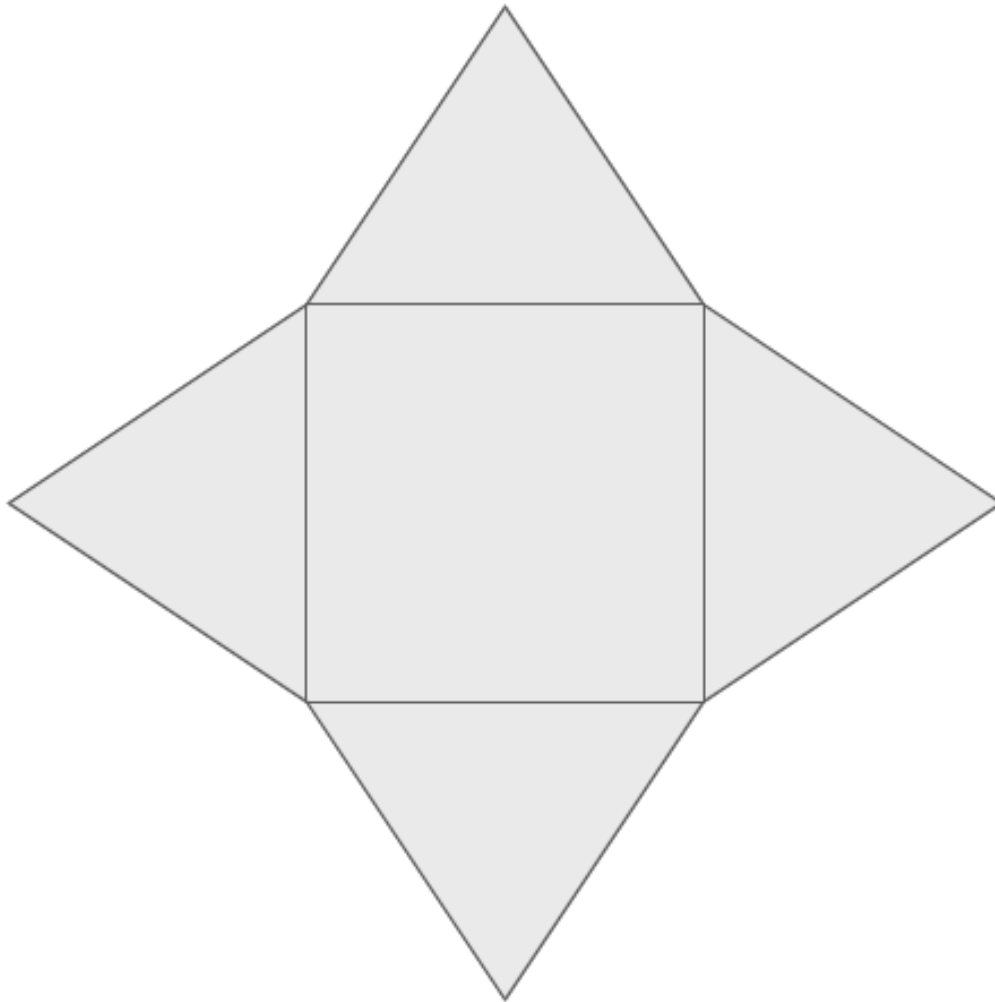
Lesson 20: Truncated Cones

Student Outcomes

- Students know that a truncated cone or pyramid is the solid obtained by removing the top portion of a cone or a pyramid above a plane parallel to its base.
- Students find the volume of truncated cones.

Lesson Notes

Finding the volume of a truncated cone is not explicitly stated as part of the eighth-grade content; however, finding the volume of a truncated cone combines two major skills learned in this grade, specifically, understanding similar triangles and their properties and calculating the volume of a cone. This topic is included because it provides an application of seemingly unrelated concepts. Furthermore, it allows students to see how learning one concept, similar triangles and their properties, can be applied to three-dimensional figures. Teaching this concept also reinforces students' understanding of similar triangles and how to determine unknown lengths of similar triangles.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

Examine the bucket below. It has a height of 9 inches and a radius at the top of the bucket of 4 inches.

    a.   **Describe the shape of the bucket. What is it similar to?**

    b.   **Estimate the volume of the bucket.**

Discussion (10 minutes)

Before beginning the discussion, have students share their thoughts about the Opening Exercise. Students may say that the bucket is cone-shaped but not a cone or that it is cylinder-shaped but tapered. Any estimate between 48π in^3 (the volume of a cone with the given dimensions) and 144π in^3 (the volume of a cylinder with the given dimensions) is reasonable. Then, continue with the discussion below.

- When the top, narrower portion of a cone is removed such that the base of the removed portion is *parallel* to the existing base, the resulting shape is what we call a truncated cone.

   Here we have a cone:

   Here we have a truncated cone:
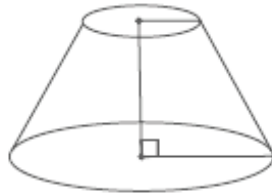
   What is the shape of the removed portion?

   - *The removed portion of the figure will look like a cone. It will be a cone that is smaller than the original.*

- Here is the cone and the part that has been removed together in one drawing:
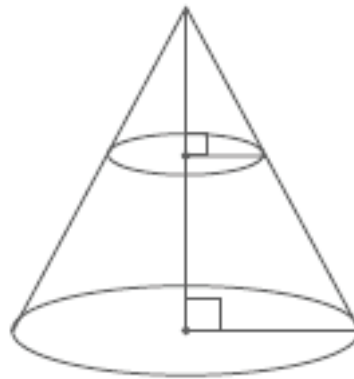
   Do you think the right triangles shown in the diagram are similar? Explain how you know.

Give students time to discuss the answer in groups, and then have them share their reasoning as to why the triangles are similar.

   - *Yes, the triangles are similar. Mark the top of the cone point O. Then, a dilation from O by scale factor r would map one triangle onto another. We also know that the triangles are similar because of the AA criterion. Each triangle has a right angle, and they have a common angle at the top of the cone (from the center of dilation).*

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- What does that mean about the lengths of the legs and the hypotenuse of each right triangle?
  - *It means that the corresponding side lengths will be equal in ratio.*
- We will use all of these facts to help us determine the volume of a truncated cone.

Example 1 (10 minutes)

- Our goal is to determine the volume of the truncated cone shown below. Discuss in your groups how we might be able to do that.

Example 1

Determine the volume of the truncated cone shown below.
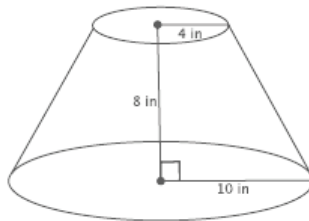
Provide students time to discuss in groups a strategy for finding the volume of the truncated cone. Use the discussion questions below to guide their thinking as needed.

- Since we know that the original cone and the portion that has been removed to make this truncated cone are similar, let's begin by drawing in the missing portion.
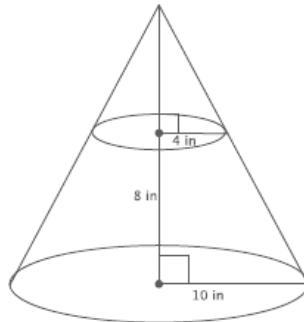
- We know the formula to find the volume of a cone. Is there enough information in the new diagram for us to find the volume? Explain.
  - *No, there's not enough information. We would have to know the height of the cone, and at this point we only know the height of the truncated cone, 8 inches.*

- Recall our conversation about the similar right triangles. We can use what we know about similarity to determine the height of the cone with the following proportion. What does each part of the proportion represent in the diagram?

  □ *Since the triangles are similar, we will let x inches represent the height of the cone that has been removed.*

  $$\frac{4}{10} = \frac{x}{x+8}$$

  The 4 is the radius of the small cone. The 10 is the radius of the large cone. The x represents the height of the small cone. The expression $x + 8$ represents the height of the large cone.

- Work in your groups to determine the height of the small cone.

  □ $4(x+8) = 10x$

  $4x + 32 = 10x$

  $32 = 6x$

  $\dfrac{32}{6} = x$

  $5.\overline{3} = x$

- Now that we know the height of the cone that has been removed, we also know the total height of the cone. How might we use these pieces of information to determine the volume of the truncated cone?

  □ *We can find the volume of the large cone, find the volume of the small cone that was removed, and then subtract the volumes. What will be left is the volume of the truncated cone.*

- Write an expression that represents the volume of the truncated cone. Use approximations for the heights since both are infinite decimals. Be prepared to explain what each part of the expression represents in the situation.

  □ *The volume of the truncated cone is given by the expression*

  $$\frac{1}{3}\pi(10)^2(13.3) - \frac{1}{3}\pi(4)^2(5.3),$$

  where $\frac{1}{3}\pi(10)^2(13.3)$ is the volume of the large cone, and $\frac{1}{3}\pi(4)^2(5.3)$ is the volume of the smaller cone. The difference in the volumes will be the volume of the truncated cone.

- Determine the volume of the truncated cone. Use the approximate value of the number $5.\overline{3}$ when you compute the volumes.

  □ *The volume of the small cone is*

  $$V \approx \frac{1}{3}\pi(4)^2(5.3)$$

  $$\approx \frac{1}{3}\pi(84.8)$$

  $$\approx \frac{84.8}{3}\pi.$$

  The volume of the large cone is

  $$V \approx \frac{1}{3}\pi(10)^2(13.3)$$

  $$\approx \frac{1330}{3}\pi.$$

The volume of the truncated cone is

$$\frac{1330}{3}\pi - \frac{84.8}{3}\pi = \left(\frac{1330}{3} - \frac{84.8}{3}\right)\pi$$
$$= \frac{1245.2}{3}\pi.$$

The volume of the truncated cone is approximately $\frac{1245.2}{3}\pi$ in³.

▪ Write an equivalent expression for the volume of a truncated cone that shows the volume is $\frac{1}{3}$ of the difference between two cylinders. Explain how your expression shows this.

▫ *The expression $\frac{1}{3}\pi(10)^2(13.3) - \frac{1}{3}\pi(4)^2(5.3)$ can be written as $\frac{1}{3}\left(\pi(10)^2(13.3) - \pi(4)^2(5.3)\right)$, where $\pi(10)^2(13.3)$ is the volume of the larger cylinder, and $\pi(4)^2(5.3)$ is the volume of the smaller cylinder. One-third of the difference is the volume of a truncated cone with the same base and height measurements as the cylinders.*

Exercises 1–5 (10 minutes)

Students work in pairs or small groups to complete Exercises 1–5.

Exercises 1–5

1. Find the volume of the truncated cone.
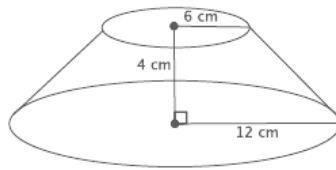
a. Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what all parts of the proportion represent.

$$\frac{6}{12} = \frac{x}{x+4}$$

Let x cm represent the height of the small cone. Then, $x + 4$ is the height of the large cone (with the removed part included). The 6 represents the base radius of the removed cone, and the 12 represents the base radius of the large cone.

b. Solve your proportion to determine the height of the cone that has been removed.

$$6(x + 4) = 12x$$
$$6x + 24 = 12x$$
$$24 = 6x$$
$$4 = x$$

c. Write an expression that can be used to determine the volume of the truncated cone. Explain what each part of the expression represents.

$$\frac{1}{3}\pi(12)^2(8) - \frac{1}{3}\pi(6)^2(4)$$

The expression $\frac{1}{3}\pi(12)^2(8)$ is the volume of the large cone, and $\frac{1}{3}\pi(6)^2(4)$ is the volume of the small cone. The difference of the volumes gives the volume of the truncated cone.

---

d. Calculate the volume of the truncated cone.

The volume of the small cone is   *The volume of the large cone is*

$$V = \frac{1}{3}\pi(6)^2(4)$$ $$V = \frac{1}{3}\pi(12)^2(8)$$
$$= \frac{144}{3}\pi.$$ $$= \frac{1152}{3}\pi.$$

The volume of the truncated cone is

$$\frac{1152}{3}\pi - \frac{144}{3}\pi = \left(\frac{1152}{3} - \frac{144}{3}\right)\pi$$
$$= \frac{1008}{3}\pi$$
$$= 336\pi.$$

The volume of the truncated cone is 336π cm^3.

2. Find the volume of the truncated cone.
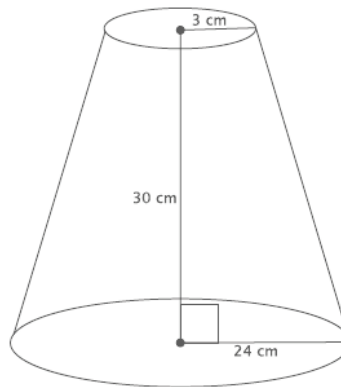
Let x cm represent the height of the small cone.

$$\frac{3}{24} = \frac{x}{x+30}$$
$$3x + 90 = 24x$$
$$90 = 21x$$
$$\frac{30}{7} = x$$
$$4.3 \approx x$$

The volume of the small cone is   *The volume of the large cone is*

$$V \approx \frac{1}{3}\pi(3)^2(4.3)$$ $$V \approx \frac{1}{3}\pi(24)^2(34.3)$$
$$\approx \frac{38.7}{3}\pi$$ $$\approx \frac{19756.8}{3}\pi$$
$$= 12.9\pi.$$ $$= 6585.6\pi.$$

The volume of the truncated cone is

$$6585.6\pi - 12.9\pi = (6585.6 - 12.9)\pi$$
$$= 6572.7\pi.$$

The volume of the truncated cone is approximately 6572.7π cm^3.
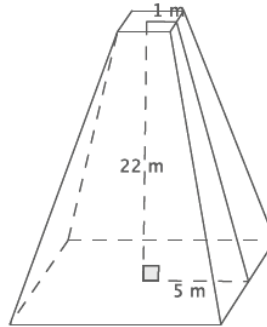
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3.   **Find the volume of the truncated pyramid with a square base.**

a.   **Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what all parts of the proportion represent.**

$$\frac{1}{5} = \frac{x}{x + 22}$$

Let x m represent the height of the small pyramid. Then $x + 22$ is the height of the large pyramid. The 1 represents half of the length of the base of the small pyramid, and the 5 represents half of the length of the base of the large pyramid.

b.   **Solve your proportion to determine the height of the pyramid that has been removed.**

$$x + 22 = 5x$$
$$22 = 4x$$
$$5.5 = x$$

c.   **Write an expression that can be used to determine the volume of the truncated pyramid. Explain what each part of the expression represents.**

$$\frac{1}{3}(100)(27.5) - \frac{1}{3}(4)(5.5)$$

The expression $\frac{1}{3}(100)(27.5)$ is the volume of the large pyramid, and $\frac{1}{3}(4)(5.5)$ is the volume of the small pyramid. The difference of the volumes gives the volume of the truncated pyramid.

d.   **Calculate the volume of the truncated pyramid.**

The volume of the small pyramid is

$$V = \frac{1}{3}(4)(5.5)$$
$$= \frac{22}{3}.$$

The volume of the large pyramid is

$$V = \frac{1}{3}(100)(27.5)$$
$$= \frac{2750}{3}.$$

The volume of the truncated pyramid is

$$\frac{2750}{3} - \frac{22}{3} = \frac{2728}{3}.$$

The volume of the truncated pyramid is $\frac{2728}{3}$ m^3.

4. A pastry bag is a tool used to decorate cakes and cupcakes. Pastry bags take the form of a truncated cone when filled with icing. What is the volume of a pastry bag with a height of 6 inches, large radius of 2 inches, and small radius of 0.5 inches?

Let x in. represent the height of the small cone.

$$\frac{x}{x+6} = \frac{0.5}{2}$$
$$2x = 0.5(x+6)$$
$$2x = \frac{1}{2}x + 3$$
$$\frac{3}{2}x = 3$$
$$x = 2$$

The volume of the small cone is

$$V = \frac{1}{3}\pi\left(\frac{1}{2}\right)^2(2)$$
$$= \frac{1}{6}\pi.$$

The volume of the large cone is

$$V = \frac{1}{3}\pi(2)^2(8)$$
$$= \frac{32}{3}\pi.$$

The volume of the truncated cone is

$$\frac{32}{3}\pi - \frac{1}{6}\pi = \left(\frac{32}{3} - \frac{1}{6}\right)\pi$$
$$= \frac{63}{6}\pi$$
$$= \frac{21}{2}\pi.$$

The volume of the pastry bag is $\frac{21}{2}\pi$ in^3 when filled.

5. Explain in your own words what a truncated cone is and how to determine its volume.

A truncated cone is a cone with a portion of the top cut off. The base of the portion that is cut off needs to be parallel to the base of the original cone. Since the portion that is cut off is in the shape of a cone, then to find the volume of a truncated cone, you must find the volume of the cone (without any portion cut off), find the volume of the cone that is cut off, and then find the difference between the two volumes. That difference is the volume of the truncated cone.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- A truncated cone or pyramid is the solid obtained by removing the top portion of a cone or a pyramid above a plane parallel to its base.
- Information about similar triangles can provide the information we need to determine the volume of a truncated figure.
- To find the volume of a truncated cone, first find the volume of the part of the cone that was removed, and then find the total volume of the cone. Finally, subtract the removed cone's volume from the total cone's volume. What is left over is the volume of the truncated cone.

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Lesson Summary

A truncated cone or pyramid is the solid obtained by removing the top portion of a cone or a pyramid above a plane parallel to its base. Shown below on the left is a truncated cone. A truncated cone with the top portion still attached is shown below on the right.
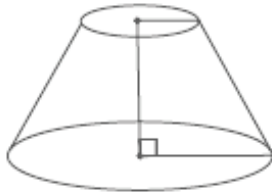
Truncated cone: **Truncated cone with top portion attached:**

To determine the volume of a truncated cone, you must first determine the height of the portion of the cone that has been removed using ratios that represent the corresponding sides of the right triangles. Next, determine the volume of the portion of the cone that has been removed and the volume of the truncated cone with the top portion attached. Finally, subtract the volume of the cone that represents the portion that has been removed from the complete cone. The difference represents the volume of the truncated cone.

Pictorially,

Exit Ticket (5 minutes)

Name _____   Date _____

Lesson 20: Truncated Cones

Exit Ticket

Find the volume of the truncated cone.

a.   Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what all parts of the proportion represent.
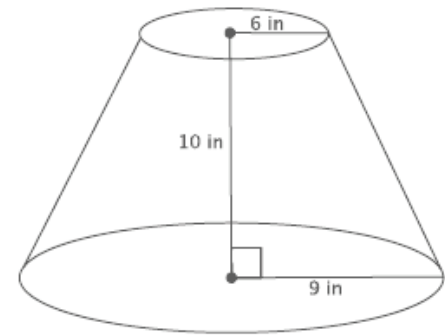
b.   Solve your proportion to determine the height of the cone that has been removed.

c.   Write an expression that can be used to determine the volume of the truncated cone. Explain what each part of the expression represents.

d.   Calculate the volume of the truncated cone.

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Exit Ticket Sample Solutions

Find the volume of the truncated cone.

a. Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what all parts of the proportion represent.

$$\frac{6}{9} = \frac{x}{x + 10}$$

Let x in. represent the height of the small cone. Then $x + 10$ is the height of the large cone. Then 6 is the base radius of the small cone, and 9 is the base radius of the large cone.
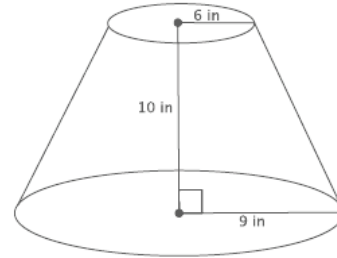
b. Solve your proportion to determine the height of the cone that has been removed.

$$6(x + 10) = 9x$$
$$6x + 60 = 9x$$
$$60 = 3x$$
$$20 = x$$

c. Write an expression that can be used to determine the volume of the truncated cone. Explain what each part of the expression represents.

$$\frac{1}{3}\pi(9)^2(30) - \frac{1}{3}\pi(6)^2(20)$$

The expression $\frac{1}{3}\pi(9)^2(30)$ represents the volume of the large cone, and $\frac{1}{3}\pi(6)^2(20)$ is the volume of the small cone. The difference in volumes represents the volume of the truncated cone.

d. Calculate the volume of the truncated cone.

The volume of the small cone is

$$V = \frac{1}{3}\pi(6)^2(20)$$
$$= \frac{720}{3}\pi.$$

The volume of the large cone is

$$V = \frac{1}{3}\pi(9)^2(30)$$
$$= \frac{2,430}{3}\pi.$$

The volume of the truncated cone is

$$\frac{2430}{3}\pi - \frac{720}{3}\pi = \left(\frac{2430}{3} - \frac{720}{3}\right)\pi$$
$$= \frac{1710}{3}\pi$$
$$= 570\pi.$$

The volume of the truncated cone is 570π in^3.

Problem Set Sample Solutions

Students use what they know about similar triangles to determine the volume of truncated cones.

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1. Find the volume of the truncated cone.
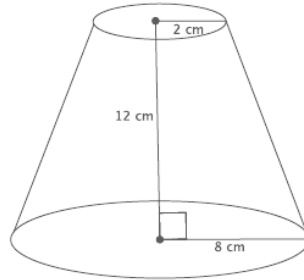
a. Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what each part of the proportion represents.

$$\frac{2}{8} = \frac{x}{x+12}$$

Let x cm represent the height of the small cone. Then $x + 12$ is the height of the large cone. The 2 represents the base radius of the small cone, and the 8 represents the base radius of the large cone.

b. Solve your proportion to determine the height of the cone that has been removed.

$$2(x+12) = 8x$$
$$2x + 24 = 8x$$
$$24 = 6x$$
$$4 = x$$

c. Show a fact about the volume of the truncated cone using an expression. Explain what each part of the expression represents.

$$\frac{1}{3}\pi(8)^2(16) - \frac{1}{3}\pi(2)^2(4)$$

The expression $\frac{1}{3}\pi(8)^2(16)$ represents the volume of the large cone, and $\frac{1}{3}\pi(2)^2(4)$ is the volume of the small cone. The difference in volumes gives the volume of the truncated cone.

d. Calculate the volume of the truncated cone.

The volume of the small cone is

$$V = \frac{1}{3}\pi(2)^2(4)$$
$$= \frac{16}{3}\pi.$$

The volume of the large cone is

$$V = \frac{1}{3}\pi(8)^2(16)$$
$$= \frac{1024}{3}\pi.$$

The volume of the truncated cone is

$$\frac{1024}{3}\pi - \frac{16}{3}\pi = \left(\frac{1024}{3} - \frac{16}{3}\right)\pi$$
$$= \frac{1008}{3}\pi$$
$$= 336\pi.$$

The volume of the truncated cone is 336π cm^3.

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2.   **Find the volume of the truncated cone.**
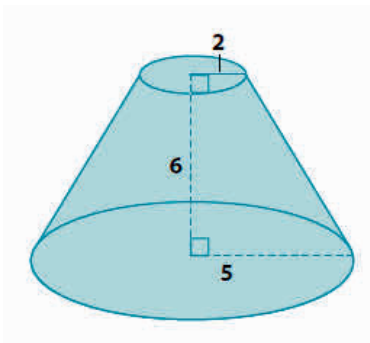
Let x represent the height of the small cone.

$$\frac{2}{5} = \frac{x}{x+6}$$

$$2(x+6) = 5x$$

$$2x + 12 = 5x$$

$$12 = 3x$$

$$4 = x$$

The volume of the small cone is

$$V = \frac{1}{3}\pi(2)^2(4)$$

$$= \frac{16}{3}\pi.$$

The volume of the large cone is

$$V = \frac{1}{3}\pi(5)^2(10)$$

$$= \frac{250}{3}\pi.$$

The volume of the truncated cone is

$$\frac{250}{3}\pi - \frac{16}{3}\pi = \left(\frac{250}{3} - \frac{16}{3}\right)\pi$$

$$= \frac{234}{3}\pi$$

$$= 78\pi.$$

The volume of the truncated cone is 78π units^3.

3.   **Find the volume of the truncated pyramid with a square base.**
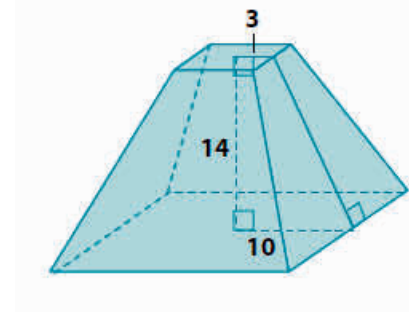
Let x represent the height of the small pyramid.

$$\frac{3}{10} = \frac{x}{x+14}$$

$$3(x+14) = 10x$$

$$3x + 42 = 10x$$

$$42 = 7x$$

$$6 = x$$

The volume of the small pyramid is

$$V = \frac{1}{3}(36)(6)$$

$$= \frac{216}{3}.$$

The volume of the large pyramid is

$$V = \frac{1}{3}(400)(20)$$

$$= \frac{8000}{3}.$$

The volume of the truncated pyramid is

$$\frac{8000}{3} - \frac{216}{3} = \frac{7784}{3}.$$

The volume of the truncated pyramid is $\dfrac{7{,}784}{3}$ units^3.

4. Find the volume of the truncated pyramid with a square base. Note: 3 mm is the distance from the center to the edge of the square at the top of the figure.
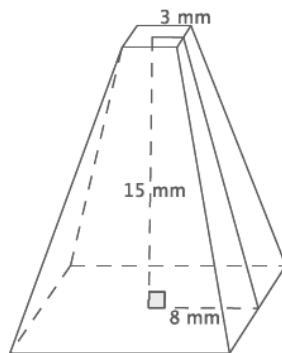
Let x mm represent the height of the small pyramid.

$$\frac{3}{8} = \frac{x}{x + 15}$$
$$3(x + 15) = 8x$$
$$3x + 45 = 8x$$
$$45 = 5x$$
$$9 = x$$

The volume of the small pyramid is

$$V = \frac{1}{3}(36)(9)$$
$$= 108.$$

The volume of the large pyramid is

$$V = \frac{1}{3}(256)(24)$$
$$= 2048.$$

The volume of the truncated pyramid is

$$2048 - 108 = 1940.$$

The volume of the truncated pyramid is $1,940$ mm^3.

5. Find the volume of the truncated pyramid with a square base. Note: 0.5 cm is the distance from the center to the edge of the square at the top of the figure.
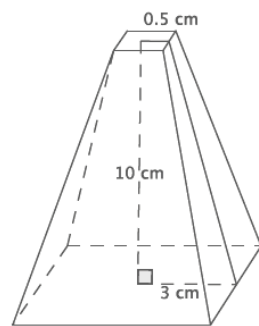
Let x cm represent the height of the small pyramid.

$$\frac{0.5}{3} = \frac{x}{x + 10}$$
$$\frac{1}{2}(x + 10) = 3x$$
$$\frac{1}{2}x + 5 = 3x$$
$$5 = \frac{5}{2}x$$
$$2 = x$$

The volume of the small pyramid is

$$V = \frac{1}{3}(1)(2)$$
$$= \frac{2}{3}.$$

The volume of the large pyramid is

$$V = \frac{1}{3}(36)(12)$$
$$= \frac{432}{3}.$$

The volume of the truncated pyramid is

$$\frac{432}{3} - \frac{2}{3} = \frac{430}{3}.$$

The volume of the truncated pyramid is $\frac{430}{3}$ cm^3.

6. Explain how to find the volume of a truncated cone.

The first thing you have to do is use the ratios of corresponding sides of similar triangles to determine the height of the cone that was removed to make the truncated cone. Once you know the height of that cone, you can determine its volume. Then, you can find the height of the cone (the truncated cone and the portion that was removed). Once you know both volumes, you can subtract the smaller volume from the larger volume. The difference is the volume of the truncated cone.

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7. **Challenge: Find the volume of the truncated cone.**

Since the height of the truncated cone is 1.2 units, we can drop a perpendicular line from the top of the cone to the bottom of the cone so that we have a right triangle with a leg length of 1.2 units and a hypotenuse of 1.3 units. Then, by the Pythagorean theorem, if b is the length of the leg of the right triangle, then

$$1.2^2 + b^2 = 1.3^2$$
$$1.44 + b^2 = 1.69$$
$$b^2 = 0.25$$
$$b = 0.5.$$
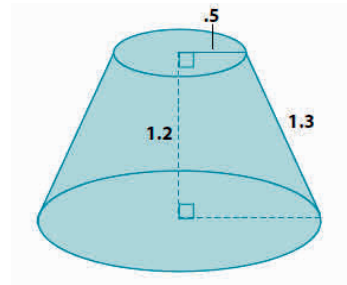
The part of the radius of the bottom base found by the Pythagorean theorem is 0.5. When we add the length of the upper radius (because if you translate along the height of the truncated cone, then it is equal to the remaining part of the lower base), then the radius of the lower base is 1.

Let x represent the height of the small cone.

$$\frac{0.5}{1} = \frac{x}{x + 1.2}$$
$$\frac{1}{2}(x + 1.2) = x$$
$$\frac{1}{2}x + 0.6 = x$$
$$0.6 = \frac{1}{2}x$$
$$1.2 = x$$

The volume of the small cone is

$$V = \frac{1}{3}\pi(0.5)^2(1.2)$$
$$= \frac{0.3}{3}\pi$$
$$= 0.1\pi.$$

The volume of the large cone is

$$V = \frac{1}{3}\pi(1)^2(2.4)$$
$$= \frac{2.4}{3}\pi$$
$$= 0.8\pi.$$

The volume of the truncated cone is

$$\frac{2.4}{3}\pi - \frac{0.3}{3}\pi = \left(\frac{2.4}{3} - \frac{0.3}{3}\right)\pi$$
$$= \frac{2.1}{3}\pi$$
$$= 0.7\pi.$$

The volume of the truncated cone is 0.7π units^3.

Lesson 21: Volume of Composite Solids

Student Outcomes

- Students find the volumes of figures composed of combinations of cylinders, cones, and spheres.

Classwork

Exploratory Challenge/Exercises 1–4 (20 minutes)

Students should know that volumes can be added as long as the solids touch only on the boundaries of their figures. That is, there cannot be any overlapping sections. Students should understand this with the first exercise. Then, allow them to work independently or in pairs to determine the volumes of composite solids in Exercises 1–4. The exercises require students to persevere with some challenging problems and compare solution methods as well as explain how the structure of their expressions relate to the diagrams from which they were created.

Exercises 1–4

1.

a. Write an expression that can be used to find the volume of the chest shown below. Explain what each part of your expression represents. (Assume the ends of the top portion of the chest are semicircular.)

$$(4 \times 15.3 \times 6) + \frac{1}{2}\big(\pi(2)^2(15.3)\big)$$

The expression $(4 \times 15.3 \times 6)$ represents the volume of the prism, and $\frac{1}{2}\big(\pi(2)^2(15.3)\big)$ is the volume of the half-cylinder on top of the chest. Adding the volumes together will give the total volume of the chest.
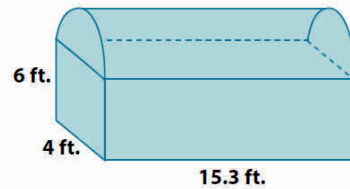
b. What is the approximate volume of the chest shown above? Use 3.14 for an approximation of π. Round your final answer to the tenths place.

The rectangular prism at the bottom has the following volume:

$$V = 4 \times 15.3 \times 6$$
$$= 367.2.$$

The half-cylinder top has the following volume:

$$V = \frac{1}{2}\big(\pi(2)^2(15.3)\big)$$
$$= \frac{1}{2}(61.2\pi)$$
$$= 30.6\pi$$
$$\approx 96.084.$$

$$367.2 + 96.084 = 463.284 \approx 463.3$$

*The total volume of the chest shown is approximately 463.3 ft³.*

Once students have finished the first exercise, ask them what they noticed about the total volume of the chest and what they noticed about the boundaries of each figure that comprised the shape of the chest. These questions illustrate the key understanding that volume is additive as long as the solids touch only at the boundaries and do not overlap.

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2.

    a.   Write an expression for finding the volume of the figure, an ice cream cone and scoop, shown below. Explain what each part of your expression represents. (Assume the sphere just touches the base of the cone.)

$$\frac{4}{3}\pi(1)^3 + \frac{1}{3}\pi(1)^2(3)$$

The expression $\frac{4}{3}\pi(1)^3$ represents the volume of the sphere, and $\frac{1}{3}\pi(1)^2(3)$ represents the volume of the cone. The sum of those two expressions gives the total volume of the figure.
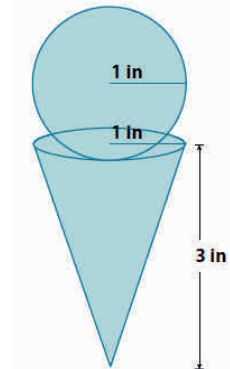
    b.   Assuming every part of the cone can be filled with ice cream, what is the exact and approximate volume of the cone and scoop? (Recall that exact answers are left in terms of π, and approximate answers use 3.14 for π). Round your approximate answer to the hundredths place.

The volume of the scoop is   *The volume of the cone is*

$$V = \frac{4}{3}\pi(1)^3 \qquad\qquad V = \frac{1}{3}\pi(1)^2(3)$$
$$= \frac{4}{3}\pi \qquad\qquad\qquad\quad = \pi$$
$$\approx 4.19. \qquad\qquad\qquad \approx 3.14.$$

The total volume of the cone and scoop is approximately $4.19 \text{ in}^3 + 3.14 \text{ in}^3$, which is 7.33 in^3. The exact volume of the cone and scoop is $\frac{4}{3}\pi \text{ in}^3 + \pi \text{ in}^3 = \frac{7}{3}\pi \text{ in}^3$.

3.

    a.   Write an expression for finding the volume of the figure shown below. Explain what each part of your expression represents.

$$(5 \times 5 \times 2) + \pi\left(\frac{1}{2}\right)^2(6) + \frac{4}{3}\pi(2.5)^3$$

The expression $(5 \times 5 \times 2)$ represents the volume of the rectangular base, $\pi\left(\frac{1}{2}\right)^2(6)$ represents the volume of the cylinder, and $\frac{4}{3}\pi(2.5)^3$ is the volume of the sphere on top. The sum of the separate volumes gives the total volume of the figure.
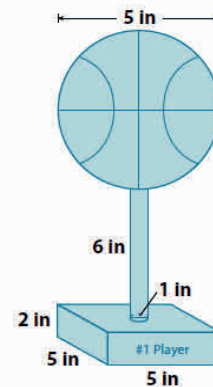
b. Every part of the trophy shown is solid and made out of silver. How much silver is used to produce one trophy? Give an exact and approximate answer rounded to the hundredths place.

The volume of the rectangular base is

$$V = 5 \times 5 \times 2$$
$$= 50.$$

The volume of the cylinder holding up the basketball is

$$V = \pi \left(\frac{1}{2}\right)^2 (6)$$
$$= \frac{1}{4}\pi(6)$$
$$= \frac{3}{2}\pi$$
$$\approx 4.71.$$

The volume of the basketball is

$$V = \frac{4}{3}\pi(2.5)^3$$
$$= \frac{4}{3}\pi(15.625)$$
$$= \frac{62.5}{3}\pi$$
$$\approx 65.42.$$

The approximate total volume of silver needed is $50 \text{ in}^3 + 4.71 \text{ in}^3 + 65.42 \text{ in}^3$, *which is* 120.13 in^3.

The exact volume of the trophy is calculated as follows:

$$V = 50 \text{ in}^3 + \frac{3}{2}\pi \text{ in}^3 + \frac{62.5}{3}\pi \text{ in}^3$$
$$= 50 \text{ in}^3 + \left(\frac{3}{2} + \frac{62.5}{3}\right)\pi \text{ in}^3$$
$$= 50 \text{ in}^3 + \frac{134}{6}\pi \text{ in}^3$$
$$= 50 \text{ in}^3 + \frac{67}{3}\pi \text{ in}^3.$$

The exact volume of the trophy is $50 \text{ in}^3 + \frac{67}{3}\pi \text{ in}^3$.

Lesson 21:   Volume of Composite Solids

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4.   Use the diagram of scoops below to answer parts (a) and (b).

    a.   Order the scoops from least to greatest in terms of their volumes. Each scoop is measured in inches.
       (Assume the third scoop is hemi-spherical.).
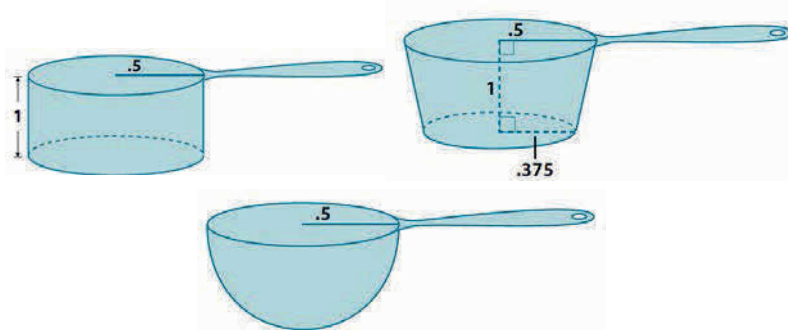
The volume of the cylindrical scoop is

$$V = \pi \left(\frac{1}{2}\right)^2 (1)$$
$$= \frac{1}{4}\pi.$$

The volume of the spherical scoop is

$$V = \frac{1}{2}\left(\frac{4}{3}\pi \left(\frac{1}{2}\right)^3\right)$$
$$= \frac{1}{2}\left(\frac{4}{3}\pi \left(\frac{1}{8}\right)\right)$$
$$= \frac{4}{48}\pi$$
$$= \frac{1}{12}\pi.$$

The volume of the truncated cone scoop is as follows.

Let x represent the height of the portion of the cone that was removed.

$$\frac{0.5}{0.375} = \frac{x+1}{x}$$
$$0.5x = 0.375(x+1)$$
$$0.5x = 0.375x + 0.375$$
$$0.125x = 0.375$$
$$x = 3$$

The volume of the small cone is

$$V = \frac{1}{3}\pi(0.375)^2(3)$$
$$= \frac{9}{64}\pi.$$

The volume of the large cone is

$$V = \frac{1}{3}\pi(0.5)^2(4)$$
$$= \frac{1}{3}\pi.$$

The volume of the truncated cone is

$$\frac{1}{3}\pi - \frac{9}{64}\pi = \left(\frac{1}{3} - \frac{9}{64}\right)\pi$$
$$= \frac{64 - 27}{192}\pi$$
$$= \frac{37}{192}\pi.$$

The three scoops have volumes of $\frac{1}{4}\pi$ in³, $\frac{1}{12}\pi$ in³, and $\frac{37}{192}\pi$ in³. In order from least to greatest, they are $\frac{1}{12}\pi$ in³, $\frac{37}{192}\pi$ in³, and $\frac{1}{4}\pi$ in³. Therefore, the spherical scoop is the smallest, followed by the truncated cone scoop, and lastly the cylindrical scoop.

b. **How many of each scoop would be needed to add a half-cup of sugar to a cupcake mixture? (One-half cup is approximately 7 in^3.) Round your answer to a whole number of scoops.**

The cylindrical scoop is $\frac{1}{4}\pi$ in^3, which is approximately 0.785 in^3. Let x be the number of scoops needed to fill one-half cup.

$$0.785x = 7$$
$$x = \frac{7}{0.785}$$
$$= 8.9171\ldots$$
$$\approx 9$$

It would take about 9 scoops of the cylindrical cup to fill one-half cup.

The spherical scoop is $\frac{1}{12}\pi$ in^3, which is approximately 0.262 in^3. Let x be the number of scoops needed to fill one-half cup.

$$0.262x = 7$$
$$x = \frac{7}{0.262}$$
$$= 26.71755\ldots$$
$$\approx 27$$

It would take about 27 scoops of the cylindrical cup to fill one-half cup.

The truncated cone scoop is $\frac{37}{192}\pi$ in^3, which is approximately 0.605 in^3. Let x be the number of scoops needed to fill one-half cup.

$$0.605x = 7$$
$$x = \frac{7}{0.605}$$
$$= 11.57024\ldots$$
$$\approx 12$$

It would take about 12 scoops of the cylindrical cup to fill one-half cup.

Discussion (15 minutes)

Ask students how they were able to determine the volume of each composite solid in Exercises 1–4. Select a student (or pair) to share their work with the class. Tell them to explain their process using the vocabulary related to the concepts needed to solve the problem. Encourage other students to critique the reasoning of their classmates and to hold them all accountable for the precision of their language.

- Is it possible to determine the volume of the solid in one step? Explain why or why not.
- What simpler problems were needed in order to determine the answer to the complex problem?
- How did your method of solving differ from the one shown?
- What did you need to do in order to determine the volume of the composite solids?
- What symbols or variables were used in your calculations, and how did you use them?
- What factors might account for minor differences in solutions?
- What expressions were used to represent the figures they model?

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- ▪ We know how to use the formulas for cones, cylinders, spheres, and truncated cones to determine the volume of a composite solid, provided no parts of the individual components overlap.

Lesson Summary

Composite solids are figures comprising more than one solid. Volumes of composite solids can be added as long as no parts of the solids overlap. That is, they touch only at their boundaries.

Exit Ticket (5 minutes)

Name _____   Date _____

Lesson 21: Volume of Composite Solids

Andrew bought a new pencil like the one shown below on the left. He used the pencil every day in his math class for a week, and now his pencil looks like the one shown below on the right. How much of the pencil, in terms of volume, did he use?

Note: Figures are not drawn to scale.
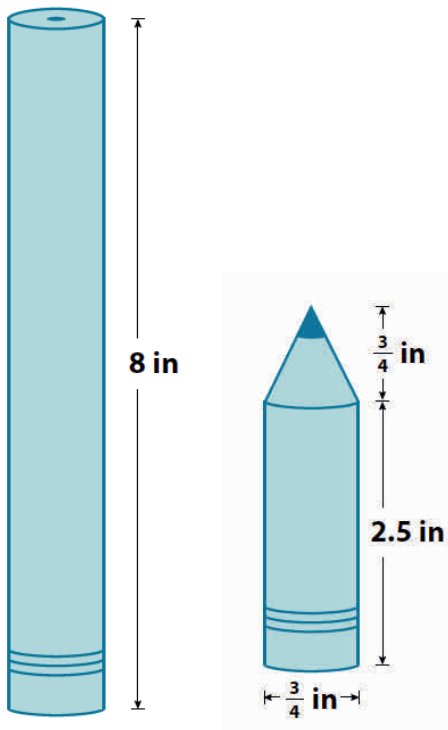
Exit Ticket Sample Solutions

Andrew bought a new pencil like the one shown below on the left. He used the pencil every day in his math class for a week, and now his pencil looks like the one shown below on the right. How much of the pencil, in terms of volume, did he use?
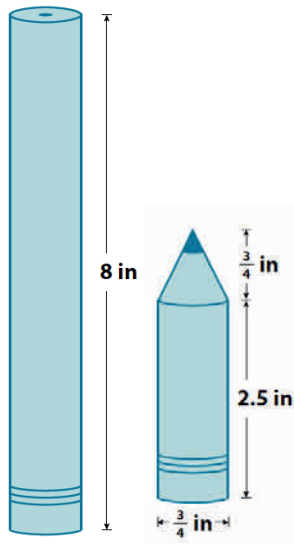
$$V = \pi(0.375)^2(8)$$
$$V = 1.125\pi$$

Volume of the pencil at the beginning of the week was 1.125π in^3.

$$V = \pi(0.375)^2(2.5)$$
$$V \approx 0.3515\pi$$

The volume of the cylindrical part of the pencil is approximately 0.3515π in^3.

$$V = \frac{1}{3}\pi(0.375)^2(0.75)$$
$$V \approx \frac{0.1054}{3}\pi$$
$$V \approx 0.0351\pi$$

The volume of the cone part of the pencil is approximately 0.0351π in^3.

$$0.3515\pi + 0.0351\pi = (0.3515 + 0.0351)\pi = 0.3866\pi$$

The total volume of the pencil after a week is approximately 0.3866π in^3.

$$1.125\pi - 0.3866\pi = (1.125 - 0.3866)\pi = 0.7384\pi$$

In one week, Andrew used approximately 0.7384π in^3 of the pencil's total volume.

8 in

$\frac{3}{4}$ in

2.5 in

$\frac{3}{4}$ in

Problem Set Sample Solutions

1. **What volume of sand is required to completely fill up the hourglass shown below? Note: 12 m is the height of the truncated cone, not the lateral length of the cone.**

   Let x m represent the height of the portion of the cone that has been removed.

   $$\frac{4}{9} = \frac{x}{x + 12}$$
   $$4(x + 12) = 9x$$
   $$4x + 48 = 9x$$
   $$48 = 5x$$
   $$\frac{48}{5} = x$$
   $$9.6 = x$$

   The volume of the removed cone is

   $$V = \frac{1}{3}\pi(4)^2(9.6)$$
   $$= \frac{153.6}{3}\pi.$$

   The volume of the cone is

   $$V = \frac{1}{3}\pi(9)^2(21.6)$$
   $$= \frac{1749.6}{3}\pi.$$

   The volume of one truncated cone is

   $$\frac{1749.6}{3}\pi - \frac{153.6}{3}\pi = \left(\frac{1749.6}{3} - \frac{153.6}{3}\right)\pi$$
   $$= \frac{1596}{3}\pi$$
   $$= 532\,\pi.$$

   The volume of sand needed to fill the hourglass is 1064π m^3.
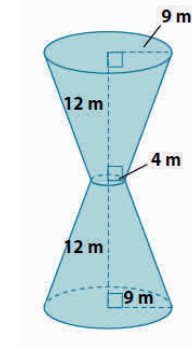
2.
   a. **Write an expression for finding the volume of the prism with the pyramid portion removed. Explain what each part of your expression represents.**
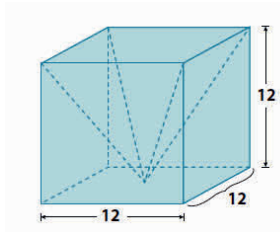
      $$(12)^3 - \frac{1}{3}(12)^3$$

      The expression $(12)^3$ represents the volume of the cube, and $\frac{1}{3}(12)^3$ represents the volume of the pyramid. Since the pyramid's volume is being removed from the cube, we subtract the volume of the pyramid from the volume of the cube.

   b. **What is the volume of the prism shown above with the pyramid portion removed?**

      The volume of the prism is

      $$V = (12)^3$$
      $$= 1728.$$

      The volume of the pyramid is

      $$V = \frac{1}{3}(1728)$$
      $$= 576.$$

      The volume of the prism with the pyramid removed is $1,152$ units^3.

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3.

a.   Write an expression for finding the volume of the funnel shown to the right. Explain what each part of your expression represents.

$$\pi(4)^2(14) + \left(\frac{1}{3}\pi(8)^2(x+16) - \frac{1}{3}\pi(4)^2 x\right)$$

The expression $\pi(4)^2(14)$ represents the volume of the cylinder. The expression $\left(\frac{1}{3}\pi(8)^2(x+16) - \frac{1}{3}\pi(4)^2 x\right)$ represents the volume of the truncated cone.
The x represents the unknown height of the smaller cone that has been removed. When the volume of the cylinder is added to the volume of the truncated cone, then we will have the volume of the funnel shown.
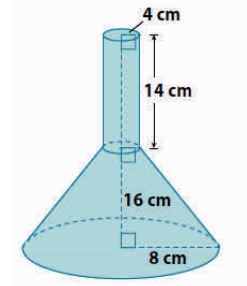
b.   Determine the exact volume of the funnel.

The volume of the cylinder is
$$V = \pi(4)^2(14)$$
$$= 224\pi.$$

Let x cm be the height of the cone that has been removed.
$$\frac{4}{8} = \frac{x}{x+16}$$
$$4(x+16) = 8x$$
$$4x + 64 = 8x$$
$$64 = 4x$$
$$16 = x$$

The volume of the small cone is   *The volume of the large cone is*
$$V = \frac{1}{3}\pi(4)^2(16)$$ $$V = \frac{1}{3}\pi(8)^2(32)$$
$$= \frac{256}{3}\pi.$$ $$= \frac{2048}{3}\pi.$$

The volume of the truncated cone is
$$\frac{2048}{3}\pi - \frac{256}{3}\pi = \left(\frac{2048}{3} - \frac{256}{3}\right)\pi$$
$$= \frac{1792}{3}\pi.$$

The volume of the funnel is 224π cm^3 $+ \frac{1792}{3}\pi$ cm^3, which is $821\frac{1}{3}\pi$ cm^3.

4.　What is the approximate volume of the rectangular prism with a cylindrical hole shown below? Use 3.14 for π. Round your answer to the tenths place.
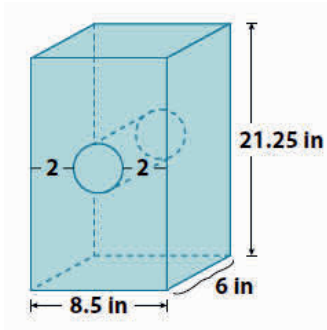
The volume of the prism is

$$V = (8.5)(6)(21.25)$$
$$= 1083.75.$$

The volume of the cylinder is

$$V = \pi(2.25)^2(6)$$
$$= 30.375\pi$$
$$\approx 95.3775.$$

The volume of the prism with the cylindrical hole is approximately $988.4 \text{ in}^3, because\ 1083.75 \text{ in}^3 - 95.3775 \text{ in}^3 = 988.3725 \text{ in}^3.$

21.25 in

2 — 2

6 in

8.5 in

5.　A layered cake is being made to celebrate the end of the school year. What is the exact total volume of the cake shown below?

4 in
2 in
4 in
4 in
4 in
8 in
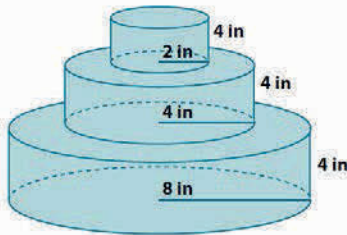
The bottom layer's volume is

$$V = (8)^2\pi(4)$$
$$= 256\pi.$$

The middle layer's volume is

$$V = (4)^2\pi(4)$$
$$= 64\pi.$$

The top layer's volume is

$$V = (2)^2\pi(4)$$
$$= 16\pi.$$

The total volume of the cake is

$$256\pi \text{ in}^3 + 64\pi \text{ in}^3 + 16\pi \text{ in}^3 = (256 + 64 + 16)\pi \text{ in}^3 = 336\pi \text{ in}^3.$$

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Lesson 22: Average Rate of Change

Student Outcomes

- Students compute the average rate of change in the height of water level when water is poured into a conical container at a constant rate.

Lesson Notes

This lesson focuses on solving one challenging problem that highlights the need to make sense of and persevere in solving problems. Working through the problem, students should reach an important conclusion about constant rate and average rate of change. Students compute the average rate of change of water-level height as a cone is filled with water at a constant rate. The rate of change of water height is not constant. Throughout the problem, students have to apply many of the concepts learned throughout the year, namely, concepts related to the volume of solids, similarity, constant rate, and rate of change.

The Opening requires a demonstration of the filling of a cone with sand or some other substance.

Classwork

Opening (5 minutes)

Teachers perform a demonstration for students pouring sand (or water or rice) into an inverted circular cone at a constant rate. Ask students to describe, intuitively, the rate at which the cone is being filled. Specifically, ask students to imagine the cone as two halves, an upper half and a lower half. Which half would fill faster and why? Teachers can contrast this with a demonstration of water filling a cylinder.
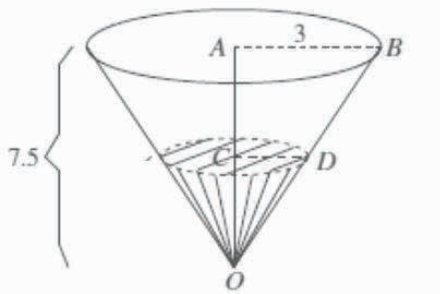
Students should be able to state that the narrower part of the cone is filled more quickly than the wider part of the cone. Therefore, they can conclude that the rate of change of the volume of the cone is not constant, and an average rate must be computed. However, the rate of change of the volume of the cylinder is constant because at each increment of the height, the size of the cylinder is exactly the same, which means that the volume increases at a constant rate.

If it is not possible to do a demonstration, there is a video of a cone being filled at the following location:
http://www.youtube.com/watch?v=VEEfHJHMQS8.

Discussion (30 minutes)

> **Exercise**
>
> The height of a container in the shape of a circular cone is 7.5 ft., and the radius of its base is 3 ft., as shown. What is the total volume of the cone?
>
>
>
> $$V = \frac{1}{3}\pi 3^2 (7.5)$$
>
> $$= 22.5\pi$$
>
> *The volume of the cone is 22.5π ft^3.*

- If we know the rate at which the cone is being filled, how could we use that information to determine how long it would take to fill the cone?
    - *We could take the total volume and divide it by the rate to determine how long it would take to fill.*
- Water flows into the container (in its inverted position) at a constant rate of 6 ft^3 per minute. Approximately when will the container be filled?

Provide students time to work in pairs on the problem. Have students share their work and their reasoning about the problem.

    - *Since the container is being filled at a constant rate, the volume must be divided by the rate at which it is being filled (using 3.14 as an approximation for π and rounding to the hundredths place):*
      $$\frac{22.5\pi}{6} \approx 11.78.$$
      It will take almost 12 minutes to fill the cone at a rate of 6 ft^3 per minute.

- Now we want to show that even though the water filling the cone flows at a constant rate, the rate of change of the volume in the cone is not constant. For example, if we wanted to know how many minutes it would take for the level in the cone to reach 1 ft., then we would have to first determine the volume of the cone when the height is 1 ft. Do we have enough information to do that?
    - *Yes, we will need to first determine the radius of the cone when the height is 1 ft.*

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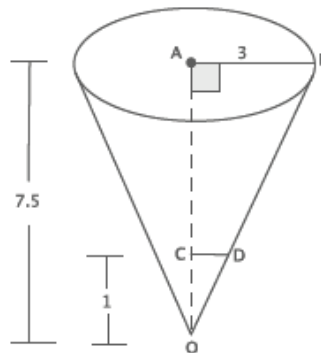
- What equation can we use to determine the radius when the height is 1 ft.? Explain how your equation represents the situation.

  □ *If we let $|CD|$ represent the radius of the cone when the height is* 1 *ft., then*

  $$\frac{3}{|CD|} = \frac{7.5}{1}.$$

  The number 3 *represents the radius of the original cone. The* 7.5 *represents the height of the original cone, and the* 1 *represents the height of the cone for which we are trying to solve.*

- Use your equation to determine the radius of the cone when the height is 1 ft.

  □ *The radius when the height is* 1 *ft. is*

  $$3 = 7.5|CD|$$
  $$\frac{3}{7.5} = |CD|$$
  $$0.4 = |CD|$$

- Now determine the volume of the cone when the height is 1 ft.

  □ *Then, we can find the volume of the cone with a height of* 1 *ft.:*

  $$V = \frac{1}{3}\pi(0.4)^2(1)$$
  $$= \frac{0.16}{3}\pi.$$

  Now we can divide the volume by the rate at which the cone is being filled to determine how many minutes it would take to fill a cone with a height of 1 *ft.:*

  $$\frac{0.16}{3}\pi \approx 0.167$$
  $$\frac{0.167}{6} \approx 0.028.$$

  It would take about 0.028 *minutes, that is, about* 1.68 *seconds (*0.028 × 60 = 1.68*) to fill a cone with a height of* 1 *ft.*

- Calculate the number of minutes it would take to fill the cone at 1 ft. intervals. Organize your data in the table below.

Provide students time to work on completing the table. They should replicate the work above by first finding the radius of the cone at the given heights, using the radius to determine the volume of the cone, and then determining the time it would take to fill a cone of that volume at the given constant rate. Once most students have finished, continue with the discussion below.

Time (in minutes)	Water Level (in feet)
0.028	1
0.22	2
0.75	3
1.78	4
3.49	5
6.03	6
9.57	7
11.78	7.5

- We know that the sand (or rice or water) being poured into the cone is poured at a constant rate, but is the level of the substance in the cone rising at a constant rate? Provide evidence to support your answer.

Provide students time to construct an argument based on the data collected to show that the substance in the cone is not rising at a constant rate. Have students share their reasoning with the class. Students should be able to show that the rate of change (slope) between any two data points is not the same using calculations like

$$\frac{2-1}{0.22-0.028} = \frac{1}{0.192} = 5.2 \text{ and } \frac{7-6}{9.57-6.03} = \frac{1}{3.54} = 0.28,$$ or by graphing the data and showing that it is not linear.
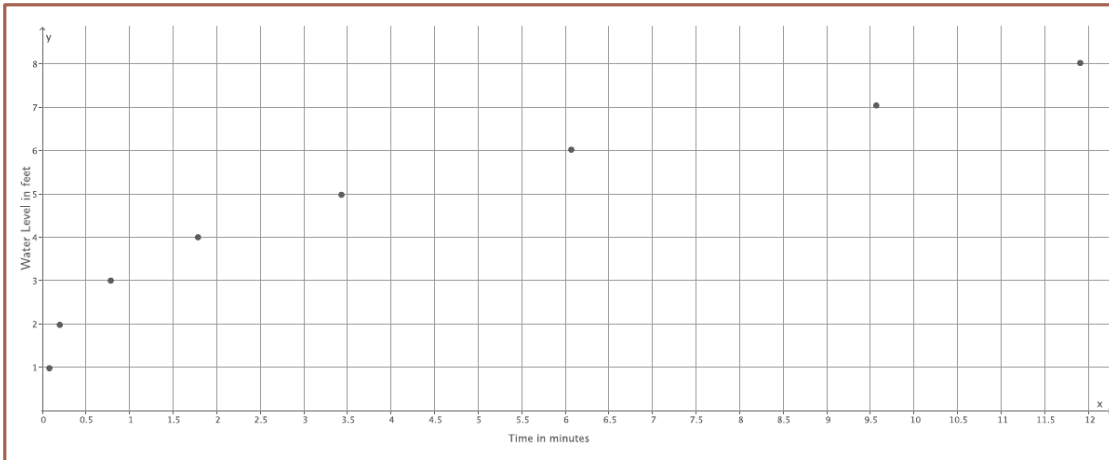
Close the discussion by reminding students of the demonstration at the Opening of the lesson. Ask students if the math supported their conjectures about average rate of change of the water level of the cone.

Closing (5 minutes)

Consider asking students to write a summary of what they learned. Prompt them to include a comparison of how filling a cone is different from filling a cylinder. Another option is to have a whole-class discussion where the teacher asks students how to interpret this information in a real-world context. For example, ask them if they were filling a cylindrical container and a conical container with the same radius and height, which would fill first. Or ask them to discuss whether the rate of change of the volume would be different if we were emptying the cone as opposed to filling it. How so? What might that look like on a graph?

Summarize, or ask students to summarize, the main points from the lesson.

- We know intuitively that the narrower part of a cone will fill up faster than the wider part of a cone.
- By comparing the time it takes for a cone to be filled to a certain water level, we can determine that the rate of filling the cone is not constant.
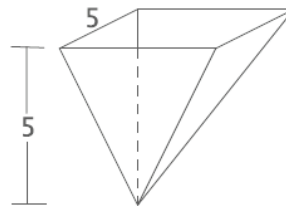
Exit Ticket (5 minutes)

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Name _____   Date _____

Lesson 22: Average Rate of Change

Exit Ticket

A container in the shape of a square base pyramid has a height of 5 ft. and a base length of 5 ft., as shown. Water flows into the container (in its inverted position) at a constant rate of 4 ft^3 per minute. Calculate how many minutes it would take to fill the cone at 1 ft. intervals. Organize your data in the table below.
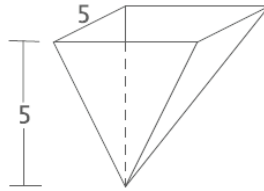

Water Level (in feet)	Area of Base (in feet²)	Volume (in feet³)	Time (in minutes)
1			
2			
3			
4			
5			

a.   How long will it take to fill up the container?

b.   Show that the water level is not rising at a constant rate. Explain.

Exit Ticket Sample Solutions

A container in the shape of a square base pyramid has a height of 5 ft. and a base length of 5 ft., as shown. Water flows into the container (in its inverted position) at a constant rate of 4 ft^3 per minute. Calculate how many minutes it would take to fill the cone at 1 ft. intervals. Organize your data in the table below.

Water Level (in feet)	Area of Base (in feet²)	Volume (in feet³)	Time (in minutes)
1	1	$\dfrac{1}{3}$	0.08
2	4	$\dfrac{8}{3}$	0.67
3	9	$\dfrac{27}{3} = 9$	2.25
4	16	$\dfrac{64}{3}$	5.33
5	25	$\dfrac{125}{3}$	10.42

a. How long will it take to fill up the container?

It will take 10.42 min. to fill up the container.

b. Show that the water level is not rising at a constant rate. Explain.

$$\frac{2-1}{0.67-0.08} = \frac{1}{0.59} \approx 1.69$$

$$\frac{5-4}{10.42-5.33} = \frac{1}{5.09} \approx 0.2$$

The rate at which the water is rising is not the same for the first foot as it is for the last foot. The rate at which the water is rising in the first foot is higher than the rate at which the water is rising in the last foot.

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## Problem Set Sample Solutions

1. Complete the table below for more intervals of water levels of the cone discussed in class. Then, graph the data on a coordinate plane.

Time (in minutes)	Water Level (in feet)
0.028	1
0.09	1.5
0.22	2
0.44	2.5
0.75	3
1.2	3.5
1.78	4
2.54	4.5
3.49	5
4.64	5.5
6.03	6
7.67	6.5
9.57	7
11.78	7.5
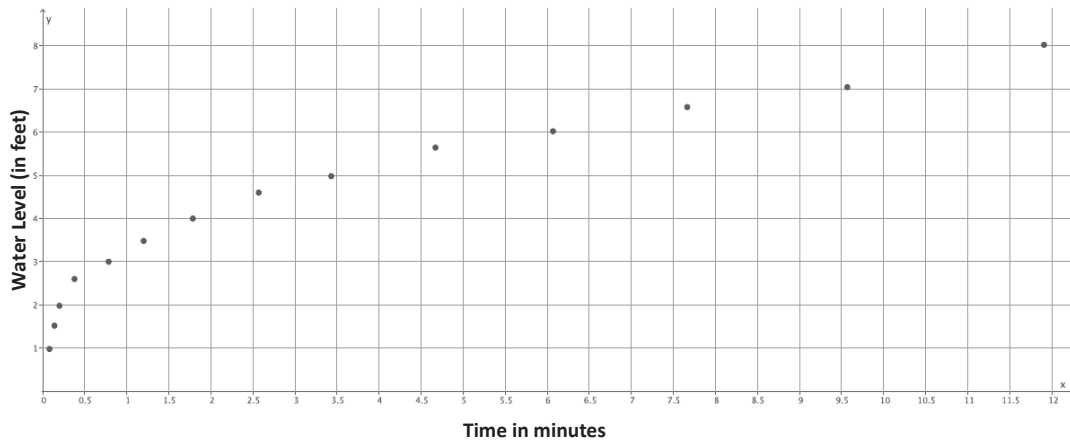
2. Complete the table below, and graph the data on a coordinate plane. Compare the graphs from Problems 1 and 2. What do you notice? If you could write a rule to describe the function of the rate of change of the water level of the cone, what might the rule include?

x	$\sqrt{x}$
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
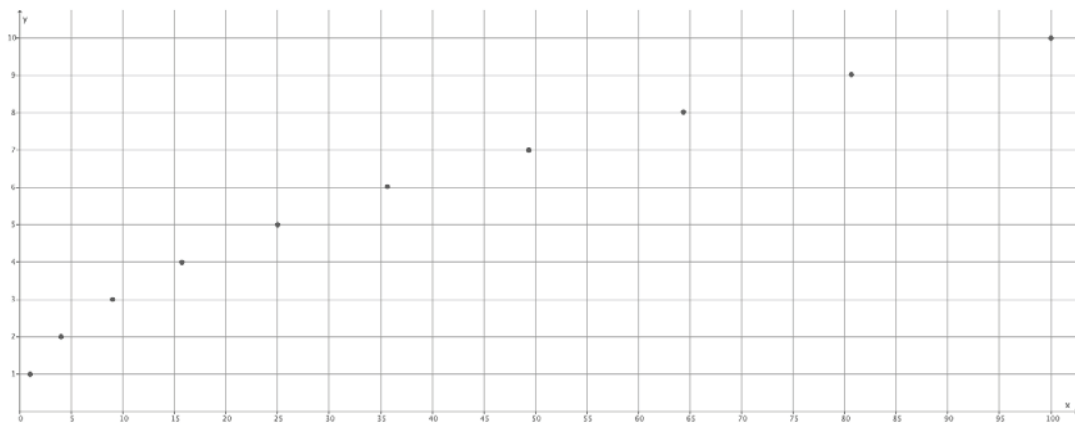
The graphs are similar in shape. The rule that describes the function for the rate of change likely includes a square root. Since the graphs of functions are the graphs of certain equations where their inputs and outputs are points on a coordinate plane, it makes sense that the rule producing such a curve would be a graph of some kind of square root.

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3. Describe, intuitively, the rate of change of the water level if the container being filled were a cylinder. Would we get the same results as with the cone? Why or why not? Sketch a graph of what filling the cylinder might look like, and explain how the graph relates to your answer.

If the container being filled were a cylinder, we would see a constant rate of change in the water level because there is no narrow or wide part like there is with a cone. Therefore, we would not see the same results as we did with the cone. The rate of change would be the same over any time interval for any given height of the cylinder. The following graph demonstrates this. If a cylinder were being filled at a constant rate, the graph would be linear as shown because the water that would flow into the cylinder would be filling up the same-sized solid throughout.
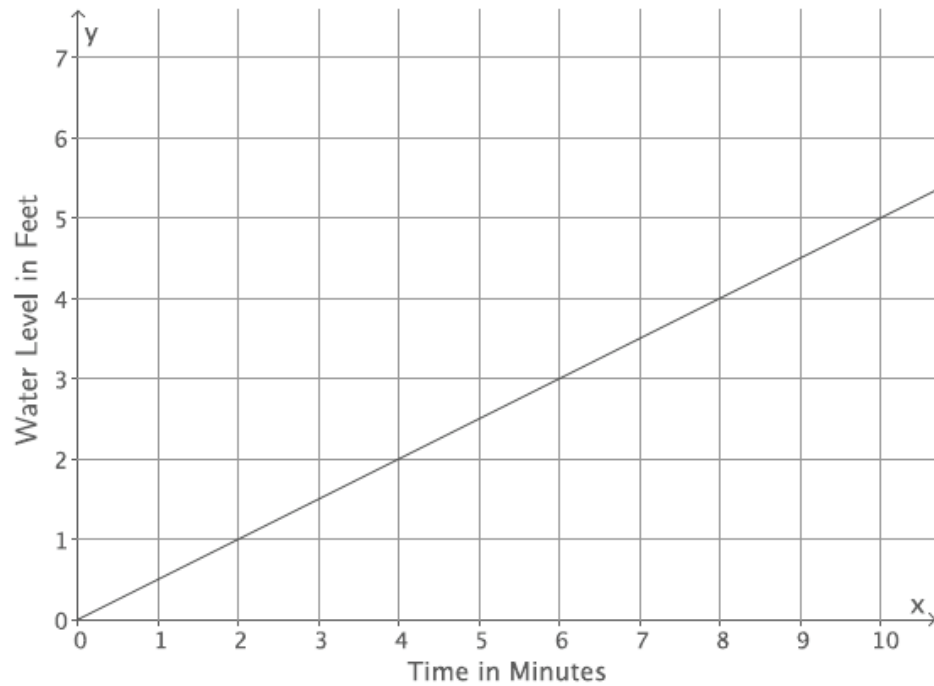
4. Describe, intuitively, the rate of change if the container being filled were a sphere. Would we get the same results as with the cone? Why or why not?

The rate of change in the water level would not be constant if the container being filled were a sphere. The water level would rise quickly at first, then slow down, and then rise quickly again because of the narrower parts of the sphere at the top and the bottom and the wider parts of the sphere around the middle. We would not get the same results as we saw with the cone, but the results would be similar in that the rate of change is nonlinear.

Lesson 23: Nonlinear Motion

Student Outcomes

- Using square roots, students determine the position of the bottom of a ladder as its top slides down a wall at a constant rate.

Lesson Notes

The purpose of this *optional* extension lesson is to incorporate the knowledge obtained throughout the year into a modeling problem about the motion at the bottom of a ladder as it slides down a wall. In this lesson, students use what they learned about solving multi-step equations from Module 4, which requires knowledge of integer exponents from Module 1. They also describe the motion of the ladder in terms of a function learned in Module 5 and use what they learned about square roots in this module. Many questions are included to guide students' thinking, but it is recommended that the teacher lead students through the discussion but allow them time to make sense of the problem and persevere in solving it throughout key points within the discussion.

Classwork

Mathematical Modeling Exercise and Discussion (35 minutes)

There are three phases of the modeling in this lesson: assigning variables, determining the equation, and analyzing results. Many questions are included to guide students' thinking, but the activity may be structured in many different ways, including students working collaboratively in small groups to make sense of and persevere in solving the problem.

Students may benefit from a demonstration of this situation. Consider using a notecard leaning against a box to show what flush means and how the ladder would slide down the wall.


Mathematical Modeling Exercise

A ladder of length L ft. leaning against a wall is sliding down. The ladder starts off being flush with (right up against) the wall. The top of the ladder slides down the vertical wall at a constant speed of v ft. per second. Let the ladder in the position L_1 slide down to position L_2 after 1 second, as shown below.

Will the bottom of the ladder move at a constant rate away from point O?

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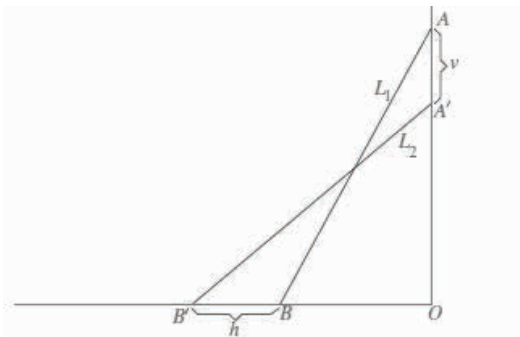
- ▪ Identify what each of the symbols in the diagram represents.
  - □ *O represents the corner where the floor and the wall intersect.*

    L_1 represents the position of the ladder after it has slid down the wall.

    L_2 represents the position of the ladder after it has slid one second after position L_1 down the wall.

    A represents the starting position of the top of the ladder.

    A′ represents the position of the top of the ladder after it has slid down the wall for one second.

    v represents the distance that the ladder slid down the wall in one second.

    B represents the starting position of the bottom of the ladder.

    B′ represents the position of the bottom of the ladder after it has slid for one second.

    h represents the distance the ladder has moved along the ground after sliding down the wall in one second.

- ▪ The distance from point A to point A' is v ft. Explain why.
  - □ *Since the ladder is sliding down the wall at a constant rate of v ft. per second, then after 1 second, the ladder moves v feet. Since we are given that the time it took for the ladder to go from position L_1 to L_2 is one second, then we know the distance between those points must be v feet.*

- ▪ The bottom of the ladder then slides on the floor to the left so that in 1 second it moves from B to B' as shown. Therefore, the average speed of the bottom of the ladder is h ft. per second in this 1-second interval. Will the bottom of the ladder move at a constant rate away from point O? In other words, if the ladder moves at a constant rate, will the distance it has moved (shown as D' in the image below) coincide with point E where $|EO|$ is the length of the ladder?
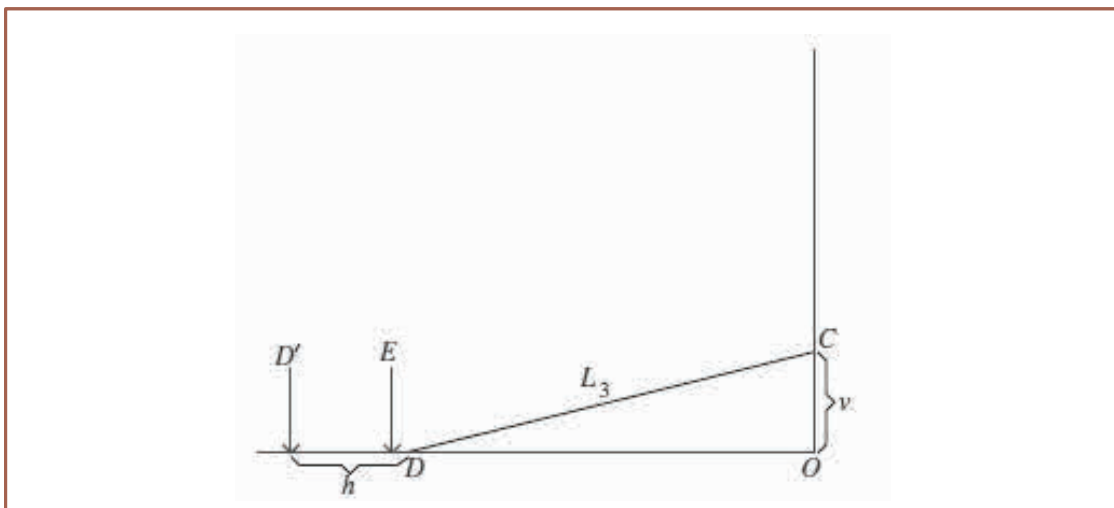
Provide time for students to discuss the answer to the question in pairs or small groups, and then have students share their reasoning. This question is the essential question of the lesson. The answer to this question is the purpose of the entire investigation.

Consider prompting their thinking with the following questions:

- ▪ Think about the distance the base of the ladder is away from the wall as a function of time t. What do you think the beginning placement of the ladder should be for time $t = 0$?
  - □ *Flush vertical against the wall*

- How far away from the wall is the base of the ladder then at time $t = 0$?

  □ *It is right up against the wall.*

- If we let d represent the distance of the base of the ladder from the wall, we know that at $t = 0$, $d = 0$.

- Can the ladder continue sliding forever? Is there a time at which it must stop sliding?

  □ *Yes, it must stop sliding when it lies flat on the floor.*

- Can we compute at what time that will occur?

Allow students time to struggle with this. They may or may not be able to develop the equation below.
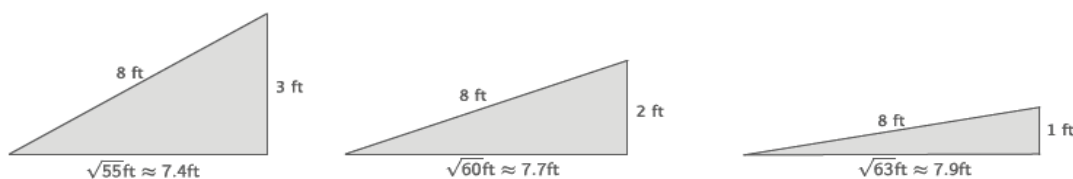
  □ $t = \dfrac{L}{v}$

- So our question is, is d a linear function of time? That is, does the value of d change by constant amounts over constant time intervals?

Students may suggest modeling the experiment using a note card against a vertical book and observing how the distance changes as the card slides down the book. Students may compare extreme cases where the base of the ladder moves a greater distance in the first second of sliding down the wall as opposed to the last second.

Students should recognize that this situation cannot be described by a linear function. Specifically, if the top of the ladder was v feet from the floor as shown below, it would reach O in one second (because the ladder slides down the wall at a constant rate of v per second). Then after 1 second, the ladder will be flat on the floor, and the foot of the ladder would be at the point where $|EO| = L$, or the length of the ladder. The discussion points below may be useful if students were unable to determine that the motion is non-linear.

- If the rate of change could be described by a linear function, then the point D would move to D' after 1 second, where $|D'D| = h$ ft. (where h is defined as the length the ladder moved from D to D' in one second). But this is impossible.

- Recall that the length of the ladder, L, is $|EO|$. When the ladder is flat on the floor, then at most, the foot of the ladder will be at point E from point O. If the rate of change of the ladder were linear, then the foot of the ladder would be at D' because the linear rate of change would move the ladder a distance of h feet every 1 second. From the picture (on the previous page) you can see that $D' \neq E$. Therefore, it is impossible that the rate of change of the ladder could be described by a linear function.

- Intuitively, if you think about when the top of the ladder, C, is close to the floor (point O), a change in the height of C would produce very little change in the horizontal position of the bottom of the ladder, D.

- Consider the three right triangles shown below. If we let the length of the ladder be 8 ft., then we can see that a constant change of 1 ft. in the vertical distance produces very little change in the horizontal distance. Specifically, the change from 3 ft. to 2 ft. produces a horizontal change of approximately 0.3 ft., and the change from 2 ft. to 1 ft. produces a horizontal change of approximately 0.2 ft. A change from 1 ft. to 0 ft., meaning that the ladder is flat on the floor, would produce a horizontal change of just 0.1 ft. (the difference between the length of the ladder, 8 ft. and 7.9 ft.)

Consider the three right triangles shown below, specifically the change in the length of the base as the height decreases in increments of 1 ft.

8 ft 3 ft $\sqrt{55}$ft $\approx$ 7.4ft

8 ft 2 ft $\sqrt{60}$ft $\approx$ 7.7ft

8 ft 1 ft $\sqrt{63}$ft $\approx$ 7.9ft

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- In particular, when the ladder is flat on the floor so that $C = 0$, then the bottom cannot be further left than the point E because $|EO| = L$, the length of the ladder. Therefore, the ladder will never reach point D', and the function that describes the movement of the ladder cannot be linear.

- We want to show that our intuitive sense of the movement of the ladder is accurate. Our goal is to derive a formula, d, for the function of the distance of the bottom of the ladder from O over time t. Because the top of the ladder slides down the wall at a constant rate of v ft. per second, the top of the ladder is now at point A, which is vt ft. below the vertical height of L feet, and the bottom of the ladder is at point B, as shown below. We want to determine $|BO|$, which by definition is the formula for the function, d.
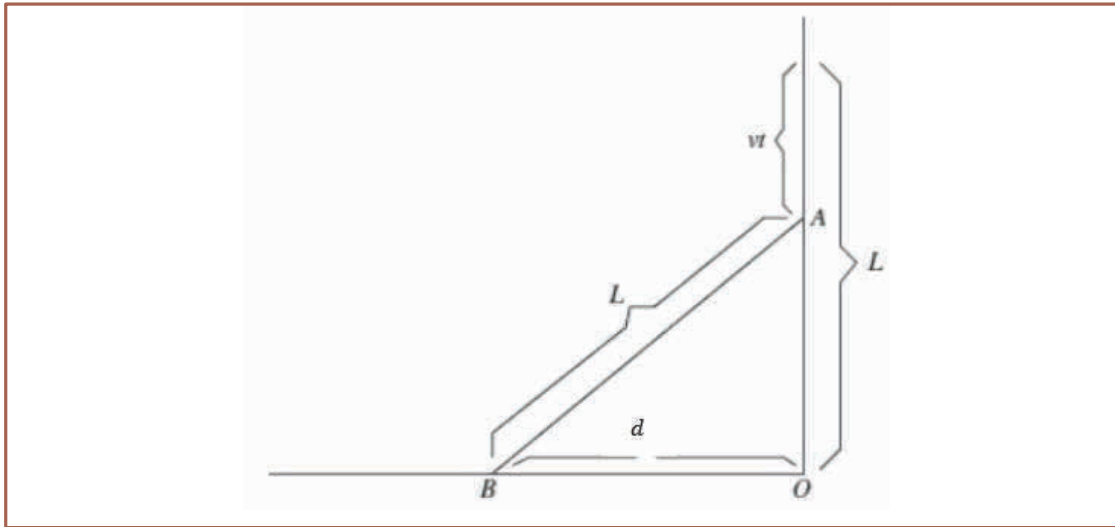
- Explain the expression vt. What does it represent?
    - *The expression vt represents the distance the ladder has slid down the wall after t seconds. Since v is the rate at which the ladder slides down the wall, then vt is the distance it slides after t seconds.*

- How can we determine $|BO|$?
    - *The shape formed by the ladder, wall, and floor is a right triangle, so we can use the Pythagorean theorem to find $|BO|$.*

- What is the length of $|AO|$?
    - *$|AO|$ is the length of the ladder L minus the distance the ladder slides down the wall after t seconds (i.e., vt). Therefore, $|AO| = L - vt$.*

- What is the length of the hypotenuse of the right triangle?
    - *The length of the hypotenuse is the length of the ladder, L.*

- Use the Pythagorean theorem to write an expression that gives $|BO|$ (i.e., d).

Provide students time to work in pairs to write the expression for $|BO|$. Give guidance as necessary.

   - *By the Pythagorean theorem,*

$$(L - vt)^2 + d^2 = L^2$$
$$d^2 = L^2 - (L - vt)^2$$
$$\sqrt{d^2} = \sqrt{L^2 - (L - vt)^2}$$
$$d = \sqrt{L^2 - (L - vt)^2}$$

Pause after deriving the equation $d = \sqrt{L^2 - (L - vt)^2}$. Ask students to explain what the equation represents. Students should recognize that the equation gives the distance the foot of the ladder is from the wall, which is $|BO|$.

- Let's return to an earlier question: Will the ladder continue to slide forever?

The goal is for students to conclude that when $d = L$, the ladder will no longer slide. That is, when $\sqrt{L^2 - (L - vt)^2} = L$, the ladder is finished sliding. If students need convincing, consider the following situation with concrete numbers.

- What would happen if t were very large? Suppose the constant rate, v, of the ladder falling down the wall is 2 feet per second, the length of the ladder, L, is 10 feet, and the time t is 100 seconds—what is d equal to?

  - *The value of d is*

$$d = \sqrt{L^2 - (L - vt)^2}$$
$$= \sqrt{10^2 - (10 - 2(100))^2}$$
$$= \sqrt{100 - (190)^2}$$
$$= \sqrt{100 - 36{,}100}$$
$$= \sqrt{-36{,}000}$$

  If the value of t were very large, then the formula would make no sense because the length of $|BO|$ would be equal to the square of a negative number.

- For this reason, we can only consider values of t so that the top of the ladder is still above the floor. Symbolically, $vt \le L$, where vt is the expression that describes the distance the ladder has moved at a specific rate v for a specific time t. We need that distance to be less than or equal to the length of the ladder.

- What happens when $t = \frac{L}{v}$? Substitute $\frac{L}{v}$ for t in our formula.

  - *Substituting $\frac{L}{v}$ for t,*

$$d = \sqrt{L^2 - (L - vt)^2}$$
$$= \sqrt{L^2 - \left(L - v\left(\frac{L}{v}\right)\right)^2}$$
$$= \sqrt{L^2 - (L - L)^2}$$
$$= \sqrt{L^2 - 0^2}$$
$$= \sqrt{L^2}$$
$$= L.$$

  When $t = \frac{L}{v}$, the top of the ladder will be at the point O, and the ladder will be flat on the floor because d represents the length of $|BO|$. If that length is equal to L, then the ladder must be on the floor.

- Back to our original concern: What kind of function describes the rate of change of the movement of the bottom of the ladder on the floor? It should be clear that by the equation $d = \sqrt{L^2 - (L - vt)^2}$, which represents $|BO|$ for any time t, that the motion (rate of change) is not one of constant speed. Nevertheless, thanks to the concept of a function, we can make predictions about the location of the ladder for any value of t as long as $t \le \frac{L}{v}$.

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▪ We will use some concrete numbers to compute the average rate of change over different time intervals. Suppose the ladder is 15 feet long, $L = 15$, and the top of the ladder is sliding down the wall at a constant speed of 1 ft. per second, $v = 1$. Then, the horizontal distance of the bottom of the ladder from the wall ($|BO|$) is given by the formula

$$d = \sqrt{15^2 - (15 - 1t)^2}$$
$$= \sqrt{225 - (15 - t)^2}.$$

Determine the outputs the function would give for the specific inputs. Use a calculator to approximate the lengths. Round to the hundredths place.

Input (in seconds) t	Output (in feet) $d = \sqrt{225 - (15 - t)^2}$
0	$\sqrt{0} = 0$
1	$\sqrt{29} \approx 5.39$
3	$\sqrt{81} = 9$
4	$\sqrt{104} \approx 10.2$
7	$\sqrt{161} \approx 12.69$
8	$\sqrt{176} \approx 13.27$
14	$\sqrt{224} \approx 14.97$
15	$\sqrt{225} = 15$

▪ Make at least three observations about what you notice from the data in the table. Justify your observations mathematically with evidence from the table.

Sample observations given below.

    ▫ *The average rate of change between* 0 *and* 1 *second is* 5.39 *feet per second.*

$$\frac{5.39 - 0}{1 - 0} = 5.39$$

    ▫ *The average rate of change between* 3 *and* 4 *seconds is* 1.2 *feet per second.*

$$\frac{10.2 - 9}{4 - 3} = 1.2$$

    ▫ *The average rate of change between* 7 *and* 8 *seconds is* 0.58 *feet per second.*

$$\frac{13.27 - 12.69}{8 - 7} = 0.58$$

    ▫ *The average rate of change between* 14 *and* 15 *seconds is* 0.03 *feet per second.*

$$\frac{15 - 14.97}{15 - 14} = 0.03$$

▪ Now that we have computed the average rate of change over different time intervals, we can make two conclusions: (1) The motion at the bottom of the ladder is not linear, and (2) there is a decrease in the average speeds; that is, the rate of change of the position of the ladder is slowing down as observed in the four 1-second intervals we computed. These conclusions are also supported by the graph of the situation shown on the next page. The data points do not form a line; therefore, the rate of change with respect to the position of the bottom of the ladder is not linear.
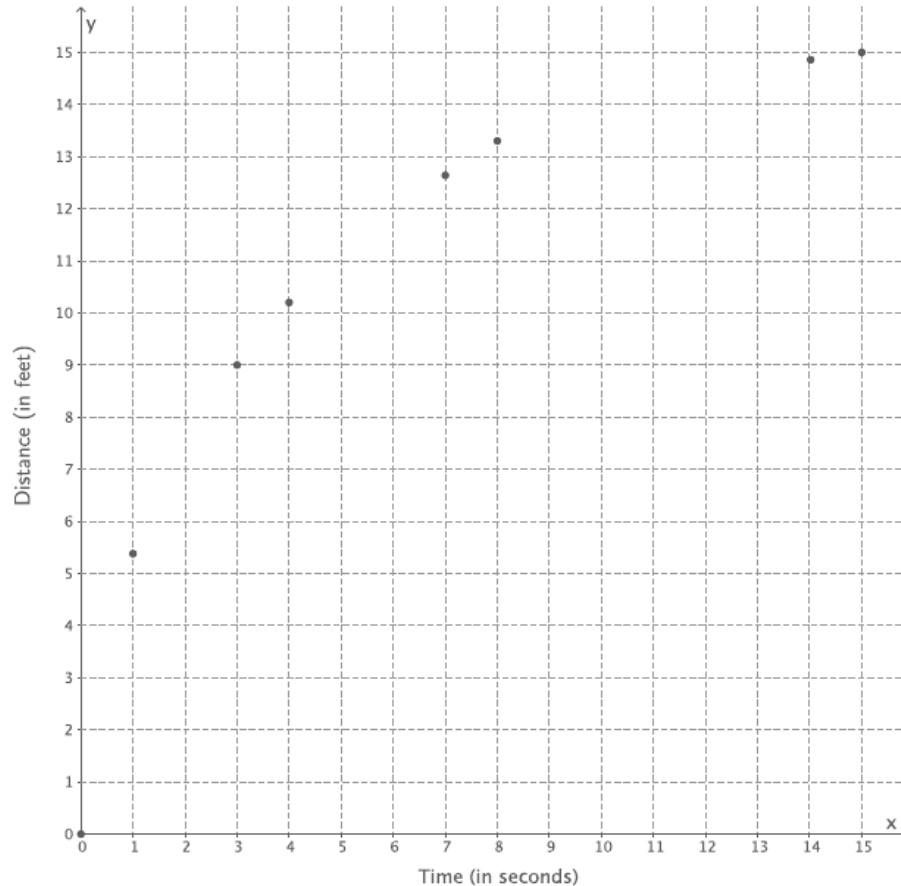
Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

▪ We know that the motion at the bottom of a ladder as it slides down a wall is not constant because the rate of change of the position of the bottom of the ladder is not constant.

▪ We have learned how to incorporate various skills to describe the rate of change in the position of the bottom of the ladder and prove that its motion is not constant by computing outputs given by a rule that describes a function and then using that data to show that the average speeds over various time intervals are not equal to the same constant.

Exit Ticket (5 minutes)

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Lesson 23: Nonlinear Motion

Exit Ticket

Suppose a book is 5.5 inches long and leaning on a shelf. The top of the book is sliding down the shelf at a rate of 0.5 in. per second. Complete the table below. Then, compute the average rate of change in the position of the bottom of the book over the intervals of time from 0 to 1 second and 10 to 11 seconds. How do you interpret these numbers?

Input (in seconds) t	Output (in inches) $d = \sqrt{30.25 - (5.5 - 0.5t)^2}$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

Exit Ticket Sample Solutions

Suppose a book is 5.5 inches long and leaning on a shelf. The top of the book is sliding down the shelf at a rate of 0.5 in. per second. Complete the table below. Then, compute the average rate of change in the position of the bottom of the book over the intervals of time from 0 to 1 second and 10 to 11 seconds. How do you interpret these numbers?

Input (in seconds) t	Output (in inches) $d = \sqrt{30.25 - (5.5 - 0.5t)^2}$
0	$\sqrt{0} = 0$
1	$\sqrt{5.25} \approx 2.29$
2	$\sqrt{10} \approx 3.16$
3	$\sqrt{14.25} \approx 3.77$
4	$\sqrt{18} \approx 4.24$
5	$\sqrt{21.25} \approx 4.61$
6	$\sqrt{24} \approx 4.90$
7	$\sqrt{26.25} \approx 5.12$
8	$\sqrt{28} \approx 5.29$
9	$\sqrt{29.25} \approx 5.41$
10	$\sqrt{30} \approx 5.48$
11	$\sqrt{30.25} = 5.50$

The average rate of change between 0 and 1 second is 2.29 inches per second.

$$\frac{2.29 - 0}{1 - 0} = \frac{2.29}{1} = 2.29.$$

The average rate of change between 10 and 11 seconds is 0.02 inches per second.

$$\frac{5.5 - 5.48}{11 - 10} = \frac{0.02}{1} = 0.02.$$

The average speeds show that the rate of change of the position of the bottom of the book is not linear. Furthermore, it shows that the rate of change of the bottom of the book is quick at first, 2.29 inches per second in the first second of motion, and then slows down to 0.02 inches per second in the one second interval from 10 to 11 seconds.

Problem Set Sample Solutions

1. Suppose the ladder is 10 feet long, and the top of the ladder is sliding down the wall at a rate of 0.8 ft. per second. Compute the average rate of change in the position of the bottom of the ladder over the intervals of time from 0 to 0.5 seconds, 3 to 3.5 seconds, 7 to 7.5 seconds, 9.5 to 10 seconds, and 12 to 12.5 seconds. How do you interpret these numbers?

Input (in seconds) t	Output (in feet) $d = \sqrt{100 - (10 - 0.8t)^2}$
0	$\sqrt{0} = 0$
0.5	$\sqrt{7.84} \approx 2.8$
3	$\sqrt{42.24} = 6.5$
3.5	$\sqrt{48.16} \approx 6.94$
7	$\sqrt{80.64} \approx 8.98$
7.5	$\sqrt{84} \approx 9.17$
9.5	$\sqrt{94.24} \approx 9.71$
10	$\sqrt{96} \approx 9.8$
12	$\sqrt{99.84} \approx 9.99$
12.5	$\sqrt{100} = 10$

The average rate of change between 0 and 0.5 seconds is 5.6 feet per second.

$$\frac{2.8 - 0}{0.5 - 0} = \frac{2.8}{0.5} = 5.6$$

The average rate of change between 3 and 3.5 seconds is 0.88 feet per second.

$$\frac{6.94 - 6.5}{3.5 - 3} = \frac{0.44}{0.5} = 0.88$$

The average rate of change between 7 and 7.5 seconds is 0.38 feet per second.

$$\frac{9.17 - 8.98}{7.5 - 5} = \frac{0.19}{0.5} = 0.38$$

The average rate of change between 9.5 and 10 seconds is 0.18 feet per second.

$$\frac{9.8 - 9.71}{10 - 9.5} = \frac{0.09}{0.5} = 0.18$$

The average rate of change between 12 and 12.5 seconds is 0.02 feet per second.

$$\frac{10 - 9.99}{12.5 - 12} = \frac{0.01}{0.5} = 0.02$$

The average speeds show that the rate of change in the position of the bottom of the ladder is not linear. Furthermore, it shows that the rate of change in the position at the bottom of the ladder is quick at first, 5.6 feet per second in the first half second of motion, and then slows down to 0.02 feet per second in the half-second interval from 12 to 12.5 seconds.

2. Will any length of ladder, L, and any constant speed of sliding of the top of the ladder, v ft. per second, ever produce a constant rate of change in the position of the bottom of the ladder? Explain.

No, the rate of change in the position at the bottom of the ladder will never be constant. We showed that if the rate were constant, there would be movement in the last second of the ladder sliding down that wall that would place the ladder in an impossible location. That is, if the rate of change were constant, then the bottom of the ladder would be in a location that exceeds the length of the ladder. Also, we determined that the distance that the bottom of the ladder is from the wall over any time period can be found using the formula $d = \sqrt{L^2 - (L - vt)^2}$, which is a non-linear equation. Since graphs of functions are equal to the graph of a certain equation, the graph of the function represented by the formula $d = \sqrt{L^2 - (L - vt)^2}$ is not a line, and the rate of change in position at the bottom of the ladder is not constant.

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When using a calculator to complete the assessment, use the π key and the full display of the calculator for computations.

1.
   a. Is a triangle with side lengths of 7 cm, 24 cm, and 25 cm a right triangle? Explain.

   b. Is a triangle with side lengths of 4 mm, 11 mm, and 15 mm a right triangle? Explain.

   c. The area of the right triangle shown below is 30 ft². The segment XY has a length of 5 ft. Find the length of the hypotenuse.
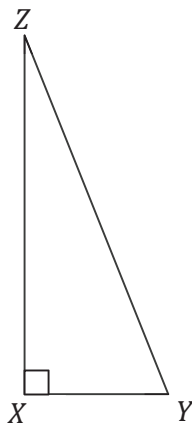
d.   Two paths from school to the store are shown below: One uses Riverside Drive, and another uses Cypress and Central Avenues. Which path is shorter? By about how much? Explain how you know.

e.   What is the distance between points A and B?
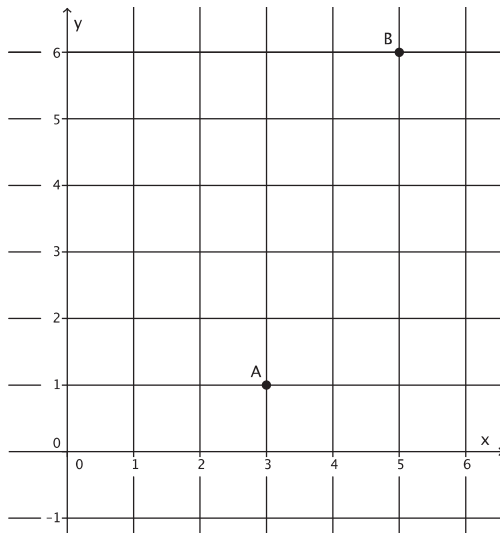
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f. Do the segments connecting the coordinates $(-1, 6)$, $(4, 2)$, and $(7, 6)$ form a right triangle? Show work that leads to your answer.
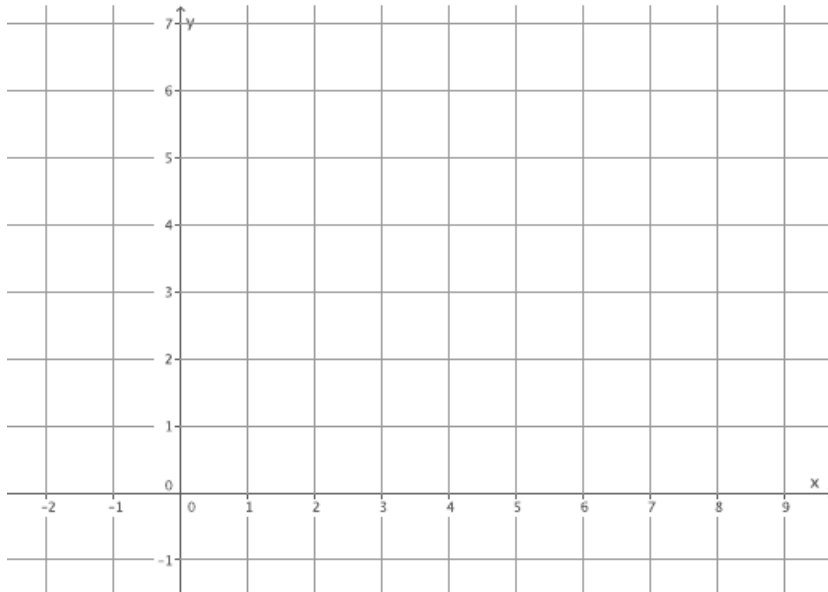
g. Using an example, illustrate and explain the Pythagorean theorem.

h. Using a different example than in part (g), illustrate and explain the converse of the Pythagorean theorem.

i. Explain a proof of the Pythagorean theorem and its converse.

2. Dorothy wants to purchase a container that will hold the most sugar. Assuming each of the containers below can be completely filled with sugar, write a note recommending a container, including justification for your choice.
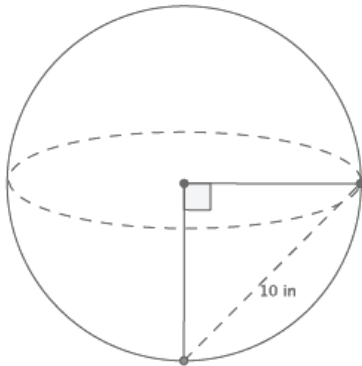
    Note: The figures are not drawn to scale.

3.
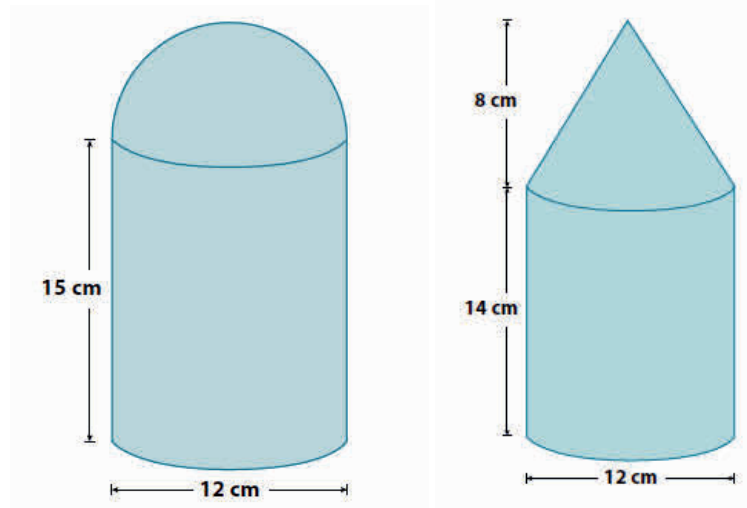
a.   Determine the volume of the cone shown below. Give an answer in terms of π and an approximate
     answer rounded to the tenths place.
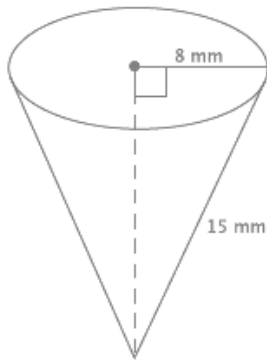
b.   The distance between the two points on the surface of the sphere shown below is 10 inches.
     Determine the volume of the sphere. Give an answer in terms of π and an approximate answer
     rounded to a whole number.

c.   A sphere has a volume of $457\frac{1}{3}\pi$ in^3. What is the radius of the sphere?

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A Progression Toward Mastery

Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	**a–b**	Student does not attempt the problem or leaves an item blank.	Student correctly responds yes or no to one of parts (a) or (b). Student may or may not provide an explanation. The explanation may show some evidence of mathematical reasoning and references the Pythagorean theorem.	Student correctly responds yes or no to one of parts (a) or (b). Student may make a mathematical error leading to an incorrect response. Student provides an explanation that references the converse of the Pythagorean theorem.	Student correctly responds (a) yes and (b) no. Student may make a mathematical error leading to an incorrect response. Student provides an explanation that references the converse of the Pythagorean theorem.
	c	Student does not attempt the problem or leaves the item blank.	Student may use the numbers 5 and 30 to determine the length of the hypotenuse. OR Student may calculate the height of the right triangle and names it as the length of the hypotenuse.	Student uses the information in the problem to determine the height of the triangle and the length of the hypotenuse. Student may make a mathematical error leading to an incorrect height and/or an incorrect hypotenuse length.	Student correctly uses the information provided to determine the height of the triangle, 12 ft., and the length of the hypotenuse, 13 ft.

	d	Student does not attempt the problem or leaves the item blank.	Student may or may not answer correctly. Student is able to calculate one of the paths correctly. OR Student is able to calculate both paths but is unable to approximate the $\sqrt{130}$. Student may or may not provide an explanation. The explanation does not make reference to the Pythagorean theorem.	Student uses the information to calculate the distance of both paths. Student may not approximate $\sqrt{130}$ correctly, leading to an incorrect answer. Student may make calculation errors that lead to an incorrect answer. Student's explanation includes the use of the Pythagorean theorem.	Student correctly uses the information provided to calculate that the shortest path is $\sqrt{130}$, or approximately 11.4, miles. Student's explanation includes the length of the other path as 16 miles. Student's explanation includes the use of the Pythagorean theorem. Student may or may not include an explanation of the approximation of $\sqrt{130}$.
	e	Student does not attempt the problem or leaves the item blank.	Student does not use the Pythagorean theorem to determine the distance between points A and B. Student may say the distance is 2 units right and 5 units up or another incorrect response.	Student uses the Pythagorean theorem to determine the distance between points A and B but may make a mathematical error leading to an incorrect answer.	Student correctly identifies the length between points A and B as $\sqrt{29}$ units by using the Pythagorean theorem.
	f	Student does not attempt the problem or leaves the item blank. Student may or may not graph the coordinates. Student finds one of the segment lengths. Student does not make use of the Pythagorean theorem.	Student may or may not answer correctly. Student may make calculation errors in using the Pythagorean theorem. Student finds one or two of the segment lengths but does not compute the third segment length. Student may make calculation errors in using the Pythagorean theorem.	Student answers correctly that the coordinates do not form a right triangle. Student makes use of the Pythagorean theorem to determine all the segment lengths of each segment. Student may make calculation errors in using the Pythagorean theorem.	Student answers correctly that the coordinates do not form a right triangle. Student makes use of the Pythagorean theorem to determine the segment lengths of each segment. Student shows the length of each segment as follows: from $(-1, 6)$ to $(7, 6)$ is 8 units, from $(-1, 6)$ to $(4, 2)$ is $\sqrt{41}$ units, and from $(4, 2)$ to $(7, 6)$ is 5 units.

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	g–i	Student does not attempt the problem or leaves an item blank. Student may use the same example to explain the Pythagorean theorem and its converse. Student's explanation does not demonstrate evidence of mathematical understanding of the Pythagorean theorem or its converse.	Student may or may not use different examples to explain the Pythagorean theorem and its converse. Student may or may not explain a proof of the Pythagorean theorem or its converse. Student's explanation lacks precision and misses many key points in the logic of the proofs. Student's explanation demonstrates some evidence of mathematical understanding of the Pythagorean theorem or its converse.	Student uses different examples to explain the Pythagorean theorem and its converse. Student explains a proof of the Pythagorean theorem and its converse. Student's explanation, though correct, may lack precision or miss a few key points in the logic of the proofs. There is substantial evidence that the student understands the proof of the Pythagorean theorem and its converse.	Student uses different examples to explain the Pythagorean theorem and its converse. Student thoroughly explains a proof of the Pythagorean theorem and its converse. Student uses appropriate mathematical vocabulary and demonstrates, with strong evidence, an understanding of the proofs. Student uses one of the proofs of the Pythagorean theorem found in M2–L15, M3–L13, or M7–L17 and the proof of the converse found in M3–L14 or M7–L18.
2		Student does not attempt the problem or leaves the item blank.	Student incorrectly applies the volume formulas leading to incorrect answers. Student may or may not identify the cylinder with the half-sphere on top as the container with the greatest volume. Student may or may not write a note with a recommendation for Dorothy.	Student correctly applies volume formulas but may make a mathematical error leading to an incorrect answer. Student may or may not identify the cylinder with the half-sphere on top as the container with the greatest volume. Student may or may not write a note with a recommendation for Dorothy.	Student correctly calculates the volume of each container, 684π and 600π. Student correctly identifies the cylinder with the half-sphere on top as the container with the greatest volume. Student writes a note with a recommendation for Dorothy.
3	**a**	Student does not attempt the problem or leaves the item blank.	Student may or may not determine the height of the cone using the Pythagorean theorem. Student may or may not apply the volume formula for a cone to determine the volume. There is some evidence that the student knows what to do but is unable to apply the correct mathematical concepts to determine the volume.	Student correctly applies the Pythagorean theorem to determine the height of the cone or correctly applies the volume formula for a cone but makes a mathematical error leading to an incorrect answer.	Student correctly calculates the volume of the cone in terms of π as $\frac{64}{3}\sqrt{161}\pi$ mm^3 and the approximate volume of the cone as 850.4 mm^3.

		Student does not attempt the problem or leaves the item blank.	Student may or may not determine the radius of the sphere using the Pythagorean theorem. Student may or may not apply the volume formula for a sphere to determine the volume. There is some evidence that the student knows what to do but is unable to apply the correct mathematical concepts to determine the volume.	Student correctly applies the Pythagorean theorem to determine the radius of the sphere or correctly applies the volume formula for a sphere but makes a mathematical error leading to an incorrect answer.	Student correctly calculates the volume of the sphere in terms of π as $\frac{4}{3}\left(\sqrt{50}\right)^3\pi$ in^3 and the approximate volume of the sphere as $1{,}481$ in^3.
b					
c		Student does not attempt the problem or leaves the item blank.	Student uses the formula for the volume of a sphere to write an equation but is unable to solve the equation to determine r. OR Student may use the wrong volume formula, leading to an incorrect answer.	Student uses the formula for the volume of a sphere to write an equation. Student may make a mathematical error leading to an incorrect answer. OR Student may leave the answer in the form of $\sqrt[3]{343}$.	Student correctly identifies the radius of the sphere as 7 in.

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When using a calculator to complete the assessment, use the π key and the full display of the calculator for computations.

1.

     a.   Is a triangle with side lengths of 7 cm, 24 cm, and 25 cm a right triangle? Explain.

$$7^2 + 24^2 = 25^2$$
$$49 + 576 = 625$$
$$625 = 625$$

Yes. The lengths 7, 24, 25 satisfy the Pythagorean theorem; therefore, it is a right triangle.

     b.   Is a triangle with side lengths of 4 mm, 11 mm, and 15 mm a right triangle? Explain.

$$4^2 + 11^2 = 15^2$$
$$16 + 121 = 225$$
$$137 \neq 225$$

No. The lengths 4, 11, 15 do not satisfy the Pythagorean theorem; therefore, it is not a right triangle.

     c.   The area of the right triangle shown below is 30 ft². The segment XY has a length of 5 ft. Find the length of the hypotenuse.
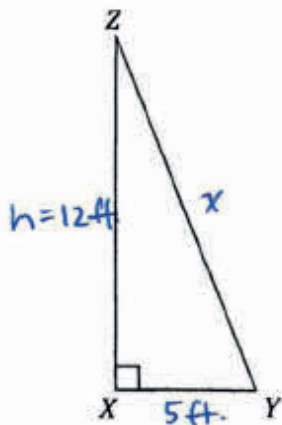
$$\frac{h(5)}{2} = 30$$
$$5h = 60$$
$$h = 12$$

$$5^2 + 12^2 = x^2$$
$$25 + 144 = x^2$$
$$169 = x^2$$
$$\sqrt{169} = x$$
$$13 = x$$

$h = 12$ ft

The length of the hypotenuse is 13 ft.

d. Two paths from school to the store are shown below: One uses Riverside Drive, and another uses Cypress and Central Avenues. Which path is shorter? By about how much? Explain how you know.
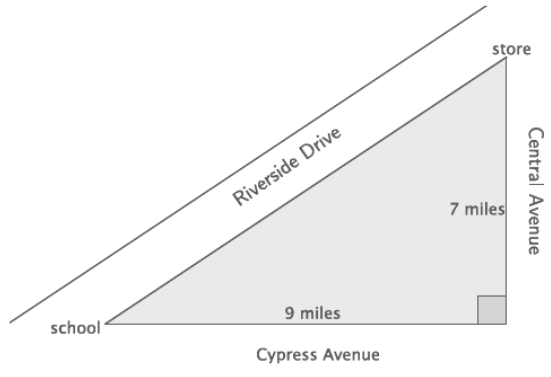
Let c be the hypotenuse in miles.

$$7^2 + 9^2 = c^2$$
$$49 + 81 = c^2$$
$$130 = c^2$$
$$\sqrt{130} = \sqrt{c^2}$$
$$\sqrt{130} = c$$
$$11.4 \approx c$$

The path along Riverside Drive is shorter, about 11.4 miles, compared to the path along Cypress and Central Avenues, 16 miles. The difference is about 4.6 miles. The Pythagorean theorem allowed me to calculate the distance along Riverside Drive because the three roads form a right triangle.

e. What is the distance between points A and B?
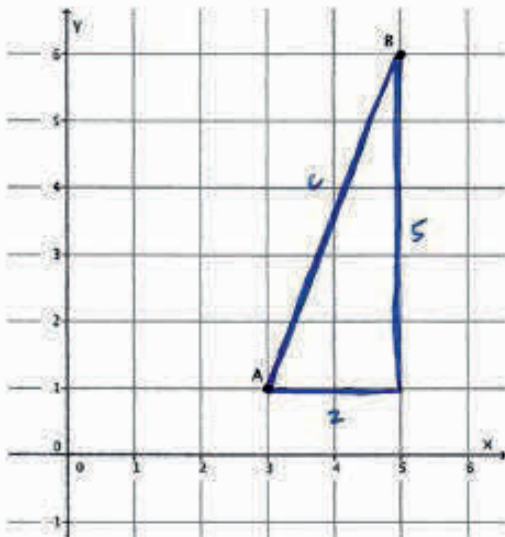
LET c BE THE DISTANCE BETWEEN POINTS A & B.

$$2^2 + 5^2 = c^2$$
$$4 + 25 = c^2$$
$$29 = c^2$$
$$\sqrt{29} = \sqrt{c^2}$$
$$\sqrt{29} = c$$
$$5.4 \approx c$$

THE DISTANCE BETWEEN POINTS A & B IS ABOUT 5.4 UNITS.

f. Do the segments connecting the coordinates $(-1, 6)$, $(4, 2)$, and $(7, 6)$ form a right triangle? Show work that leads to your answer.
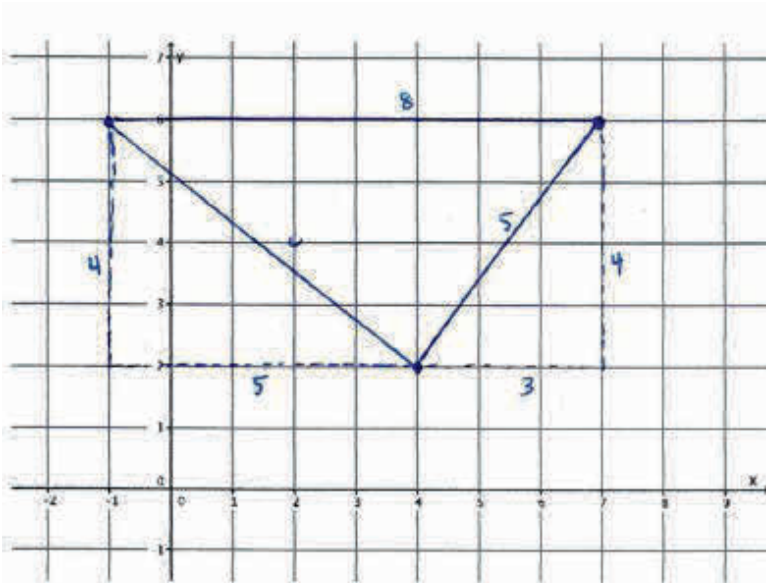
$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

$25 = 25$

$4^2 + 5^2 = c^2$

$16 + 25 = c^2$

$41 = c^2$

$\sqrt{41} = \sqrt{c^2}$

$\sqrt{41} = c$

$6 < \sqrt{41} < 7$, so side 8 units is the longest.

$5^2 + (\sqrt{41})^2 = 8^2$

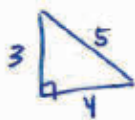$25 + 41 = 64$

$66 \neq 64$

No. The segments connecting $(-1, 6)$, $(4, 2)$, and $(7, 6)$ do not form a right triangle because their lengths do not satisfy the Pythagorean theorem.

g. Using an example, illustrate and explain the Pythagorean theorem.

Given a right triangle ABC, the sides a, b, c (where c is the hypotenuse) satisfy $a^2 + b^2 = c^2$.
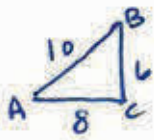
$a = 3$, $b = 4$, $c = 5$

$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

$25 = 25$

h. Using a different example than in part (g), illustrate and explain the converse of the Pythagorean theorem.

Given a triangle ABC with side lengths a, b, c (where c is the hypotenuse) that satisfies $a^2 + b^2 = c^2$, then triangle ABC is a right triangle.

$$6^2 + 8^2 = 10^2$$
$$36 + 64 = 100$$
$$100 = 100$$

Therefore, triangle ABC is a right triangle because it satisfies the converse of the Pythagorean theorem.

i. Explain a proof of the Pythagorean theorem and its converse.

See rubric to locate proofs of the theorem and its converse within the module.

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2. Dorothy wants to purchase a container that will hold the most sugar. Assuming each of the containers below can be completely filled with sugar, write a note recommending a container, including justification for your choice.

Note: The figures are not drawn to scale.
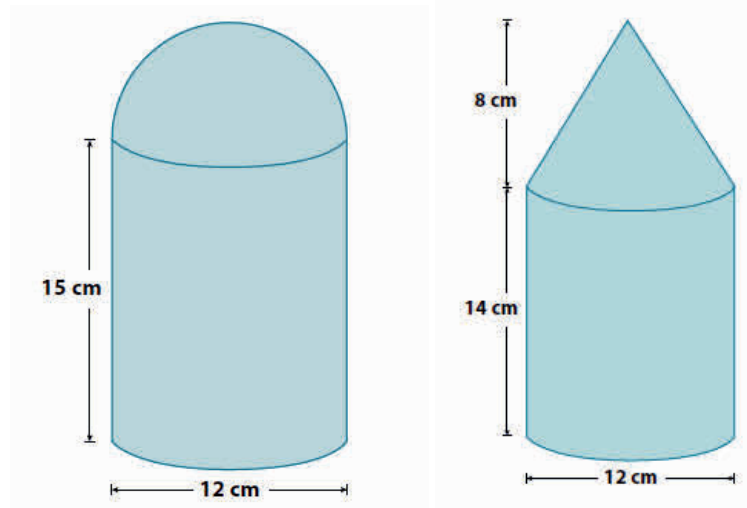
Cylinder:
$$V = 6^2 \pi (15)$$
$$= 540 \pi$$

$\frac{1}{2}$ Sphere:
$$V = \frac{1}{2} \left(\frac{4}{3}\right) \pi \, 6^3$$
$$= \frac{2}{3} (216) \pi$$
$$= 144 \pi$$

Total Volume:
$$540\pi + 144\pi = 684\pi$$

Cylinder:
$$V = 6^2 \pi (14)$$
$$= 504 \pi$$

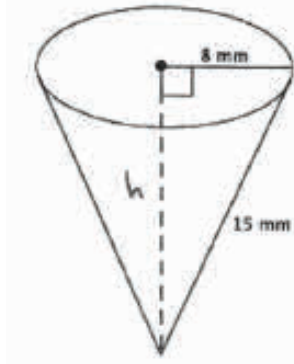Cone:
$$V = \frac{1}{3} \pi (6^2) 8$$
$$= 96 \pi$$

Total Volume:
$$504\pi + 96\pi = 600\pi$$

Dorothy,

    You should choose the container with the half sphere on top because it has a greater volume than the container with the cone on top. The containers have volumes of 684π cm³ and 600π cm³, respectively. Since 684π is greater than 600π, then the container with the half sphere will hold more sugar compared to the container with the cone on top.

3.

a. Determine the volume of the cone shown below. Give an answer in terms of π and an approximate answer rounded to the tenths place.
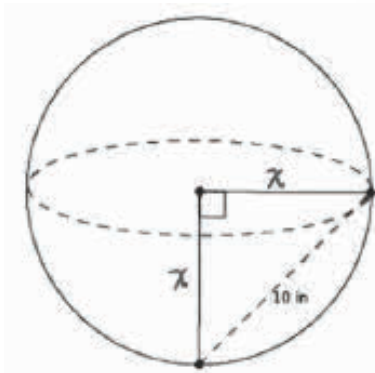

$8^2 + h^2 = 15^2$

$64 + h^2 = 225$

$h^2 = 161$

$\sqrt{h^2} = \sqrt{161}$

$h = \sqrt{161}$

$V = \frac{1}{3}\pi(64)(\sqrt{161})$

$= \frac{64}{3}\sqrt{161}\ \pi$

≈ 850.4

THE VOLUME OF THE CONE IS EXACTLY $\frac{64}{3}\sqrt{161}\ \pi$ mm³

AND APPROXIMATELY 850.4 mm³.

b. The distance between the two points on the surface of the sphere shown below is 10 inches. Determine the volume of the sphere. Give an answer in terms of π and an approximate answer rounded to a whole number.

$x^2 + x^2 = 10^2$

$2x^2 = 100$

$x^2 = 50$

$\sqrt{x^2} = \sqrt{50}$

$x = \sqrt{50}$

$V = \frac{4}{3}\pi(\sqrt{50})^3$

≈ 1481

THE VOLUME OF THE SPHERE IS EXACTLY $\frac{4}{3}\pi(\sqrt{50})^3$ in³

AND APPROXIMATELY 1481 in³.

c. A sphere has a volume of $457\frac{1}{3}\pi$ in³. What is the radius of the sphere?

$V = 457\frac{1}{3}\pi$

$\dfrac{\frac{4}{3}\pi r^3}{\pi} = \dfrac{457\frac{1}{3}\pi}{\pi}$

$\frac{4}{3}r^3 = 457\frac{1}{3}$

$r^3 = 457\frac{1}{3} \times \frac{3}{4}$

$r^3 = \frac{1372}{3} \times \frac{3}{4}$

$r^3 = \frac{1372}{4}$

$r^3 = 343$

$\sqrt[3]{r^3} = \sqrt[3]{343}$

$r = 7$

The radius of the sphere is 7 in.

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