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Teacher Edition

Eureka Math
Algebra I
Module 4

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Polynomial and Quadratic Expressions, Equations, and Functions

[1]Each lesson is ONE day, and ONE day is considered a 45-minute period.

Algebra I • Module 4

Polynomial and Quadratic Expressions, Equations, and Functions

OVERVIEW

By the end of middle school, students are familiar with linear equations in one variable and have applied graphical and algebraic methods to analyze and manipulate equations in two variables. They use expressions and equations to solve real-life problems. They have experience with square and cube roots, irrational numbers, and expressions with integer exponents.

In Algebra I, students have been analyzing the process of solving equations and developing fluency in writing, interpreting, and translating among various forms of linear equations (Module 1) and linear and exponential functions (Module 3). These experiences, combined with modeling with data (Module 2), set the stage for Module 4. Here, students continue to interpret expressions, create equations, rewrite equations and functions in different but equivalent forms, and graph and interpret functions using polynomial functions—more specifically quadratic functions as well as square root and cube root functions.

Topic A introduces polynomial expressions. In Module 1, students learn the definition of a polynomial and how to add, subtract, and multiply polynomials. Here, their work with multiplication is extended and connected to factoring polynomial expressions and solving basic polynomial equations. They analyze, interpret, and use the structure of polynomial expressions to multiply and factor polynomial expressions. They understand factoring as the reverse process of multiplication. In this topic, students develop the factoring skills needed to solve quadratic equations and simple polynomial equations by using the zero product property. Students transform quadratic expressions from standard form, $ax^2 + bx + c$, to factored form, $a(x - m)(x - n)$, and then solve equations involving those expressions. They identify the solutions of the equation as the zeros of the related function. Students apply symmetry to create and interpret graphs of quadratic functions. They use average rate of change on an interval to determine where the function is increasing or decreasing. Using area models, students explore strategies for factoring more complicated quadratic expressions, including the product-sum method and rectangular arrays. They create one- and two-variable equations from tables, graphs, and contexts and use them to solve contextual problems represented by the quadratic function. Students then relate the domain and range for the function to its graph and the context.

Students apply their experiences from Topic A as they transform quadratic functions from standard form to vertex form, $f(x) = a(x - h)^2 + k$, in Topic B. The strategy known as *completing the square* is used to solve quadratic equations when the quadratic expression cannot be factored. Students recognize that this form reveals specific features of quadratic functions and their graphs, namely the *minimum* or *maximum of the function* (i.e., the vertex of the graph) and the line of symmetry of the graph.

Students derive the quadratic formula by completing the square for a general quadratic equation in standard form, $y = ax^2 + bx + c$, and use it to determine the nature and number of solutions for equations when y equals zero. For quadratic equations with irrational roots, students use the quadratic formula and explore the properties of irrational numbers. With the added technique of completing the square in their toolboxes, students come to see the structure of the equations in their various forms as useful for gaining insight into the features of the graphs of equations. Students study business applications of quadratic functions as they create quadratic equations and graphs from tables and contexts and then use them to solve problems involving profit, loss, revenue, cost, etc. In addition to applications in business, students solve physics-based problems involving objects in motion. In doing so, students also interpret expressions and parts of expressions in context and recognize when a single entity of an expression is dependent or independent of a given quantity.

In Topic C, students explore the families of functions that are related to the parent functions, specifically for quadratic ($f(x) = x^2$), square root ($f(x) = \sqrt{x}$), and cube root ($f(x) = \sqrt[3]{x}$), to perform horizontal and vertical translations as well as shrinking and stretching. They recognize the application of transformations in vertex form for a quadratic function and use it to expand their ability to efficiently sketch graphs of square and cube root functions. Students compare quadratic, square root, or cube root functions in context and represent each in different ways (verbally with a description, numerically in tables, algebraically, or graphically). In the final two lessons, students examine real-world problems of quadratic relationships presented as a data set, a graph, a written relationship, or an equation. They choose the most useful form for writing the function and apply the techniques learned throughout the module to analyze and solve a given problem, including calculating and interpreting the rate of change for the function over an interval.

Focus Standards

Use properties of rational and irrational numbers.

- Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Interpret the structure of expressions.

- Interpret expressions that represent a quantity in terms of its context.★
  - Interpret parts of an expression, such as terms, factors, and coefficients.[2]
  - Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

[2]The "such as" listed are not the only parts of an expression students are expected to know; others include, but are not limited to, degree of a polynomial, leading coefficient, constant term, and the standard from of a polynomial (descending exponents).

- Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.[3]

Write expressions in equivalent forms to solve problems.

- Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★
  - Factor a quadratic expression to reveal the zeros of the function it defines.[4]
  - Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

Perform arithmetic operations on polynomials.

- Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

- Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.[5]

Create equations that describe numbers or relationships.

- Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions*.★[6]
- Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

[3]In Algebra I, tasks are limited to numerical expressions and polynomial expressions in one variable. Examples: Recognize that $53^2 - 47^2$ is the difference of squares, and see an opportunity to rewrite it in the easier-to-evaluate form $(53 - 47)(53 + 47)$. See an opportunity to rewrite $a^2 + 9a + 14$ as $(a + 7)(a + 2)$. This does not include factoring by grouping and factoring the sum and difference of cubes.
[4]Includes trinomials and leading coefficients other than 1.
[5]In Algebra I, tasks are limited to quadratic and cubic polynomials, in which linear and quadratic factors are available. For example, find the zeros of $(x - 2)(x^2 - 9)$.
[6]In Algebra I, tasks are limited to linear, quadratic, or exponential equations with integer exponents.

Solve equations and inequalities in one variable.

- Solve quadratic equations in one variable.[7]
  - Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
  - Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.[8]

Represent and solve equations and inequalities graphically.

- Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★[9]

Interpret functions that arise in applications in terms of the context.

- For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★[10]
- Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★
- Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★[11]

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[7] Solutions may include simplifying radicals.
[8] Tasks do not require students to write solutions for quadratic equations that have roots with nonzero imaginary parts. However, tasks can require that students recognize cases in which a quadratic equation has no real solutions.
[9] In Algebra I, tasks that assess conceptual understanding of the indicated concept may involve any of the function types mentioned except exponential and logarithmic functions. Finding the solutions approximately is limited to cases where $f(x)$ and $g(x)$ are polynomial functions.
[10] Tasks have a real-world context. In Algebra I, tasks are limited to linear functions, quadratic functions, square root functions, cube root functions, piecewise-defined functions (including step functions and absolute value functions), and exponential functions with domains in the integers. The focus in this module is on linear and exponential functions.
[11] Tasks have a real-world context. In Algebra I, tasks are limited to linear functions, quadratic functions, square root functions, cube root functions, piecewise-defined functions (including step functions and absolute value functions), and exponential functions with domains in the integers. The focus in this module is on linear and exponential functions.

Analyze functions using different representations.

- Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
  - Graph linear and quadratic functions and show intercepts, maxima, and minima.
  - Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
- Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
  - Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
- Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*[12]

Build new functions from existing functions.

- Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*[13]

Foundational Standards

Know that there are numbers that are not rational, and approximate them by rational numbers.

- Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

[12]In Algebra I, tasks are limited to linear functions, quadratic functions, square root functions, cube root functions, piecewise-defined functions (including step functions and absolute value functions), and exponential functions with domains in the integers.

[13]In Algebra I, identifying the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative) is limited to linear and quadratic functions. Experimenting with cases and illustrating an explanation of the effects on the graph using technology is limited to linear functions, quadratic functions, square root functions, cube root functions, piecewise-defined functions (including step functions and absolute value functions), and exponential functions with domains in the integers. Tasks do not involve recognizing even and odd functions.

Work with radicals and integer exponents.

- Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example,* $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.

Reason quantitatively and use units to solve problems.

- Define appropriate quantities for the purpose of descriptive modeling.★[14]
- Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★

Create equations that describe numbers or relationships.

- Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R.★

Understand solving equations as a process of reasoning and explain the reasoning.

- Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

- Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Represent and solve equations and inequalities graphically.

- Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Understand the concept of a function and use function notation.

- Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
- Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

[14]This will be assessed in Algebra I by ensuring that some modeling tasks (involving Algebra I content or securely held content from Grades 6–8) require the student to create a quantity of interest in the situation being described.

Build a function that models a relationship between two quantities.

- Write a function that describes a relationship between two quantities.★ [15]
  - Determine an explicit expression, a recursive process, or steps for calculation from a context.

Focus Standards for Mathematical Practice

- **Make sense of problems and persevere in solving them.** Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. In Module 4, students make sense of problems by analyzing the critical components of the problem, a verbal description, data set, or graph and persevere in writing the appropriate function to describe the relationship between two quantities.
- **Reason abstractly and quantitatively.** Mathematically proficient students make sense of quantities and their relationships in problem situations. This module alternates between algebraic manipulation of expressions and equations and interpretation of the quantities in the relationship in terms of the context. Students must be able to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own without necessarily attending to their referents, and then to *contextualize*—to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning requires the habit of creating a coherent representation of the problem at hand, considering the units involved, attending to the meaning of quantities (not just how to compute them), knowing different properties of operations, and using them with flexibility.
- **Model with mathematics.** Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In this module, students create a function from a contextual situation described verbally, create a graph of their function, interpret key features of both the function and the graph (in the terms of the context), and answer questions related to the function and its graph. They also create a function from a data set based on a contextual situation. In Topic C, students use the full modeling cycle. They model quadratic functions presented mathematically or in a context. They explain the reasoning used in their writing or by using appropriate tools, such as graphing paper, graphing calculator, or computer software.

[15] Tasks have a real-world context. In Algebra I, tasks are limited to linear functions, quadratic functions, and exponential functions with domains in the integers.

- **Use appropriate tools strategically.** Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. Throughout the entire module, students must decide whether to use a tool to help find a solution. They must graph functions that are sometimes difficult to sketch (e.g., cube root and square root functions) and functions that are sometimes required to perform procedures that, when performed without technology, can be tedious and distract students from thinking mathematically (e.g., completing the square with non-integer coefficients). In such cases, students must decide when to use a tool to help with the calculation or graph so they can better analyze the model.
- **Attend to precision.** Mathematically proficient students try to communicate precisely to others. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. When calculating and reporting quantities in all topics of Module 4, students must be precise in choosing appropriate units and use the appropriate level of precision based on the information as it is presented. When graphing, they must select an appropriate scale.
- **Look for and make use of structure.** Mathematically proficient students look closely to discern a pattern or structure. They can see algebraic expressions as single objects, or as a composition of several objects. In this module, students use the structure of expressions to find ways to rewrite them in different but equivalent forms. For example, in the expression $x^2 + 9x + 14$, students must see the 14 as 2×7 and the 9 as $2 + 7$ to find the factors of the quadratic. In relating an equation to a graph, they can see $y = -3(x-1)^2 + 5$ as 5 added to a negative number times a square and realize that its value cannot be more than 5 for any real domain value.

Terminology

New or Recently Introduced Terms

- **Axis of symmetry of the graph of a quadratic function** (Given a quadratic function in standard form, $f(x) = ax^2 + bx + c$, the vertical line given by the graph of the equation, $x = -\frac{b}{2a}$, is called the *axis of symmetry* of the graph of the quadratic function.)
- **Cube root function** (The parent function $f(x) = \sqrt[3]{x}$.)
- **Cubic function** (A polynomial function of degree 3.)
- **Degree of a monomial term** (The *degree of a monomial term* is the sum of the exponents of the variables that appear in a term of a polynomial.)
- **Degree of a polynomial** (The *degree of a polynomial in one variable in standard form* is the highest degree of the terms in the polynomial.)

- **Discriminant** (The *discriminant* of a quadratic function in the form $ax^2 + bx + c = 0$ is $b^2 - 4ac$. The nature of the roots of a quadratic equation can be identified by determining if the discriminant is positive, negative, or equal to zero.)
- **End behavior of a quadratic function** (Given a quadratic function in the form $f(x) = ax^2 + bx + c$ (or $f(x) = a(x - h)^2 + k$), the quadratic function is said to *open up* if $a > 0$ and *open down* if $a < 0$.)
- **Factored form for a quadratic function** (A quadratic function written in the form $f(x) = a(x - n)(x - m)$.)
- **Leading coefficient** (The *leading coefficient* of a polynomial is the coefficient of the term of highest degree.)
- **Parent function** (A *parent function* is the simplest function in a "family" of functions that can each be formed by one or more transformations of another.)
- **Quadratic formula** (The *quadratic formula* is the formula that emerges from solving the general form of a quadratic equation by completing the square, $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. It can be used to solve any quadratic equation.)
- **Quadratic function** (A polynomial function of degree 2.)
- **Roots of a polynomial function** (The domain values for a polynomial function that make the value of the polynomial function equal zero when substituted for the variable.)
- **Square root function** (The parent function $f(x) = \sqrt{x}$.)
- **Standard form for a quadratic function** (A quadratic function written in the form $f(x) = ax^2 + bx + c$.)
- **Standard form of a polynomial in one variable** (A polynomial expression with one variable symbol x is in *standard form* if it is expressed as,

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a non-negative integer, and $a_0, a_1, a_2, \ldots, a_n$ are constant coefficients with $a_n \neq 0$.)

- **Vertex form** (Completed-square form for a quadratic function; in other words, written in the form $f(x) = a(x - h)^2 + k$.)
- **Vertex of the graph of a quadratic function** (The point where the graph of a quadratic function and its axis of symmetry intersect is called the *vertex*. The vertex is either a maximum or a minimum of the quadratic function, depending on whether the leading coefficient of the function in standard form is negative or positive, respectively.)

Familiar Terms and Symbols[16]

- Average rate of change
- Binomial
- Closed
- Closure
- Coefficient
- Cube root
- Cubic
- Degree of a polynomial
- Domain and range
- Explicit expression
- Factor
- Integers
- Irrational numbers
- Monomial
- Parabola
- Power
- Quadratic
- Rational numbers
- Real numbers
- Recursive process
- Solution set
- Solutions (solution set) of an equation
- Square root
- Term
- Trinomial
- Zeros of a function

Suggested Tools and Representations

- Coordinate plane
- Equations
- Graphing calculator
- Graph paper

[16]These are terms and symbols students have seen previously.

Assessment Summary

Assessment Type	Administered	Format
Mid-Module Assessment Task	After Topic A	Constructed response with rubric
End-of-Module Assessment Task	After Topic C	Constructed response with rubric

Topic A

Quadratic Expressions, Equations, Functions, and Their Connection to Rectangles

Focus Standards:

- Interpret expressions that represent a quantity in terms of its context.★
  - Interpret parts of an expression, such as terms, factors, and coefficients.
  - Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.*
- Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*
- Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★
  - Factor a quadratic expression to reveal the zeros of the function it defines.
- Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
- Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★
- Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

- Solve quadratic equations in one variable.
  - Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
- Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★
- For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity*.★
- Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★
- Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
- Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
  - Graph linear and quadratic functions and show intercepts, maxima, and minima.

Instructional Days: 10

Lessons 1–2: Multiplying and Factoring Polynomial Expressions (P, P)[1]
Lessons 3–4: Advanced Factoring Strategies for Quadratic Expressions (P, P)
Lesson 5: The Zero Product Property (E)
Lesson 6: Solving Basic One-Variable Quadratic Equations (P)
Lesson 7: Creating and Solving Quadratic Equations in One Variable (P)
Lesson 8: Exploring the Symmetry in Graphs of Quadratic Functions (E)
Lesson 9: Graphing Quadratic Functions from Factored Form, $f(x) = a(x - m)(x - n)$ (P)
Lesson 10: Interpreting Quadratic Functions from Graphs and Tables (P)

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Deep conceptual understanding of operations with polynomials is the focus of this topic. The emphasis is on using the properties of operations for multiplying and factoring quadratic trinomials, including the connections to numerical operations and rectangular geometry, rather than using common procedural gimmicks such as FOIL. In Topic A, students begin by using the distributive property to multiply monomials by polynomials. They relate binomial expressions to the side lengths of rectangles and find area by multiplying binomials, including those whose expanded form is the difference of squares and perfect squares. They analyze, interpret, and use the structure of polynomial expressions to factor, with the understanding that factoring is the reverse process of multiplication. There are two exploration lessons in Topic A. The first is Lesson 6, in which students explore all aspects of solving quadratic equations, including using the zero product property. The second is Lesson 8, where students explore the unique symmetric qualities of quadratic graphs. Both explorations are revisited and extended throughout this topic and the module.

In Lesson 3, students encounter quadratic expressions for which extracting the GCF is impossible (the leading coefficient, a, is not 1 and is not a common factor of the terms). They discover the importance of the product of the leading coefficient and the constant (ac) and become aware of its use when factoring expressions such as $6x^2 + 5x - 6$. In Lesson 4, students explore other factoring strategies strongly associated with the area model, such as using the area method or a table to determine the product-sum combinations. In Lesson 5, students discover the zero product property and solve for one variable by setting factored expressions equal to zero. In Lesson 6, they decontextualize word problems to create equations and inequalities that model authentic scenarios addressing area and perimeter.

Finally, students build on their prior experiences with linear and exponential functions and their graphs to include interpretation of quadratic functions and their graphs. Students explore and identify key features of quadratic functions and calculate and interpret the average rate of change from the graph of a function. Key features include x-intercepts (zeros of the function), y-intercepts, the vertex (minimum or maximum values of the function), end behavior, and intervals where the function is increasing or decreasing. It is important for students to use these features to understand how functions behave and to interpret a function in terms of its context.

A focus of this topic is to develop a deep understanding of the symmetric nature of a quadratic function. Students use factoring to reveal its zeros and then use these values and their understanding of quadratic function symmetry to determine the axis of symmetry and the coordinates of the vertex. Often, students are asked to use $x = -\frac{b}{2a}$ as an efficient way of finding the axis of symmetry or the vertex. (Note: Students learn to use this formula without understanding that this is a generalization for the average of the domain values for the x-intercepts.) Only after students develop an understanding of symmetry is $x = -\frac{b}{2a}$ explored as a general means of finding the axis of symmetry.

Lesson 1: Multiplying and Factoring Polynomial Expressions

Student Outcomes

- Students use the distributive property to multiply a monomial by a polynomial and understand that factoring reverses the multiplication process.
- Students use polynomial expressions as side lengths of polygons and find area by multiplying.
- Students recognize patterns and formulate shortcuts for writing the expanded form of binomials whose expanded form is a perfect square or the difference of perfect squares.

Lesson Notes

Central to the concepts in this lesson is understanding the system and operations of polynomial expressions, specifically multiplication and factoring of polynomials. Lengths of time suggested for the examples and exercises in this lesson assume that students remember some of what is presented in the examples from work in earlier modules and earlier grades. Students may need more or less time with this lesson than is suggested. The teacher should make decisions about how much time is needed for these concepts based on students' needs.

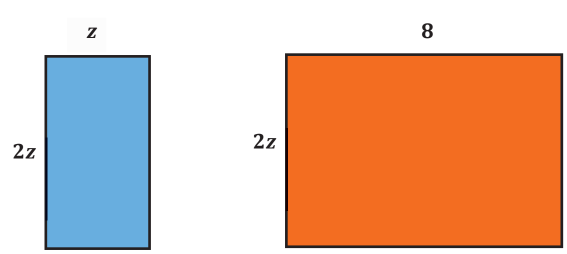
This lesson asks students to use geometric models to demonstrate their understanding of multiplication of polynomials.

Classwork

Opening Exercise (4 minutes)

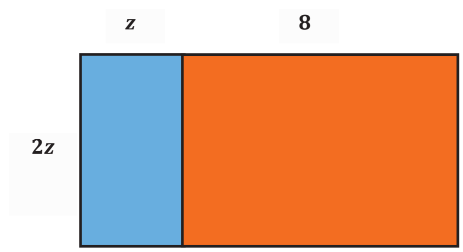
Opening Exercise

Write expressions for the areas of the two rectangles in the figures given below.



Blue: $2z(z) = 2z^2$; *Orange:* $2z(8) = 16z$

Now write an expression for the area of this rectangle:



$\mathbf{2z^2 + 16z}$

- How did you find your answer for the second rectangle?
  - *Add the two areas in part (a), or multiply the length of the rectangle by the width* $2z(z+8) = 2z^2 + 16z$. *Both give the same final result.*
- If you find the area by multiplying the total length times total width, what property of operations are you using?
  - *The distributive property*
- What would be another way to find the total area?
  - *Finding the area of the two separate rectangles and adding their areas:* $2z(z) + 2z(8) = 2z^2 + 16z$

Scaffolding:

Model the following examples of polynomial multiplication for students who need to review.
Multiply two monomials: Point out to students that because multiplication is both commutative and associative, factors may be reordered to group the numerical factors together and then the variable factors together.
$5ab \cdot 4c = (5 \cdot 4)(ab \cdot c) = 20abc$

$3x^2 \cdot 4x^3y \cdot 5y^2 = (3 \cdot 4 \cdot 5)(x^2 \cdot x^3 \cdot y^1 \cdot y^2) = 60x^{2+3}\, y^{1+2} = 60x^5y^3$

Multiply a polynomial by a monomial: Some students may benefit from relating multiplication of polynomials to multiplication of numbers in base 10. In the example below, the multiplication process is represented vertically (like a base 10 product of a 2-digit number by a 1-digit number) and then horizontally, using the distributive property.

Multiply $(5a + 7b)$ by $3c$. To find this product vertically, follow the same procedure as you would with place values for whole numbers. Just be sure to follow the rules for combining like terms. Show how to multiply vertically.

$$\begin{array}{r} 5a + 7b \\ \underline{\times \quad\quad 3c} \\ 15ac\ + 21bc \end{array}$$

Now, multiply the polynomial by the monomial horizontally, using the distributive property for multiplication over addition. Make sure each term of the first binomial is distributed over both terms of the second.

$(5a + 7b)3c = (5a \cdot 3c) + (7b \cdot 3c) = 15ac + 21bc$
(The associative property for multiplication allows us to group the numbers and the variables together.)

Example 1 (3 minutes)

Have students work on this example with a partner or in small groups. Have the groups share their processes and their findings and discuss the differences in processes used (if there are any).

Example 1

Jackson has given his friend a challenge:

The area of a rectangle, in square units, is represented by $3a^2 + 3a$ for some real number a. Find the length and width of the rectangle.

How many possible answers are there for Jackson's challenge to his friend? List the answer(s) you find.

$3a^2 + 3a$ square units

$3a^2 + 3a = 3a(a + 1)$ so the width of the rectangle could be $3a$ units and the length could be $(a + 1)$ units.

(Students may opt to factor only a or 3 or even $\frac{1}{2}a$: $a(3a + 3)$ or $3(a^2 + a)$ or $\frac{1}{2}a(6a + 6)$.)

There are infinite representations for the dimensions of the rectangle.

If students try to use 1 as the common factor for two or more numbers, point out that, while 1 is indeed a factor, factoring out a 1 does not help in finding the factors of an expression. If this issue arises, it may be necessary to discuss the results when factoring out a 1.

Factoring out the Greatest Common Factor (GCF)

Students now revisit factoring out the greatest common factor as was introduced in Grade 6, Module 2.

- When factoring a polynomial, we first look for a monomial that is the greatest common factor (GCF) of all the terms of the polynomial. Then, we reverse the distribution process by factoring the GCF out of each term and writing it on the outside of the parentheses.
- In the example above, we factored out the GCF: $3a$.

Exercises 1–3 (3 minutes)

For the exercises below, have students work with a partner or in small groups to factor out the GCF for each expression.

Exercises 1–3

Factor each by factoring out the greatest common factor:

1. $10ab + 5a$

   $5a(2b + 1)$

> **2.** $3g^3h - 9g^2h + 12h$
>
> $3h(g^3 - 3g^2 + 4)$
>
> **3.** $6x^2y^3 + 9xy^4 + 18y^5$
>
> $3y^3(2x^2 + 3xy + 6y^2)$ ***(Students may find this one to be more difficult. It is used as an example in a scaffolding suggestion below.)***

Discussion (4 minutes): The Language of Polynomials

Make sure students have a clear understanding of the following terms and use them appropriately during instruction. The scaffolding suggestion below may be used to help students understand the process of factoring out the GCF. Begin the discussion by reviewing the definition of prime and composite numbers given in the student materials.

> **Discussion: The Language of Polynomials**
>
> **PRIME NUMBER: A *prime number* is a positive integer greater than 1 whose only positive integer factors are 1 and itself.**
>
> **COMPOSITE NUMBER: A *composite number* is a positive integer greater than 1 that is not a prime number.**
>
> **A composite number can be written as the product of positive integers with at least one factor that is not 1 or itself.**
>
> **For example, the prime number 7 has only 1 and 7 as its factors. The composite number 6 has factors of 1, 2, 3, and 6; it could be written as the product $2 \cdot 3$.**

- Factoring is the reverse process of multiplication (through multiple use of the distributive property). We factor a polynomial by reversing the distribution process—factoring the GCF out of each term and writing it on the outside of the parentheses. To check whether the polynomial's factored form is equivalent to its expanded form, you can multiply the factors to see if the product yields the original polynomial.
- $4(x + 3)$ is called a *factored form* of $4x + 12$.

> **A nonzero polynomial expression with integer coefficients is said to be prime or irreducible over the integers if it satisfies two conditions:**
>
> **(1) It is not equivalent to 1 or -1, and**
>
> **(2) If the polynomial is written as a product of two polynomial factors, each with integer coefficients, then one of the two factors must be 1 or -1.**

- Note that this definition actually specifies prime numbers and their negatives as well (the case when the polynomial has degree 0).
- For example: $4x + 9$ is irreducible over the integers.

> **Given a polynomial in standard form with integer coefficients, let c be the greatest common factor of all of the coefficients. The polynomial is *factored completely over the integers* when it is written as a product of c and one or more prime polynomial factors, each with integer coefficients.**

- In the future, we learn to factor over the rationals and reals.

Scaffolding:

For students who struggle with factoring the GCF from a more complicated polynomial, suggest they use a chart to organize the terms and factors. Here is an example using Exercise 3 above:

Stack the three terms (monomial expressions) on the far left of a table, and then write each of the terms of the polynomial in prime factor form across the row, stacking those that are the same. Then, shade or circle the columns that have the same factor for all three terms.

$6x^2y^3$	2	3		x	x	y	y	y			$2x^2$
$9xy^4$	3	3		x		y	y	y	y		$3xy$
$18y^5$	2	3	3			y	y	y	y	y	$6y^2$
		3				y	y	y			→ GCF: $3y^3$

Now, look down the columns to find which factors are in all three rows. The blue columns show those common factors, which are shared by all three terms. So, the greatest common factor (GCF) for the three terms is the product of those common factors: $3y^3$. This term is written on the outside of the parentheses. Then reversing the distributive property, we write the remaining factors inside the parentheses for each of the terms that are not in the blue shaded columns. (It may be helpful to point out that factoring out the GCF is the same as dividing each term by the GCF.) In this example, it is $3y^3(2x^2 + 3xy + 6y^2)$. You can find the GCF by multiplying the factors across the bottom, and you can find the terms of the other factor by multiplying the remaining factors across each row.

Example 2 (4 minutes): Multiply Two Binomials

Demonstrate that the product can be found by applying the distributive property (twice) where the first binomial distributes over each of the second binomial's terms, and relate the result to the area model as was used in Module 1 and as is shown below. Note that while the order of the partial products shown corresponds with the well-known FOIL method (Firsts, Outers, Inners, Lasts), teachers are discouraged from teaching polynomial multiplication as a procedure or with mnemonic devices such as FOIL. Instead, foster understanding by relating the process to the distributive property and the area model.

Since side lengths of rectangles cannot be negative, it is not directly applicable to use the area model for multiplying general polynomials. (We cannot be certain that each term of the polynomial represents a positive quantity.) However, we can use a tabular method that resembles the area model to track each partial product as we use the distributive property to multiply the polynomials.

Example 2: Multiply Two Binomials

<u>Using a Table as an Aid</u>

You have seen the geometric area model used in previous examples to demonstrate the multiplication of polynomial expressions for which each expression was known to represent a measurement of length.

Without a context such as length, we cannot be certain that a polynomial expression represents a positive quantity.

Therefore, an area model is not directly applicable to all polynomial multiplication problems. However, a table can be used in a similar fashion to identify each partial product as we multiply polynomial expressions. The table serves to remind us of the area model even though it does not represent area.

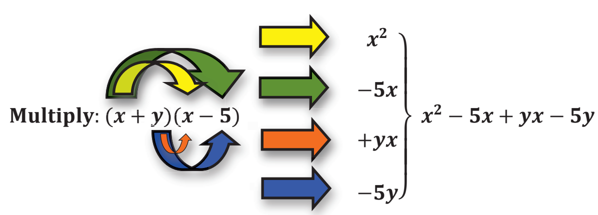
For example, fill in the table to identify the partial products of $(x+2)(x+5)$. Then, write the product of $(x+2)(x+5)$ in standard form.

	x	$+\ 5$
x	x^2	$5x$
$+\ 2$	$2x$	10

$x^2 + 7x + 10$

Without the Aid of a Table

Regardless of whether or not we make use of a table as an aid, the multiplying of two binomials is an application of the distributive property. Both terms of the first binomial distribute over the second binomial. Try it with $(x+y)(x-5)$. In the example below, the colored arrows match each step of the distribution with the resulting partial product.



Example 3 (4 minutes): The Difference of Squares

Example 3: The Difference of Squares

Find the product of $(x+2)(x-2)$. Use the distributive property to distribute the first binomial over the second.

With the Use of a Table:

	x	$+\ 2$
x	x^2	$2x$
$+\ -2$	$-2x$	-4

$x^2 - 4$

Without the Use of a Table:

$(x)(x) + (x)(-2) + (2)(x) + (2)(-2) = x^2 - 2x + 2x - 4 = x^2 - 4$

- Do you think the linear terms are always opposites when we multiply the sum and difference of the same two terms? Why?
  - *Yes. When we multiply the first term of the first binomial by the last term of the second, we get the opposite of what we get when we multiply the second term of the first binomial by the first term of the second.*
- So, $x^2 - 4$ is the difference of two perfect squares. Factoring the difference of two perfect squares reverses the process of finding the product of the sum and difference of two terms.

Exercise 4 (6 minutes)

The following can be used as a guided practice or as independent practice.

Exercise 4

Factor the following examples of the difference of perfect squares.

a. $t^2 - 25$ $(t-5)(t+5)$

b. $4x^2 - 9$ $(2x-3)(2x+3)$

c. $16h^2 - 36k^2$ $(4h-6k)(4h+6k)$

d. $4 - b^2$ $(2-b)(2+b)$

e. $x^4 - 4$ $(x^2-2)(x^2+2)$

f. $x^6 - 25$ $(x^3-5)(x^3+5)$

Point out that any even power can be a perfect square and that 1 is always a square.

Write a General Rule for Finding the Difference of Squares

Write $a^2 - b^2$ in factored form.

$(a+b)(a-b)$

Exercises 5–7 (4 minutes)

The following exercises may be guided or modeled, depending on how well students did on the previous example.

Exercises 5–7

Factor each of the following differences of squares completely:

5. $9y^2 - 100z^2$

   $(3y+10z)(3y-10z)$

6. $a^4 - b^6$

   $(a^2+b^3)(a^2-b^3)$

7. $r^4 - 16s^4$ **(Hint: This one factors twice.)**

   $(r^2+4s^2)(r^2-4s^2) = (r^2+4s^2)(r-2s)(r+2s)$

Example 4 (4 minutes): The Square of a Binomial

It may be worthwhile to let students try their hands at finding the product before showing them how. If students struggle to include every step in the process, pause at each step so that they have time to absorb the operations that took place.

Example 4: The Square of a Binomial

To square a binomial, such as $(x+3)^2$, multiply the binomial by itself.

$$\begin{aligned}(x+3)(x+3) &= (x)(x)+(x)(3)+(3)(x)+(3)(3)\\ &= x^2+3x+3x+9\\ &= x^2+6x+9\end{aligned}$$

Square the following general examples to determine the general rule for squaring a binomial:

a. $(a+b)^2$

$(a+b)(a+b)=a^2+ab+ba+b^2=a^2+2ab+b^2$

b. $(a-b)^2$

$(a-b)(a-b)=a^2-ab-ba+b^2=a^2-2ab+b^2$

Point out that the process used in squaring the binomial is called *expanding*; in general, *expanding* means rewriting a product of sums as a sum of products through use of the distributive property.

- How are the answers to the two general examples similar? How are they different? What is the cause of the difference between the two?
  - *Both results are quadratic expressions with three terms. The first and second examples both have an a^2, b^2, and a $2ab$ term. In part b, the $2ab$ term is negative, while it is positive in part a. The negative (subtraction) in part b causes the middle term to be negative.*

Exercises 8–9 (3 minutes)

Exercises 8–9

Square the binomial.

8. $(a+6)^2$

   $a^2+12a+36$

9. $(5-w)^2$

   $25-10w+w^2$

Closing (2 minutes)

- Factoring is the reverse process of multiplication.
- Look for a GCF first when you are factoring a polynomial.
- Keep factoring until all the factors are prime.
- Factor the difference of squares $a^2 - b^2$ as $(a - b)(a + b)$.

Lesson Summary

Factoring is the reverse process of multiplication. When factoring, it is always helpful to look for a GCF that can be pulled out of the polynomial expression. For example, $3ab - 6a$ can be factored as $3a(b - 2)$.

Factor the difference of perfect squares $a^2 - b^2$:

$$(a - b)(a + b).$$

When squaring a binomial $(a + b)$,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

When squaring a binomial $(a - b)$,

$$(a - b)^2 = a^2 - 2ab + b^2.$$

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 1: Multiplying and Factoring Polynomial Expressions

Exit Ticket

When you multiply two terms by two terms, you should get four terms. Why is the final result when you multiply two binomials sometimes only three terms? Give an example of how your final result can end up with only two terms.

Exit Ticket Sample Solutions

When you multiply two terms by two terms, you should get four terms. Why is the final result when you multiply two binomials sometimes only three terms? Give an example of how your final result can end up with only two terms.

Often when you multiply two binomials, each has a term with the same variable, say x, and two of the terms combine to make one single x-term. If the two terms combine to make zero, there will be only two of the four terms left. For example, $(x+3)(x-3)=x^2-9$.

Problem Set Sample Solutions

1. **For each of the following, factor out the greatest common factor:**
   a. $6y^2+18$

      $6(y^2+3)$

   b. $27y^2+18y$

      $9y(3y+2)$

   c. $21b-15a$

      $3(7b-5a)$

   d. $14c^2+2c$

      $2c(7c+1)$

   e. $3x^2-27$

      $3(x^2-9)$

2. **Multiply.**
   a. $(n-5)(n+5)$

      n^2-25

   b. $(4-y)(4+y)$

      $16-y^2$

   c. $(k+10)^2$

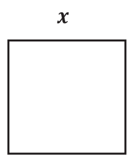
      $k^2+20k+100$

   d. $(4+b)^2$

      $16+8b+b^2$

3. The measure of a side of a square is x units. A new square is formed with each side 6 units longer than the original square's side. Write an expression to represent the area of the new square. (Hint: Draw the new square and count the squares and rectangles.)

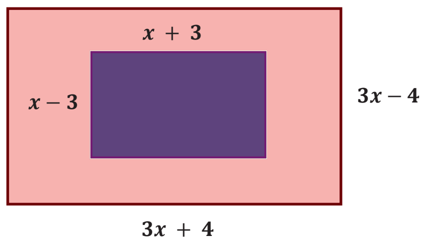
Original square



New square:

$(x+6)^2$ *or* $x^2+12x+36$

4. In the accompanying diagram, the width of the inner rectangle is represented by $x-3$ and the length by $x+3$. The width of the outer rectangle is represented by $3x-4$ and the length by $3x+4$.



a. Write an expression to represent the area of the larger rectangle.

Find the area of the larger (outer) rectangle by multiplying the binomials:

$(3x-4)(3x+4) = 9x^2 - 16.$

b. Write an expression to represent the area of the smaller rectangle.

Find the area of the smaller (inner) rectangle by multiplying the binomials:

$(x-3)(x+3) = x^2 - 9.$

c. Express the area of the region inside the larger rectangle but outside the smaller rectangle as a polynomial in terms of x. (Hint: You will have to add or subtract polynomials to get your final answer.)

Subtract the area of the smaller rectangle from the area of the larger rectangle:

$(9x^2-16)-(x^2-9) = 9x^2 - 16 - x^2 + 9 = 8x^2 - 7.$

Lesson 2: Multiplying and Factoring Polynomial Expressions

Student Outcomes

- Students understand that factoring reverses the multiplication process as they find the linear factors of basic, factorable quadratic trinomials.

Throughout this lesson, students represent multiplication of binomials and factoring quadratic polynomials using geometric models.

Lesson Notes

This lesson continues to emphasize the understanding of the system and operations of polynomial expressions, specifically multiplication and factoring of polynomials. Factoring quadratic expressions can unlock their secrets and reveal key features of the function to facilitate graphing. The reverse relationship between multiplication and factoring is explored, emphasizing the structure of the quadratic expressions. The lesson offers some different strategies for factoring and begins to build a toolbox for students faced with factoring quadratic expressions for which the factors may not be immediately apparent. In future lessons, students' factoring skills are used to find values of variables that make the polynomial expression evaluate to 0.

Classwork

Example 1 (5 minutes): Using a Table as an Aid

Example 1: Using a Table as an Aid

Use a table to assist in multiplying $(x+7)(x+3)$.

	x	$+\ 7$
x	x^2	$7x$
$+\ 3$	$3x$	21

$x^2 + 10x + 21$

- Are there like terms in the table that can be combined?
  - *Yes, the terms in the diagonal can be added to give* $10x$.
- After combining like terms, what is the simplified product of the two binomials?
  - $x^2 + 7x + 3x + 21$ *or* $x^2 + 10x + 21$

Exercise 1 (4 minutes)

Exercise 1

Use a table to aid in finding the product of $(2x+1)(x+4)$.

	$2x$	$+\ 1$
x	$2x^2$	x
$+\ 4$	$8x$	4

$(2x+1)(x+4) = 2x^2 + x + 8x + 4 = 2x^2 + 9x + 4$

Discussion (4 minutes)

- What is the constant term of the polynomial $x - 7$?

Students may respond with 7 or -7; in any case, encourage a discussion to acknowledge that the expression $x - 7$ is equivalent to the expression $x + -7$ and to recall the definition of a polynomial expression (first presented in Algebra I, Module 1 and given again here in the student materials).

POLYNOMIAL EXPRESSION: A *polynomial expression* is either:

(1) A numerical expression or a variable symbol, or

(2) The result of placing two previously generated polynomial expressions into the blanks of the addition operator (__+__) or the multiplication operator (__× __).

- What does the definition of a polynomial expression tell us, then, about the constant term of the polynomial $x - 7$?
  - *That the constant term is actually -7.*
- While we may write the polynomial as $x - 7$, the terms of this polynomial are actually x and -7.

Exercises 2–6 (9 minutes)

Students may benefit from doing the first few of the exercises as a group.

Exercises 2–6

Multiply the following binomials; note that every binomial given in the problems below is a polynomial in one variable, x, with a degree of one. Write the answers in standard form, which in this case takes the form $ax^2 + bx + c$, where a, b, and c are constants.

2. $(x+1)(x-7)$

$x^2 - 6x - 7$

3. $(x+9)(x+2)$

$\mathbf{x^2+11x+18}$

4. $(x-5)(x-3)$

$\mathbf{x^2-8x+15}$

5. $\left(x+\frac{15}{2}\right)(x-1)$

$\mathbf{x^2+\frac{13}{2}x-\frac{15}{2}}$

6. $\left(x-\frac{5}{4}\right)\left(x-\frac{3}{4}\right)$

$\mathbf{x^2-2x+\frac{15}{16}}$

Allow early finishers to record their answers to the next question independently, and then host a class discussion on the question below. Scaffold as needed until students are able to see and verbalize the patterns (as described by the sample solution below).

Describe any patterns you noticed as you worked.

All the coefficients for the x^2 term are 1. The constant term for the resulting trinomial is the product of constant terms of the two binomials. There are always two terms that are like terms and can be combined; the coefficients of those terms after they are combined is the sum of the constant terms from the two binomials.

- If the coefficients of one of the x-terms in the binomials above were not 1, would the observations that we just made still hold true?
  - *No*
- Can you give an example that proves these patterns would not hold in that case?

Optionally, note the following:

- The trinomial that results from the multiplication in Exercise 6 is factorable over the rationals.

Exercises 7–10 (8 minutes)

- A polynomial expression of degree 2 is often referred to as a *quadratic expression*. Why do you suppose that is? What does the prefix "quad" have to do with a polynomial of degree 2?
- The term quadratic relates to quadrangles (rectangles and squares); quadratic expressions and equations are useful for solving problems involving quadrangles. Further, we come to learn about a quadrangle method for working with quadratic expressions and equations called *completing the square*.

Scaffolding:

Choose one or more of the exercises, and use a tabular model to reinforce the conceptual understanding of the use of the distributive property to multiply polynomials.

- In this module, we spend a substantial amount of time exploring quadratic expressions in one variable.

Consider beginning these exercises with guidance and slowly remove support, moving students toward independence.

Exercises 7–10

Factor the following quadratic expressions.

7. $x^2 + 8x + 7$

   $(x+7)(x+1)$

8. $m^2 + m - 90$

   $(m+10)(m-9)$

9. $k^2 - 13k + 40$

   $(k-8)(k-5)$

10. $v^2 + 99v - 100$

    $(v-1)(v+100)$

Have students check their results by multiplying to see if the product is the original quadratic expression.

Example 3 (5 minutes): Quadratic Expressions

Scaffolding:

Strategically grouping or pairing students for these examples might help struggling students to get support and emerging students to deepen their understanding.

Example 3: Quadratic Expressions

If the leading coefficient for a quadratic expression is not 1, the first step in factoring should be to see if all the terms in the expanded form have a common factor. Then, after factoring out the greatest common factor, it may be possible to factor again.

For example, to factor to $2x^3 - 50x$ completely, begin by finding the GCF.
The GCF of the expression is $2x$: $2x(x^2 - 25)$.
Now, factor the difference of squares: $2x(x-5)(x+5)$.

Another example: Follow the steps to factor $-16t^2 + 32t + 48$ completely.

a. **First, factor out the GCF. (Remember: When you factor out a negative number, all the signs on the resulting factor change.)**

   The GCF is -16. Hint: Do not leave the negative 1 as the leading coefficient. Factor it out with the 16.

   $-16(t^2 - 2t - 3)$

b. **Now look for ways to factor further. (Notice the quadratic expression factors.)**

   $-16(t-3)(t+1)$

- Are all the factors prime?
  - *No, -16 is not prime, but the other factors, $(t-3)$ and $(t+1)$, are prime, so we can say that the polynomial has been factored completely over the integers.*

Closing (2 minutes)

- A deep understanding of factoring quadratic trinomials relies on a full and deep understanding of multiplication of binomials and the reverse relationship between the two.

Lesson Summary

Multiplying binomials is an application of the distributive property; each term in the first binomial is distributed over the terms of the second binomial.

The area model can be modified into a tabular form to model the multiplication of binomials (or other polynomials) that may involve negative terms.

When factoring trinomial expressions (or other polynomial expressions), it is useful to look for a GCF as your first step.

Do not forget to look for these special cases:

- **The square of a binomial**
- **The product of the sum and difference of two expressions.**

Exit Ticket (8 minutes)

Name ______________________________ Date ________________

Lesson 2: Multiplying and Factoring Polynomial Expressions

Exit Ticket

1. Factor completely: $2a^2 + 6a + 18$

2. Factor completely: $5x^2 - 5$

3. Factor completely: $3t^3 + 18t^2 - 48t$

4. Factor completely: $4n - n^3$

Exit Ticket Sample Solutions

1. Factor completely: $2a^2 + 6a + 18$

   Factor out the GCF: $\mathbf{2(a^2 + 3a + 9)}$.

2. Factor completely: $5x^2 - 5$

   Factor out the GCF: $\mathbf{5(x^2 - 1)}$.

   Now, factor the difference of perfect squares: $\mathbf{5(x + 1)(x - 1)}$.

3. Factor completely: $3t^3 + 18t^2 - 48t$

   The GCF of the terms is $\mathbf{3t}$.

   Factor out $3t$: $\mathbf{3t^3 + 18t^2 - 48t = 3t(t^2 + 6t - 16)}$.

   To factor further, find the pair of integers whose product is -16 *and whose sum is* $+6$.

   $(+8)(-2) = -16$ *and* $(+8) + (-2) = 6$, *so the factors have* -2 *and* $+8$.

   So, the final factored form is $\mathbf{3t(t + 8)(t - 2)}$.

4. Factor completely: $4n - n^3$

   Factor out the GCF: $\mathbf{n(4 - n^2)}$.

   Then factor the difference of squares: $\mathbf{n(2 - n)(2 + n)}$.

Problem Set Sample Solutions

1. Factor these trinomials as the product of two binomials, and check your answer by multiplying.

   a. $x^2 + 3x + 2$

   The pair of integers whose product is $+2$ *and whose sum is* $+3$ *are* $+1$ *and* $+2$.

   So, the factored form is $(x + 1)(x + 2)$.

   Check: $(x + 1)(x + 2) = x^2 + 2x + x + 2 = x^2 + 3x + 2$

   b. $x^2 - 8x + 15$

   The pair of integers whose product is $+15$ *and whose sum is* -8 *are* -3 *and* -5.

   So, the factored form is $(x - 3)(x - 5)$.

   Check: $(x - 3)(x - 5) = x^2 - 5x - 3x + 15 = x^2 - 8x + 15$

   c. $x^2 + 8x + 15$

   The pair of integers whose product is $+15$ *and whose sum is* $+8$ *are* $+3$ *and* $+5$.

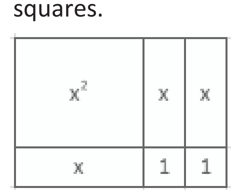
   So, the factored form is $(x + 3)(x + 5)$.

   Check: $(x + 3)(x + 5) = x^2 + 5x + 3x + 15 = x^2 + 8x + 15$

Scaffolding:

Use area models to reinforce the connection between quadratic expressions and rectangles.

- Example for part (a): We use the larger square to represent the x^2 (x by x square units), three 1 by x smaller rectangles, and two 1 by 1 unit squares.



Ask: What are the dimensions of this rectangle?

Answer: $(x + 2)$ *by* $(x + 1)$

Factor completely.

d. $4m^2 - 4n^2$

The GCF of the terms is 4.

Factor out 4*:* $4(m^2 - n^2)$.

Factor the difference of squares: $4(m - n)(m + n)$.

e. $-2x^3 - 2x^2 + 112x$

The GCF of the terms is $-2x$.

Factor out $-2x$*:* $-2x^3 - 2x^2 + 112x = -2x(x^2 + x - 56)$.

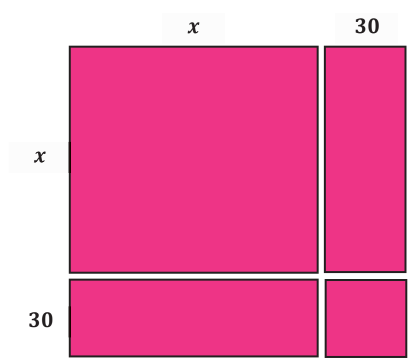
Factor the quadratic trinomial: $-2x(x - 7)(x + 8)$.

f. $y^8 - 81x^4$

Factor the difference of squares: $y^8 - 81x^4 = (y^4 + 9x^2)(y^4 - 9x^2)$.

Factor the difference of squares: $(y^4 + 9x^2)(y^2 + 3x)(y^2 - 3x)$.

2. **The square parking lot at Gene Simon's Donut Palace is going to be enlarged so that there will be an additional 30 ft. of parking space in the front of the lot and an additional 30 ft. of parking space on the side of the lot, as shown in the figure below. Write an expression in terms of x that can be used to represent the area of the new parking lot.**



We know that the original parking lot is a square. We can let x represent the length of each side, in feet, of the original square. We can represent each side of the new parking lot as $x + 30$. Using the area formula for a square, $\text{Area} = s^2$*, we can represent this as* $(x + 30)^2$.

$$(x + 30)^2 = (x + 30)(x + 30)$$
$$= x^2 + 60x + 900$$

Explain how your solution is demonstrated in the area model.

The original square in the upper left corner is x by x, which results in an area of x^2 square feet; each smaller rectangle is 30 by x, which results in an area of $30x$ square feet; there are 2 of them, giving a total of $60x$ square feet. The smaller square is 30 by 30 square feet, which results in an area of 900 square feet. That gives us the following expression for the area of the new parking lot: $x^2 + 60x + 900$.

Lesson 3: Advanced Factoring Strategies for Quadratic Expressions

Student Outcomes

- Students develop strategies for factoring quadratic expressions that are not easily factorable, making use of the structure of the quadratic expression.

Throughout this lesson students are asked to make use of the structure of an expression, seeing part of a complicated expression as a single entity in order to factor quadratic expressions and to compare the areas in using geometric and tabular models.

Lesson Notes

This lesson and Lesson 4 expand on what students have previously learned and include some new tools to use for factoring quadratic expressions efficiently. These techniques and tools are useful when perseverance is required in solving more complicated equations in subsequent lessons. We continue to focus on the structure of quadratic expressions as we explore more complex quadratic expressions.

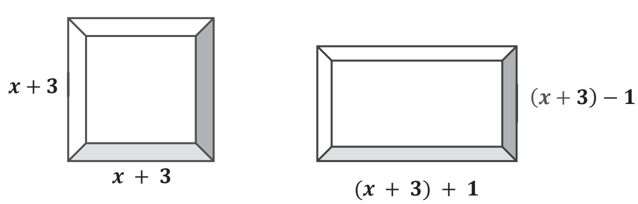
Classwork

Use the problem below to introduce how to use the structure of an expression to compare the areas of the two figures and to factor more complicated expressions. Read the prompt aloud and have students work in pairs or small groups to answer the questions one at a time.

Opening Exercise (12 minutes)

Opening Exercise

Carlos wants to build a sandbox for his little brother. He is deciding between a square sandbox with side lengths that can be represented by $x + 3$ units and a rectangular sandbox with a length 1 unit more than the side of the square and width 1 unit less than the side of the square.



Scaffolding:

- Strategically pairing or grouping students can help those who are struggling with the concepts of this lesson.
- Using graphic diagrams benefits visual learners.
- This task lends itself well to using algebra manipulatives.

Carlos thinks the areas should be the same because one unit is just moved from one side to the other.

a. **Do you agree that the two areas should be the same? Why or why not?**

Allow some class discussion of this question. Some students may agree, but it is unlikely that any students will have evidence to support their claim. Others may disagree but will also be unsure why they instinctively feel there will be a difference.

b. **How would you write the expressions that represent the length and width of the rectangular sandbox in terms of the side length of the square?**

Students are likely to perform the calculation and give $x+4$ and $x+2$ as the length and width. Lead them to using $(x+3)$ as a separate entity with the length of the rectangle as $(x+3)+1$ and the width as $(x+3)-1$.

c. **If you use the expressions for length and width represented in terms of the side length of the square, can you then write the area of the rectangle in the same terms?**

Area of the rectangle: $[(x+3)+1][(x+3)-1]$

d. **How can this expression be seen as the product of a sum and difference: $(a+b)(a-b)$?**

This is $(a+b)(a-b)$, where $a=(x+3)$ and $b=1$.

e. **Can you now rewrite the area expression for the rectangle as the difference of squares: $(a+b)(a-b)=a^2-b^2$?**

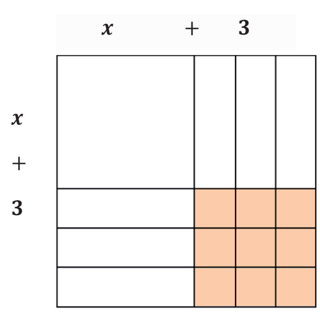
$(a+b)(a-b)=a^2-b^2$*, so* $[(x+3)+1][(x+3)-1]=(x+3)^2-1$.

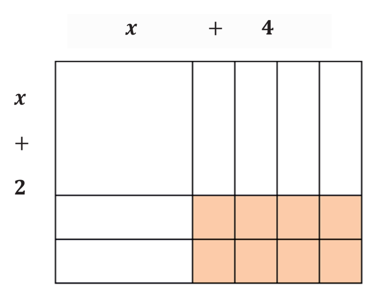
f. **Look carefully at your answer to the last question. What does it tell you about the areas of the two shapes?**

Since the area of the square is $(x+3)^2$ and the area of the rectangle is $(x+3)^2-1$, this shows the area of the square $(x+3)^2$ is one more square unit than the area of the rectangle.

g. **Can you verify that our algebra is correct using a diagram or visual display?**

Use the visual diagram below in the same configuration to demonstrate the difference visually. Notice that there is the same number of smaller shapes for x^2 and for x square units: 1 square each with x^2 square units and 6 small rectangles each with dimensions 1 by x, or x square units. However, if you count the smaller 1 by 1 square unit, the square has 9 and the rectangle has 8, making it 1 square unit less in area.





- What if you start with a square of a given side length, say 4, and increase one side while decreasing the other by one unit to form the related rectangle? Will the areas have the same relationship as those of Carlos's sandboxes?
  - *Yes, the 4 by 4 square has an area of 16 square units, and the dimensions of the related rectangle will be 5 by 3, which has an area of 15 square units.*
- Try it with a square of side length 7. Does it still work?
  - *Yes, the 7 by 7 square will have an area of 49 square units, and the related rectangle will be 6 by 8, which has an area of 48 square units.*
- Do you think this will be true for any square and its "modified rectangle"? Explain how you know.
  - *Yes, if a is the side length for the square, then $(a+1)$ and $(a-1)$ are the rectangle side measures. The area of the square is a^2, and the area of the rectangle will be $(a+1)(a-1) = a^2 - 1$.*

Example 1 (4 minutes)

- Recall that a polynomial expression of degree 2 is often referred to as a *quadratic expression*.

Example 1

In Lesson 2, we saw that factoring is the reverse process of multiplication. We factor a polynomial by reversing the distribution process.

Consider the following example of multiplication:

$$(x+3)(x+5) \rightarrow x^2 + 5x + 3x + 15 \rightarrow x^2 + 8x + 15.$$

When we compare the numbers in the factored form with the numbers in the expanded form, we see that 15 is the product of the two numbers $(3 \cdot 5)$, and 8 is their sum $(3+5)$. The latter is even more obvious when we look at the expanded form before the like terms are combined.

When the x-term coefficient is 1, we usually do not write it out algebraically, but it is actually there as a coefficient. Pointing this out here prepares students for the next example where the x-term coefficients in the factors are not both 1.

Can you explain why that relationship exists between the numbers in the factors and the numbers in the final expanded form?

The coefficient of the term with a variable to the first degree (the linear term), in this case 8, is the sum of the two constant terms of the binomials. The constant term of the quadratic expression is the product of the two constant terms of the binomials. (Make sure students observe a connection rather than merely memorizing a procedure.)

Example 2 (4 minutes)

Scaffolding:

- Visual learners may benefit from seeing this expansion in a tabular model.

	$2x$	3
$1x$	$2x^2$	$3x$
5	$10x$	15

Example 2

Now compare the expansion of this binomial product to the one above:

$$(2x+3)(1x+5) \rightarrow 2x^2+10x+3x+15 \rightarrow 2x^2+13x+15.$$

In the expression lying between the two arrows (before the like terms are combined), we can see the coefficients of the "split" linear terms $(+10x+3x)$. Also notice that for this example, we have coefficients on both x-terms in the factors and that one of the coefficients is not 1. We have 2 and 1 as the factors of the leading coefficient in the expanded form and 3 and 5 as the factors of the constant term. Get ready for quadratic expressions in factored form where neither of the x-term coefficients are 1.

a. **How is this product different from the first example? How is it similar?**

When comparing the factored forms, we see that they are the same except for the coefficient of the first factor in Example 2 (i.e., 2). (It is important to point out that the other factor has an x term with a coefficient of 1. There will be examples in the future where both x terms have coefficients other than 1.) When comparing the expanded forms of both examples (the green numbers), the quadratic term (i.e., $2x^2$) and one of the linear terms (i.e., $10x$) are different in this example because of the new leading coefficient of 2. The other two terms in the expanded form (i.e., $3x$ and 15) were not affected by the leading coefficient. Point out that the differences are by a factor of 2.

b. **Why are the "split" linear terms different in the two examples?**

In the first example, the linear coefficients are 5 and 3. For the second example, they are 10 and 3. (Most students will notice that both products have a $3x$, but one has a $5x$ and the other a $10x$, or $2(5x)$, for the other linear term. The difference is because of the leading coefficient, which doubled two of the four terms of the expanded expression between the two arrows.)

c. **Now that we have four different numbers (coefficients) in each form of the expression, how can we use the numbers in the expanded form of the quadratic expression on the right to find the numbers in the factors on the left?**

We still know that the factor pairs of 15 are the only possibilities for two of the constant terms (1 and 15 or 3 and 5). However, we now have to use the 2 as a factor with one of them to find the sum. Here are all the possibilities:

$(+2)(+1)+(+1)(+15);$

$(+2)(+15)+(+1)(+1);$

$(+2)(+5)+(+1)(+3);$

$(+2)(+3)+(+1)(+5).$

Only one of these gives 13 for the middle-term coefficient: $2(5)+(1)(3)=13.$

d. **Now we need to place those numbers into the parentheses for the factors so that the product matches the expanded form of the quadratic expression. Here is a template for finding the factors using what we call the product-sum method:**

(__x $\pm$ __)(__x $\pm$ __) [We have four number places to fill in this factor template.]

(__x $\pm$ 3)(__x $\pm$ 5) [We know that the 3 and 5 are the correct factors for 15, so we start there.]

($2x$ $\pm$ 3)($1x$ $\pm$ 5) [We know that 2 and 1 are the only factors of 2, with the 2 opposite the 5 so that the distribution process gives us $10x$ for one product.]

($2x$ $+$ 3)(x $+$ 5) [Finally, we know, at least for this example, that all the numbers are positive.]

Remind students to keep the sign for the coefficients with the numbers so that they can keep track of the negatives that eventually can become subtraction.

Example 3 (4 minutes)

Now try factoring a quadratic expression with some negative coefficients: $3x^2 - x - 4$.

Example 3

($\_\_x \pm \_\_$)($\_\_x \pm \_\_$) **[We have four number places to fill in this factor template.]**

($\_\_x \pm 1$)($\_\_x \pm 4$) **[We know that ± 1 and ± 4 or ± 2 and ± 2 are the only possible factors for the constant term, -4, so we start there. Try 1 and 4 to start, and if that does not work, go back and try ± 2 and ± 2. We know that only one of the numbers can be negative to make the product negative.]**

($1x \pm 1$)($3x \pm 4$) **[We know that 3 and 1 are the only factors of 3. We also know that both of these are positive (or both negative). But we do not know which positions they should take, so we will try both ways to see which will give a sum of -1.]**

($x + 1$)($3x - 4$) **[Finally, we determine the two signs needed to make the final product $3x^2 - x - 4$.]**

Exercises 1–6 (8 minutes)

Have students work independently to factor the following quadratic expressions. If time is short, select three or four of the following and assign the remaining for homework.

Exercises

For Exercises 1–6, factor the expanded form of these quadratic expressions. Pay particular attention to the negative and positive signs.

1. $3x^2 - 2x - 8$

   $(3x + 4)(x - 2)$

2. $3x^2 + 10x - 8$

   $(x + 4)(3x - 2)$

3. $3x^2 + x - 14$ **[Notice that there is a 1 as a coefficient in this one.]**

   $(3x + 7)(x - 2)$

4. $2x^2 - 21x - 36$ **[This might be a challenge. If it takes too long, try the next one.]**

   $(2x + 3)(x - 12)$

5. $-2x^2 + 3x + 9$ **[This one has a negative on the leading coefficient.]**

   $(2x + 3)(-x + 3)$

6. $r^2 + \frac{6}{4}r + \frac{9}{16}$ [We need to try one with fractions, too.]

$\left(r + \frac{3}{4}\right)\left(r + \frac{3}{4}\right)$

Exercises 7–10 (8 minutes)

Complicated expressions can be factored by taking advantage of structure; this is sometimes called *chunking*. For example, $49x^2 + 35x + 6$ seems difficult (or impossible) to factor until we notice that $49x^2$ and $35x$ share $7x$ as a factor. If we let, for example, $A = 7x$, we can rewrite the expression as $A^2 + 5A + 6$, which is factorable as $(A + 3)(A + 2)$. Substituting $7x$ back in for A yields $(7x + 3)(7x + 2)$ as the factored form of the original expression. Thus, seeing $49x^2 + 35x + 6$ as $(7x)^2 + 5(7x) + 2$ is of great value; this is a skill with wide applications throughout algebra. Consider challenging students to apply this strategy in the following four exercises, modeling as necessary.

For Exercises 7–10, use the structure of these expressions to factor completely.

7. $100x^2 - 20x - 63$

$(10x)^2 - 2(10x) - 63$

Factored form: $(10x + 7)(10x - 9)$

8. $y^4 + 2y^2 - 3$

$(y^2)^2 + 2(y^2) - 3$

$(y^2 - 1)(y^2 + 3)$

Factored form: $(y - 1)(y + 1)(y^2 + 3)$

9. $9x^2 - 3x - 12$

$(3x)^2 - (3x) - 12$

$(3x + 3)(3x - 4)$

Factored form: $3(x + 1)(3x - 4)$

10. $16a^2b^4 + 20ab^2 - 6$

$(4ab^2)^2 + 5(4ab^2) - 6$

$(4ab^2 + 6)(4ab^2 - 1)$

Factored form: $2(2ab^2 + 3)(4ab^2 - 1)$

Closing (1 minute)

- The method we learned for finding the factors of a quadratic trinomial is called the product-sum method.

Lesson Summary

QUADRATIC EXPRESSION: A polynomial expression of degree 2 is often referred to as a *quadratic expression*.

Some quadratic expressions are not easily factored. The following hints will make the job easier:

- **In the difference of squares $a^2 - b^2$, either of these terms a or b could be a binomial itself.**
- **The product-sum method is useful but can be tricky when the leading coefficient is not 1.**
- **Trial and error is a viable strategy for finding factors.**
- **Check your answers by multiplying the factors to ensure you get back the original quadratic expression.**

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 3: Advanced Factoring Strategies for Quadratic Expressions

Exit Ticket

1. Use algebra to explain how you know that a rectangle with side lengths one less and one more than a square will always be 1 square unit smaller than the square.

2. What is the difference in the areas of a square and rectangle if the rectangle has side lengths 2 less and 2 more than a square? Use algebra or a geometric model to compare the areas and justify your answer.

3. Explain why the method for factoring shown in this lesson is called the product-sum method.

Exit Ticket Sample Solutions

1. **Use algebra to explain how you know that a rectangle with side lengths one less and one more than a square will always be 1 square unit smaller than the square.**

   If a is the length of the side of the square, then a^2 is the area of the square. The rectangle's side lengths will be $(a-1)$ and $(a+1)$. That product, which represents the area of the rectangle, is a^2-1, or 1 square unit less than the area of the square.

2. **What is the difference in the areas of a square and rectangle if the rectangle has side lengths 2 less and 2 more than a square? Use algebra or a geometric model to compare the areas and justify your answer.**

   Using the same logic as for Problem 1, the rectangle dimensions will be $(a+2)$ and $(a-2)$ with an area of a^2-4. Therefore, the area of the rectangle is 4 square units less than the area of the original square.

3. **Explain why the method for factoring shown in this lesson is called the product-sum method.**

   It is called the product-sum method because you look for the two numbers with a product equal to the constant term of the quadratic expression and a sum equal to the coefficient of the linear term.

Problem Set Sample Solutions

Factor the following quadratic expressions.

1. $x^2+9x+20$

   $(x+4)(x+5)$

2. $3x^2+27x+60$

   $3(x+4)(x+5)$

3. $4x^2+9x+5$

   $(4x+5)(x+1)$

4. $3x^2-2x-5$

   $(3x-5)(x+1)$

5. $-2x^2+5x$

   $x(-2x+5)$ *or* $-x(2x-5)$

6. $-2x^2+5x-2$

   $(x-2)(-2x+1)$ *or* $-(2x-1)(x-2)$

7. $5x^2 + 19x - 4$

   $(5x - 1)(x + 4)$

8. $4x^2 - 9$

   $(2x + 3)(2x - 3)$

9. $4x^2 - 12x + 9$ **[This one is tricky, but look for a special pattern.]**

   $(2x - 3)(2x - 3)$ *or* $(2x - 3)^2$

10. $3x^2 - 13x + 12$

    $(x - 3)(3x - 4)$

Factor each expression completely.

11. $a^4 - b^4$

    $a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$

12. $16a^4 - b^4$

    $16a^4 - b^4 = (4a^2)^2 - (b^2)^2 = (4a^2 + b^2)(4a^2 - b^2) = (4a^2 + b^2)(2a + b)(2a - b)$

13. $a^2 - 5a + 4$

    $a^2 - 5a + 4 = (a - 4)(a - 1)$

14. $a^4 - 5a^2 + 4$

    $a^4 - 5a^2 + 4 = (a^2)^2 - 5(a^2) + 4 = (a^2 - 4)(a^2 - 1) = (a + 2)(a - 2)(a + 1)(a - 1)$

15. $9a^2 - 15a + 4$

    $9a^2 - 15a + 4 = (3a)^2 - 5(3a) + 4 = (3a - 4)(3a - 1)$

Lesson 4: Advanced Factoring Strategies for Quadratic Expressions

Student Outcomes

- Students factor quadratic expressions that cannot be easily factored and develop additional strategies for factorization, including splitting the linear term, using graphing calculators, and using geometric or tabular models.

In this lesson, students look to discern a pattern or structure in order to rewrite a quadratic trinomial in an equivalent form.

Lesson Notes

This lesson is a continuation of Lesson 3, offering tools and techniques for more efficient factoring of quadratic expressions that are difficult to factor. These tools are of assistance when perseverance is required in solving more complicated equations in future lessons. We continue to focus on the structure of quadratic expressions as we explore quadratic expressions that are difficult to factor. They have leading coefficients other than 1 and factors that are rational but may be tricky.

Classwork

Opening Exercise (5 minutes)

Have students work in pairs or small groups to factor the following two quadratic expressions using the product-sum method and discuss their similarities and differences.

Opening Exercise

Factor the following quadratic expressions.

a. $2x^2 + 10x + 12$

$2(x^2 + 5x + 6) = 2(x + 2)(x + 3)$

b. $6x^2 + 5x - 6$

$(3x - 2)(2x + 3)$

Scaffolding:

Provide students with a graphic organizer that includes the steps below on one side and space for their work toward the solution on the other.

Process	Solution
1. Multiply a and c.	
2. List all possible factor pairs of ac.	
3. Find the pair that satisfies the requirements of the product-sum method.	
4. Rewrite the expression with the same first and last term but with an expanded b term using that pair of factors as coefficients.	
5. We now have four terms and can enter them into a rectangle or factor by pairs.	
6. The common binomial factor presents itself. Rewrite by combining the coefficients of said common binomial factors and multiplying by the common binomial factor.	

- In what ways do these expressions differ?
  - *The first is easily factorable after factoring out a common factor of* 2*, making it possible to rewrite the expression with a leading coefficient* 1*. In Exercise 1, there is only one possibility for factoring the leading term coefficient* a*, and in Exercise 2, there are two.*
- How does this difference affect your process?
  - *We cannot rewrite the expression so that the leading coefficient is* 1*, so we have to work with the* 6 *as the leading coefficient. That means the number of possible factors increases, and we have more possibilities to test.*
- Why is the trial-and-error method so time consuming?
  - *There are multiple possibilities that we may have to test before arriving at the correct answer.*
- In the following, we explore a more efficient way for factoring quadratic expressions.

Example (15 minutes): Splitting the Linear Term

Introduce the following strategy (i.e., *splitting the linear term into two terms and regrouping*), and apply it to the second problem above: $6x^2 + 5x - 6$. This strategy works for factoring any quadratic expression that is factorable over the integers but is especially useful when the leading coefficient is not 1 and has multiple factor pairs.

Example: Splitting the Linear Term

How might we find the factors of $6x^2 + 5x - 6$?

1. **Consider the product $(a)(c)$: $(6)(-6) = -36$.**
2. **Discuss the possibility that a and c are also multiplied when the leading coefficient is 1.**
3. **List all possible factor pairs of $(a)(c)$: $(1, -36)$, $(-1, 36)$, $(2, -18)$, $(-2, 18)$, $(3, -12)$, $(-3, 12)$, $(4, -9)$, $(-4, 9)$, and $(-6, 6)$.**
4. **Find the pair that satisfies the requirements of the product-sum method (i.e., a pair of numbers whose product equals ac and whose sum is b): $(-4) + 9 = 5$.**
5. **Rewrite the expression with the same first and last term but with an expanded b term using that pair of factors as coefficients: $6x^2 - 4x + 9x - 6$.**
6. **We now have four terms that can be entered into a tabular model or factored by grouping.**
7. **Factoring by grouping: Take the four terms above and pair the first two and the last two; this makes two *groups.***

   $[6x^2 - 4x] + [9x - 6]$ **[Form two groups by pairing the first two and the last two.]**

   $[2x(3x - 2)] + [3(3x - 2)]$ **[Factor out the GCF from each pair.]**

   The common binomial factor is now visible as a common factor of each group. Now rewrite by carefully factoring out the common factor, $3x - 2$, from each group: $(3x - 2)(2x + 3)$.

Note that we can factor difficult quadratic expressions, such as $6x^2 + 5x - 6$, using a tabular model or by splitting the linear term algebraically. Try both ways to see which one works best for you.

Have students try switching the $-4x$ and the $+9x$ in step 5. A common error when factoring out a negative number is to mix up the signs on the final result.

- Does it work? Check your answer by multiplying the binomials.
  - *Yes, it works.* $(3x - 2)(2x + 3) = 6x^2 - 4x + 9x - 6 = 6x^2 + 5x - 6$

We can factor difficult quadratic expressions using the tabular model method or by splitting the linear term algebraically. Demonstrate the two methods below for factoring $6x^2 + 5x - 6$, and ask students to compare the two methods.

For each example, we start with the original trinomial, $6x^2 + 5x - 6$, and find the two numbers whose product is $(a)(c)$ and whose sum is b. In this case, the numbers are $(+9)$ and (-4). (Hint: Always keep the associated sign with the numbers.)

Tabular Model—Example:

	$3x$	-2
$2x$	$6x^2$	$-4x$
$+3$	$+9x$	-6

Fill in the table's cells starting with $6x^2$ and -6 in the left to right diagonal and the split linear term in the right to left diagonal. Working backward, you may have to try a few combinations since the upper left cell could have been formed from $(2x)(3x)$ or $(6x)(x)$, and the lower right could come from $(\pm 2)(\pm 3)$ or $(\pm 1)(\pm 6)$. Just look for combinations that also give you the linear terms in the other diagonal.

The final answer is $(3x - 2)(2x + 3)$.

Splitting the Linear Term—Example:

Using the two numbers we found as coefficients on the linear term (sometimes called the *middle term*), split into two parts:

$6x^2 - 4x \quad + \quad 9x - 6.$

Grouping by pairs (i.e., putting the first two together and the second two together) and factoring out the GCF from each, shows that one of the factors is visible as the common factor.

$2x(3x - 2) \quad + \quad 3(3x - 2)$

Do you see the common factor in the two groups?

$\underline{(3x - 2)(2x + 3)}$

You can always check your answers by multiplying the factors:

$$(3x + 2)(2x - 3) = 6x^2 + 4x - 9x - 6 = 6x^2 - 5x - 6.$$

- What are the advantages of using the tabular model? What are the disadvantages?
  - *Responses should vary and reflect personal choices. Some students may see the tabular model's visual element as an advantage. Some may see it as more complicated than the algebraic method.*
- What are the advantages of using the second method, splitting the linear term? What are the disadvantages?
  - *Responses should vary and reflect personal choices. Some students may prefer the algebraic method for its simplicity. Some may prefer the visual aspect of the tabular model.*
- How is using the tabular model similar to using the product-sum method?
  - *Both look for two numbers that equal the product, $(a)(c)$, and whose sum is b.*

Exercises (25 minutes)

Have students work independently on the following three exercises. A scaffolded task may be helpful if students still struggle with the factoring strategies they have been practicing. Consider working through one more task on the board or screen or having students work through one with a partner before starting these independent exercises.

For example: Factor $6x^2 + 13x + 6$ using a tabular model or by splitting the linear term.

Solution: $(2x + 3)(3x + 2)$

Exercises

Factor the following expressions using your method of choice. After factoring each expression completely, check your answers using the distributive property. Remember to always look for a GCF prior to trying any other strategies.

1. $2x^2 - x - 10$

   Find two numbers such that the product is -20 *and the sum is* -1: $(-5)(+4)$.

   Now reverse the tabular model or split the linear term, grouping by pairs:

   $x(2x-5) + 2(2x-5) = \underline{(2x-5)(x+2)}$.

2. $6x^2 + 7x - 20$

   Find two numbers such that the product is -120 *and the sum is* $+7$: $(-8)(+15)$.

   Now reverse the tabular model or split the linear term, grouping by pairs:

   $2x(3x-4) + 5(3x-4) = \underline{(3x-4)(2x+5)}$.

Exercise 3 is tricky for two reasons: The leading coefficient is negative and the second group has no common factors. Students might want to work with a partner or have a class discussion as they begin to think about this problem. Remind students to factor out the negative with the common factor on the first pair and not to forget that 1 is a common factor for all prime polynomials.

3. $-4x^2 + 4x - 1$

   Find the two numbers such that the product is $+4$ *and the sum is* $+4$: $(+2)(+2)$.

   Now reverse the tabular model or split the linear term, grouping by pairs:

   $-4x^2 + 2x + 2x - 1 = -2x(2x-1) + 1(2x-1) = (2x-1)(-2x+1)$.

Number 4 requires that students understand how to deal with the $\frac{1}{2}$ in the formula for the area of a triangle. A class discussion of how to organize this problem may be in order.

4. **The area of a particular triangle can be represented by** $x^2 + \frac{3}{2}x - \frac{9}{2}$. **What are its base and height in terms of** x**?**

   Factoring out the $\frac{1}{2}$ *first gives:* $\frac{1}{2}(2x^2 + 3x - 9)$. *Now we are looking for a pair of numbers with product* -18 *and sum* $+3$: $(-3)(+6)$.

   Now split the linear term: $\frac{1}{2}(2x^2 - 3x + 6x - 9) = \frac{1}{2}\big(x(2x-3) + 3(2x-3)\big) = \frac{1}{2}(2x-3)(x+3)$.

   So, the dimensions of the triangle would be $2x-3$ *and* $x+3$ *for the base and the height. (There is not enough information to tell which is which.)*

Closing (3 minutes)

- We learned a method today for factoring difficult to factor quadratic expressions using a tabular model and splitting the linear term.
- How does it relate to the product-sum method?
  - *It still requires looking for the two numbers with the product to match $(a)(c)$ and the sum to match the coefficient of the linear term.*

Lesson Summary

While there are several steps involved in splitting the linear term, it is a relatively more efficient and reliable method for factoring trinomials in comparison to simple guess-and-check.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 4: Advanced Factoring Strategies for Quadratic Expressions

Exit Ticket

1. Explain the importance of recognizing common factors when factoring complicated quadratic expressions.

2. Factor: $8x^2 + 6x + 1$.

Exit Ticket Sample Solutions

1. **Explain the importance of recognizing common factors when factoring complicated quadratic expressions.**

   Students should see the importance of factoring out a GCF before attempting to apply the reverse tabular model or splitting the linear term. In every case, the quadratic expressions are much easier to handle if the common factors are out of the way.

2. **Factor: $8x^2 + 6x + 1$.**

   $(4x + 1)(2x + 1)$

Problem Set Sample Solutions

1. **Factor completely.**

   a. $9x^2 - 25x$

   $x(9x - 25)$

   b. $9x^2 - 25$

   $(3x + 5)(3x - 5)$

   c. $9x^2 - 30x + 25$

   $(3x - 5)(3x - 5)$ *or* $(3x - 5)^2$

   d. $2x^2 + 7x + 6$

   $(2x + 3)(x + 2)$

   e. $6x^2 + 7x + 2$

   $(3x + 2)(2x + 1)$

   f. $8x^2 + 20x + 8$

   GCF is 4: $4(2x^2 + 5x + 2) = 4(2x + 1)(x + 2)$

   g. $3x^2 + 10x + 7$

   $(3x + 7)(x + 1)$

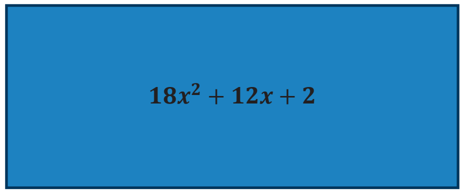
   h. $x^2 + \frac{11}{2}x + \frac{5}{2}$

   $\frac{1}{2}(2x^2 + 11x + 5) = \frac{1}{2}(2x + 1)(x + 5$

i. $6x^3 - 2x^2 - 4x$ **[Hint: Look for a GCF first.]**

$2x(3x^2 - 1x - 2) = 2x(3x + 2)(x - 1)$

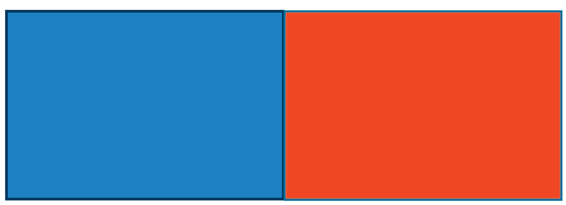
2. **The area of the rectangle below is represented by the expression $18x^2 + 12x + 2$ square units. Write two expressions to represent the dimensions, if the length is known to be twice the width.**



If we factor out the 2 first (GCF), we can use that to double one of the dimensions after we finish factoring to give us $2(9x^2 + 6x + 1) = 2(3x + 1)(3x + 1) = (6x + 2)(3x + 1)$. So, the length is $(6x + 2)$, and the width is $(3x + 1)$.

In the following task, students must solve a problem related to finding dimensions of a geometric figure when area is represented as an expression that is not easily factorable. This question is open-ended with multiple correct answers. Students may question how to begin and should persevere in solving.

3. **Two mathematicians are neighbors. Each owns a separate rectangular plot of land that shares a boundary and has the same dimensions. They agree that each has an area of $2x^2 + 3x + 1$ square units. One mathematician sells his plot to the other. The other wants to put a fence around the perimeter of his new combined plot of land. How many linear units of fencing does he need? Write your answer as an expression in x.**



Note: This question has two correct approaches and two different correct solutions. Can you find them both?

The dimensions of each original plot can be found by factoring the expression for area given in the prompt, $2x^2 + 3x + 1$, which gives us $(2x + 1)(x + 1)$ as the dimensions. Selecting which boundary is common affects the solution because the length of the common side is not included when finding the perimeter of the combined plot. Those measures could be either $(2x + 1)$ or $(x + 1)$.

If the first: $P = 2(2x + 1) + 4(x + 1) = 4x + 2 + 4x + 4 = 8x + 6$.

If the second: $P = 2(x + 1) + 4(2x + 1) = 2x + 2 + 8x + 4 = 10x + 6$.

Lesson 5: The Zero Product Property

Student Outcomes

- Students solve increasingly complex one-variable equations, some of which need algebraic manipulation, including factoring as a first step and using the zero product property.

Lesson Notes

In this lesson, we discover ways to use the factoring skills honed in the last four lessons to solve equations involving quadratic expressions. We begin with an exploration of the zero product property and a discussion how and why it is used. Students then use the property to solve basic one-variable equations, many of which are presented in the context of area.

Classwork

Opening Exercise (3 minutes)

> **Opening Exercise**
>
> **Consider the equation $a \cdot b \cdot c \cdot d = 0$. What values of a, b, c, and d would make the equation true?**
>
> ***Any set of values where one of the four variables was equal to zero, and also values where two, three, or even all four of the variables were equal to zero.***

State or show the zero product property. Make connections between the example above and the given property.

> **<u>Zero Product Property</u>**
>
> If $ab = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$.

Scaffolding:

- If students struggle with the concept of the zero product property, ask them to try substituting numbers into the equation until they find one or more that make the equation true. Point out that in every case, at least one factor must be 0.
- Go back one step further by asking several students in the class for a product of any two numbers that equals 0. Then, ask what they notice about the numbers used in the products. Relate those numerical products to the equations in this example.

Exercises 1–4 (8 minutes)

Exercises 1–4

Find values of c and d that satisfy each of the following equations. (There may be more than one correct answer.)

1. $cd = 0$

   Either c or d must be zero, but the other can be any number, including zero (i.e., both c and d MIGHT be equal to zero at the same time).

2. $(c - 5)d = 2$

   There are an infinite number of correct combinations of c and d, but each choice of c leads to only one choice for d and vice versa. For example, if $d = 2$, then c must be 6, and if $c = 4$, then d must be -2.

3. $(c - 5)d = 0$

   Since the product must be zero, there are only two possible solution scenarios that make the equation true, $c = 5$ (and d can be anything) or $d = 0$ (and c can be anything); specifically, one solution would be $c = 5$ and $d = 0$.

- Why can we easily pinpoint a number that must be substituted for c to make $(c - 5)d = 0$ a true statement but not for $(c - 5)d = 2$?

Refer to the zero product property. Discuss that if the expression is set equal to zero, we know that at least one factor must be equal to zero, limiting the number of possible solutions. Therefore, it is more convenient to set an expression equal to zero when solving.

4. $(c - 5)(d + 3) = 0$

   $c = 5$ or $d = -3$. Either makes the product equal zero; they could both be true, but both do not have to be true. However, at least one must be true.

Example 1 (17 minutes)

Ask students to read the problem's prompt in their materials and to take notes while the teacher reads the questions below aloud. They should work in pairs or small groups to find the answers.

Example 1

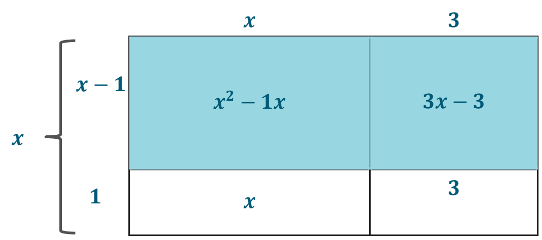
For each of the related questions below, use what you know about the zero product property to find the answers.

a. **The area of a rectangle can be represented by the expression $x^2 + 2x - 3$. If the dimensions of the rectangle are known to be the linear factors of the expression, write each dimension of this rectangle as a binomial. Write the area in terms of the product of the two binomials.**

   The factors of $x^2 + 2x - 3$ are $(x + 3)(x - 1)$, so the dimensions of the rectangle are $(x - 1)$ on the shorter side (width) and $(x + 3)$ on the longer side (length).

b. **Draw and label a diagram that represents the rectangle's area.**

The diagram below models the problem in a way that avoids negative areas.



Let students try using a diagram for this problem, but suggest that it is not easy to do so in a way that avoids negative areas.

Try drawing the diagram shown or ask a student who did a particularly good job to display hers for the class to examine.

It is also *very important* to point out that an area model represents the relationship between the areas and is not to be set as an actual size. In fact, every student in the class could draw a different scale diagram with the length of x looking like a different number of units in length, and all could be a correct model.

c. **Suppose the rectangle's area is 21 square units. Can you find the dimensions of the rectangle?**

Yes. By setting the area expression equal to 21, I can solve for x and from there find the dimensions.

(Use the following questions to guide students through solving this part of the problem.)

- What is the problem asking us to do?
  - *Start by asking students if they can find a number to substitute for x that makes the statement true. Substituting numbers into the quadratic expression to find a way to equal 21 should prove daunting. Still, making a table of values and seeing if we can get close to 21 might be a viable strategy. Remind students that they need to find both solutions. (The correct solution is $x = 4$ or -6.)*
- Should we use the factors we already know for the quadratic expression on the left of the equation? Here is what that would look like: $(x + 3)(x - 1) = 21$. Is this easier to see the solutions?
  - *When looking at the factored form, students may find one of the correct answers right away since 21 has only two factor pairs to check. Even so, students should see that it is harder to find the solution(s) when the product is given as 21 than it is when we know the product is 0. Also, remind students that there are two correct solutions, and one may be easier to find than the other.*
- Is there a way to set the equation equal to zero rather than to 21? What is the benefit of setting an equation equal to zero? What strategies have we used previously when dealing with similar equations to solve for the variable?
  - *Yes, we can set an expression equal to any number. However, setting it equal to 0 makes our solutions easier to find. In some cases, it is more efficient to leave the number the expression equals alone. An example of this is when the structure of the expression makes it possible to simply take the square root of each side of the equation (e.g., $(x + 2)^2 = 9$).*

- Read part (d). What do you need to do first?
  - *We need to subtract* 21 *from both sides of the equation and put the quadratic equation in standard form (expanded form).*
- Now solve for x in the new form of the equation.

> **d. Rewrite the equation so that it is equal to zero and solve.**
>
> $x^2 + 2x - 3 - 21 = 0$ ***becomes*** $x^2 + 2x - 24 = 0$
>
> $(x + 6)(x - 4) = 0$***, which leads to*** $x = -6$ ***or*** 4

- Once we have solved for x, how do we make sense of those numerical values? What do those values mean in the context of this problem? Why does only one value of x work in the context?
  - *A numerical solution only makes sense if it yields dimensions that are positive. If we try* $x = -6$ *as a solution and substitute it into the original expressions for each side of the rectangle, we arrive at dimensions of* -3 *and* -7*.*

    $$(x + 3)(x - 1) = 21 \rightarrow (-6 + 3)(-6 - 1) = (-3)(-7) = 21$$

    However, having negative lengths is not viable for the rectangle. When we try the other solution, $x = 4$*, we arrive at dimensions of* 7 *and* 3*.*

    $$(x + 3)(x - 1) = 21 \rightarrow (4 + 3)(4 - 1) = (7)(3) = 21$$

    Therefore, $x = 4$ *is the only solution that is useful in the context of the original problem.*

Discuss substituting values into the original factored form of the expression to check.

> **e. What are the actual dimensions of the rectangle?**
>
> ***The dimensions of the rectangle are*** $(x - 1)$ ***by*** $(x + 3)$***, so the dimensions would be*** $(4 - 1) = 3$ ***by*** $(4 + 3) = 7$***.***
>
> **f. A smaller rectangle can fit inside the first rectangle, and it has an area that can be represented by the expression** $x^2 - 4x - 5$**. If the dimensions of the rectangle are known to be the linear factors of the expression, what are the dimensions of the smaller rectangle in terms of** x**?**
>
> ***Factoring the quadratic expression, we get*** $(x - 5)(x + 1)$***.***
>
> **g. What value for** x **would make the smaller rectangle have an area of** $\frac{1}{3}$ **that of the larger?**
>
> $\frac{1}{3}$ ***of the larger area is*** $\frac{1}{3}$ ***times*** 21***, which is*** 7 ***square units.*** $x^2 - 4x - 5 = 7$ ***is easier to solve if we subtract*** 7 ***from each side to get*** $x^2 - 4x - 12 = 0$***. Factoring the left side gives us*** $(x - 6)(x + 2) = 0$***. So,*** $x = 6$ ***or*** -2***. Discuss which answer is correct based on this context.***

Emphasize the importance of striking or crossing out the rejected solution.

Exercises 5–8 (10 minutes)

The following exercises might be modeled by the teacher, used as guided practice, or assigned to be solved independently based on the needs of students.

Exercises 5–8

Solve. Show your work.

5. $x^2 - 11x + 19 = -5$

   $x^2 - 11x + 19 = -5$
   $x^2 - 11x + 24 = 0$
   $(x - 3)(x - 8) = 0$
   $x = 3$ *or* 8

6. $7x^2 + x = 0$

   $7x^2 + x = 0$
   $x(7x + 1) = 0$
   $x = 0$ *or* $-\frac{1}{7}$

   (This problem has two points to remember: All terms have a factor of 1, and sometimes the solution is a fraction.)

7. $7r^2 - 14r = -7$

   $7r^2 - 14r = -7$
   $7r^2 - 14r + 7 = 0$
   $7(r^2 - 2r + 1) = 0$
   $7(r - 1)(r - 1) = 0$ *or* $7(r - 1)^2 = 0$
   $r = 1$

   (There is only one solution; rather, both solutions are 1 in this case.)

8. $2d^2 + 5d - 12 = 0$

   $2d^2 + 5d - 12 = 0$

   Two numbers for which the product is -24 and the sum is $+5$: -3 and $+8$.

   So, we split the linear term: $2d^2 - 3d + 8d - 12 = 0$.

   And group by pairs: $d(2d - 3) + 4(2d - 3) = 0$.

   Then factor: $(2d - 3)(d + 4) = 0$.

   So, $d = -4$ or $\frac{3}{2}$.

Closing (1 minute)

- The *zero product property* tells us that if a product equals zero, then at least one factor must be zero. Thus, the product of zero and any monomial, polynomial, or constant is always equal to zero.

> Lesson Summary
>
> **Zero Product Property**
>
> **If $ab = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$.**
>
> **When solving for the variable in a quadratic equation, rewrite the quadratic expression in factored form and set equal to zero. Using the zero product property, you know that if one factor is equal to zero, then the product of all factors is equal to zero.**
>
> **Going one step further, when you have set each binomial factor equal to zero and have solved for the variable, all of the possible solutions for the equation have been found. Given the context, some solutions may not be viable, so be sure to determine if each possible solution is appropriate for the problem.**

Exit Ticket (6 minutes)

Name ______________________________ Date ______________

Lesson 5: The Zero Product Property

Exit Ticket

1. Factor completely: $3d^2 + d - 10$.

2. Solve for d: $3d^2 + d - 10 = 0$.

3. In what ways are Problems 1 and 2 similar? In what ways are they different?

Exit Ticket Sample Solutions

1. Factor completely: $3d^2 + d - 10$.

   $(3d - 5)(d + 2)$

2. Solve for d: $3d^2 + d - 10 = 0$.

   $(3d - 5)(d + 2) = 0$, *so* $d = \frac{5}{3}$ *or* -2.

3. In what ways are Problems 1 and 2 similar? In what ways are they different?

   Both involve the same quadratic expression. The first is just an expression (no equal sign), so it cannot be solved, but there are two factors that are irreducible over the integers. The second is an equation in d and has two solutions that are related to those factors.

Problem Set Sample Solutions

Solve the following equations.

1. $(2x - 1)(x + 3) = 0$

   $x = \frac{1}{2}$ *or* -3

2. $(t - 4)(3t + 1)(t + 2) = 0$

   $t = 4$ *or* $-\frac{1}{3}$ *or* -2

3. $x^2 - 9 = 0$

   $(x + 3)(x - 3) = 0$

   $x = -3$ *or* 3

4. $(x^2 - 9)(x^2 - 100) = 0$

   $(x + 3)(x - 3)(x + 10)(x - 10) = 0$

   $x = -3$ *or* 3 *or* -10 *or* 10

5. $x^2 - 9 = (x - 3)(x - 5)$

   $(x + 3)(x - 3) - (x - 3)(x - 5) = 0$

   $(x - 3)\big(x + 3 - (x - 5)\big) = 0$

   $(x - 3)(8) = 0$

   $x = 3$

6. $x^2 + x - 30 = 0$

$(x+6)(x-5) = 0$
$x = -6 \text{ or } 5$

7. $p^2 - 7p = 0$

$p(p-7) = 0$
$p = 0 \text{ or } 7$

8. $p^2 - 7p = 8$

$p^2 - 7p - 8 = 0$
$(p-8)(p+1) = 0$
$p = 8 \text{ or } -1$

9. $3x^2 + 6x + 3 = 0$

$3(x+1)(x+1) = 0$
$x = -1$

10. $2x^2 - 9x + 10 = 0$

$(2x-5)(x-2) = 0$
$x = \frac{5}{2} \text{ or } 2$

11. $x^2 + 15x + 40 = 4$

$x^2 + 15x + 36 = 0$
$(x+12)(x+3) = 0$
$x = -12 \text{ or } -3$

12. $7x^2 + 2x = 0$

$x(7x+2) = 0$
$x = 0 \text{ or } -\frac{2}{7}$

13. $7x^2 + 2x - 5 = 0$

$(7x-5)(x+1) = 0$
$x = \frac{5}{7} \text{ or } -1$

14. $b^2 + 5b - 35 = 3b$

$b^2 + 2b - 35 = 0$
$(b+7)(b-5) = 0$
$b = -7 \text{ or } 5$

15. $6r^2 - 12r = 18$

$6r^2 - 12r - 18 = 0$
$6(r-3)(r+1) = 0$
$r = 3$ *or* -1

16. $2x^2 + 11x = x^2 - x - 32$

$x^2 + 12x + 32 = 0$
$(x+8)(x+4) = 0$
$x = -8$ *or* -4

17. **Write an equation (in factored form) that has solutions of $x = 2$ or $x = 3$.**

$c(x-2)(x-3) = 0$ *where c is any constant*

18. **Write an equation (in factored form) that has solutions of $a = 0$ or $a = -1$.**

$ca(a+1) = 0$ *where c is any constant*

19. **Quinn looks at the equation $(x-5)(x-6) = 2$ and says that since the equation is in factored form it can be solved as follows:**

$$(x-5)(x-6) = 2$$
$$x - 5 = 2 \text{ or } x - 6 = 2$$
$$x = 7 \text{ or } x = 8.$$

Explain to Quinn why this is incorrect. Show her the correct way to solve the equation.

This method is incorrect because the factored expression is equal to 2 and not 0. Only zero has the special property that if the product of two numbers equals zero then one (or both) of the numbers must equal zero. In order to solve this equation, we need to rewrite it as a factored expression equal to zero.

$$x^2 - 11x + 30 = 2$$
$$x^2 - 11x + 28 = 0$$
$$(x-7)(x-4) = 0$$
$$x = 7 \text{ or } 4$$

Lesson 6: Solving Basic One-Variable Quadratic Equations

Student Outcomes

- Students use appropriate and efficient strategies to find solutions to basic quadratic equations.
- Students interpret the verbal description of a problem and its solutions in context and then justify the solutions using algebraic reasoning.

Lesson Notes

Up to this point, students have practiced factoring and using the *zero product property* to solve basic quadratic equations using area models. In this lesson, we expand our contextual applications to include problems involving objects in motion. We continue to explore efficient and elegant ways to solve quadratic equations by factoring, this time for those involving expressions of the form: ax^2 and $a(x-b)^2$.

The next two examples use a series of questions to help students decontextualize a verbal description of a problem situation to solve basic one-variable quadratic equations. Students then contextualize their solutions to fully answer the questions posed.

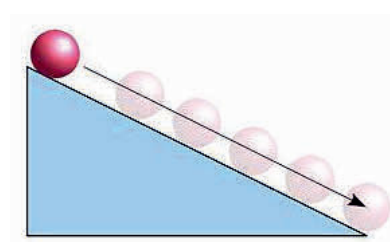
Classwork

Example 1 (10 minutes)

Ask students to read the prompt in their student materials and to take notes individually while the teacher reads the questions below.

Example 1

A physics teacher put a ball at the top of a ramp and let it roll down toward the floor. The class determined that the height of the ball could be represented by the equation $h = -16t^2 + 4$, where the height, h, is measured in feet from the ground and time, t, is measured in seconds.



a. **What do you notice about the structure of the quadratic expression in this problem?**

There is no linear term, just a square and constant.

b. **In the equation, explain what the 4 represents.**

The height when the time is 0 (i.e., the initial height of the top of the ramp is 4 feet).

c. **Explain how you would use the equation to determine the time it takes the ball to reach the floor.**

The ball reaches the ground when the height is zero, so set the expression equal to zero.

Show the chart below or work through the two methods on the board or screen. Have students compare the two methods and discuss them with a partner or small group before asking the next three questions.

- I came up with two methods for finding the seconds it takes the ball to reach the ground. Compare the two solutions. Which solution uses the structure of the expression? How is the structure used? Which solution is the most efficient?

Method 1	Method 2
Set the expression equal to zero: $-16t^2 + 4 = 0.$ Add -4 to both sides of the equation: $-16t^2 = -4.$ Divide both sides by -16 and take the square root: $t^2 = \frac{1}{4}$, so $t = -\frac{1}{2}$ or $\frac{1}{2}$. (Discuss the need for both positive and negative values and the use of the $\pm$ symbol. Advise students to remember that the symbol represents two *different* numbers.)	Set the expression equal to zero: $-16t^2 + 4 = 0.$ Factor out the GCF: $-4(4t^2 - 1) = 0.$ (You could also factor out -16 to give $-16\left(t^2 - \frac{1}{4}\right)$. Do you see the difference of perfect squares? However, it is usually advisable to work with integers rather than fractions when you have a choice.) Factor the difference of squares: $-4(2t + 1)(2t - 1) = 0.$ Now use the zero product property: (Point out that it is only necessary to consider the two variable factors since -4 cannot equal 0.) $2t - 1 = 0$, so $t = \frac{1}{2}$ OR $2t + 1 = 0$, so $t = -\frac{1}{2}$. Therefore, $t = -\frac{1}{2}$ or $\frac{1}{2}$.

Some students may prefer the second method. It is important to point out that neither is wrong. But, also point out that method 2 takes more steps and involves two types of factoring. Method 1 may seem more straightforward, but it does not work for every quadratic equation. It only works when there is no linear term. It is the mathematician's goal to find the most efficient, and sometimes most elegant, path to a solution. (Remind students to always keep in mind the negative solution when they take the square root of a square.)

d. Now consider the two solutions for t. Which one is reasonable? Does the final answer make sense based on this context? Explain.

Only $t = +\frac{1}{2}$ makes sense since time cannot be negative in this context. This means that it took $\frac{1}{2}$ sec. for the ball to travel to the end of the ramp. That would make sense if the ramp was pretty short.

When finishing this problem, be sure to cross out the negative solution to emphasize that it is not applicable for this context.

Example 2 (10 minutes)

Have students read the first part of the prompt in the student materials. Then, read and answer the questions that follow. Students might work with a partner or small group to answer the questions initially, but they should be ready to work independently for the exercises that follow. Encourage independent thinking.

> *Scaffolding:*
>
> - This example is highly scaffolded for struggling students.
> - For an extension, consider having students draw the design for the garden plots that uses closest to the 300 sq m allotted but also includes a narrow walkway between or around the individual plots so that there is access on at least two sides. Then, they should determine how much more land they need to accommodate the walkway or by how much the plots need to be reduced to incorporate the walkway in the original 300 sq m area.

Example 2

Lord Byron is designing a set of square garden plots so some peasant families in his kingdom can grow vegetables. The minimum size for a plot recommended for vegetable gardening is at least 2 m on each side. Lord Byron has enough space around the castle to make bigger plots. He decides that each side should be the minimum (2 m) plus an additional $x\text{ m}$.

a. **What expression can represent the area of one individual garden based on the undecided additional length x?**

$(x+2)^2$

b. **There are 12 families in the kingdom who are interested in growing vegetables in the gardens. What equation can represent the total area, A, of the 12 gardens?**

$A = 12(x+2)^2$

c. **If the total area available for the gardens is 300 sq m, what are the dimensions of each garden?**

$12(x+2)^2 = 300$

$(x+2)^2 = 25$ *(Consider and discuss why we divide both sides of the equation by 12 BEFORE we take the square root.)*

$(x+2) = 5$ *or* -5. *The side length for the garden is 5 m. (Note: Make sure to emphasize the rejection of the -5 in this context (the length of the side is given as $x+2$, which cannot be negative) but also to point out that not ALL negative solutions are rejected for ALL problems in a context.)*

d. **Find both values for x that make the equation in part (c) true (the solution set). What value of x does Lord Byron need to add to the 2 m?**

$(x+2) = 5$ *or* -5, *so* $x = 3$ *or* -7.

He needs to add 3 m to the minimum measurement of 2 m.

The examples above involve perfect square numbers with solutions that are whole numbers. Try Exercises 1–6 for practice with problems, some of which have radical solutions. Remind students that quadratic equations can have two solutions.

These exercises are designed to highlight the structure of the expressions. Students see three different types of solutions even though each of these exercises has $(x-3)^2$ as part of the expression. Students are asked to analyze the structure of each equation and why each yielded a different type of answer.

Exercises (15 minutes)

Exercises

Solve each equation. Some of them may have radicals in their solutions.

1. $3x^2 - 9 = 0$

   $3x^2 = 9 \rightarrow x^2 = 3 \rightarrow x = \pm\sqrt{3}$

2. $(x-3)^2 = 1$

   $(x-3) = \pm 1 \rightarrow x = 3 \pm 1 \rightarrow x = 2$ *or* 4

3. $4(x-3)^2 = 1$

   $(x-3)^2 = \frac{1}{4} \rightarrow (x-3) = \pm\frac{1}{2} \rightarrow x = 3 \pm \frac{1}{2} \rightarrow x = \frac{7}{2}$ *or* $\frac{5}{2}$

4. $2(x-3)^2 = 12$

   $(x-3)^2 = 6 \rightarrow (x-3) = \pm\sqrt{6} \rightarrow x = 3 \pm \sqrt{6}$ *(As estimated decimals:* 5.45 *or* 0.55*)*

5. **Analyze the solutions for Exercises 2–4. Notice how the questions all had $(x-3)^2$ as a factor, but each solution was different (radical, mixed number, whole number). Explain how the structure of each expression affected each problem-solution pair.**

   Question 2: In the equation, $(x-3)^2$ equals a perfect square (1). When the square root is taken, we get $x - 3 = \pm 1$, which yields whole-number solutions.

   Question 3: After both sides are divided by 4, $4(x-3)^2 = 1$ becomes $(x-3)^2 = \frac{1}{4}$, which is a fraction that has a perfect square for the numerator and denominator. Therefore, when the square root is taken, we get $x - 3 = \pm\frac{1}{2}$, which yields fraction solutions.

   Question 4: In this equation, $(x-3)^2$ does not equal a perfect square or a fraction whose denominator and numerator are perfect squares after both sides are divided by 2. Instead, $x - 3$ equals an irrational number after we take the square root of both sides.

6. **Peter is a painter, and he wonders if he would have time to catch a paint bucket dropped from his ladder before it hits the ground. He drops a bucket from the top of his 9-foot ladder. The height, h, of the bucket during its fall can be represented by the equation, $h = -16t^2 + 9$, where the height is measured in feet from the ground, and the time since the bucket was dropped, t, is measured in seconds. After how many seconds does the bucket hit the ground? Do you think he could catch the bucket before it hits the ground?**

   $-16t^2 + 9 = 0 \rightarrow -16t^2 = -9 \rightarrow t^2 = -\frac{9}{-16} \rightarrow t^2 = \frac{9}{16} \rightarrow t = \pm\frac{3}{4}$

   The bucket will hit the ground after $\frac{3}{4}$ seconds. I do not think he could catch the bucket before it hits the ground. It would be impossible for him to descend the 9-foot ladder and catch the bucket in $\frac{3}{4}$ seconds.

Closing (5 minutes)

- Look at the structure of the quadratic equation to determine the best method for solving it.
- Missing linear terms, perfect squares, and factored expressions are examples of the types of structures to look at when trying to come up with a method to solve a quadratic equation.
- Height functions for problems involving falling objects have time as the domain and the height of the object at a specific time as the range.
- In falling object problems, the object has hit the ground when the output of a height function is zero.

Lesson Summary

By looking at the structure of a quadratic equation (missing linear terms, perfect squares, factored expressions), you can find clues for the best method to solve it. Some strategies include setting the equation equal to zero, factoring out the GCF or common factors, and using the zero product property.

Be aware of the domain and range for a function presented in context, and consider whether answers make sense in that context.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 6: Solving Basic One-Variable Quadratic Equations

Exit Ticket

1. Solve the equations.

   a. $4a^2 = 16$

   b. $5b^2 - 25 = 0$

   c. $8 - c^2 = 5$

2. Solve the equations.

   a. $(x - 2)^2 = 9$

   b. $3(x - 2)^2 = 9$

   c. $6 = 24(x + 1)^2$

Exit Ticket Sample Solutions

1. **Solve the equations.**

   a. $4a^2 = 16$

   $a^2 = 4$

   $a = 2 \text{ or } -2$

   b. $5b^2 - 25 = 0$

   $5b^2 = 25$

   $b^2 = 5$

   $b = \pm\sqrt{5}$

   c. $8 - c^2 = 5$

   $-c^2 = -3$

   $c^2 = 3$

   $c = \pm\sqrt{3}$

2. **Solve the equations.**

   a. $(x-2)^2 = 9$

   $(x-2) = \pm 3$

   $x = 2 \pm 3 = -1 \text{ or } 5$

   b. $3(x-2)^2 = 9$

   $(x-2)^2 = 3$

   $x - 2 = \pm\sqrt{3}$

   $x = 2 \pm \sqrt{3}$

   c. $6 = 24(x+1)^2$

   $(x+1)^2 = \frac{6}{24} = \frac{1}{4}$

   $x + 1 = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

   $x = -1 \pm \frac{1}{2}$

   $x = -\frac{1}{2} \text{ or } -\frac{3}{2}$

Problem Set Sample Solutions

1. Factor completely: $15x^2 - 40x - 15$.

   GCF is 5: $5(3x^2 - 8x - 3) = 5(3x + 1)(x - 3)$.

Solve each equation.

2. $4x^2 = 9$

   $x^2 = \frac{9}{4} \rightarrow x = \pm\frac{3}{2}$

3. $3y^2 - 8 = 13$

   $3y^2 = 21 \rightarrow y^2 = 7 \rightarrow y = \pm\sqrt{7}$

4. $(d + 4)^2 = 5$

   $d + 4 = \pm\sqrt{5} \rightarrow d = -4 \pm \sqrt{5}$

5. $4(g - 1)^2 + 6 = 13$

   $4(g - 1)^2 = 7 \rightarrow (g - 1)^2 = \frac{7}{4} \rightarrow g - 1 = \pm\frac{\sqrt{7}}{2} \rightarrow g = 1 \pm \frac{\sqrt{7}}{2}$

6. $12 = -2(5 - k)^2 + 20$

   $-8 = -2(5 - k)^2 \rightarrow 4 = (5 - k)^2 \rightarrow (5 - k) = \pm 2 \rightarrow -k = -5 \pm 2 = -3$ *or* -7*, so* $k = 3$ *or* 7

7. Mischief is a toy poodle that competes with her trainer in the agility course. Within the course, Mischief must leap through a hoop. Mischief's jump can be modeled by the equation $h = -16t^2 + 12t$, where h is the height of the leap in feet and t is the time since the leap, in seconds. At what values of t does Mischief start and end the jump?

   To find the start and end of the jump, we need to find where height, h*, is zero and solve the resulting equation.*

   $$-16t^2 + 12t = 0$$
   $$-4t(4t - 3) = 0$$
   $$t = 0 \text{ or } \frac{3}{4} \text{ seconds}$$

   The leap starts at 0 *seconds and ends at* $\frac{3}{4}$ *seconds.*

   (Students may decide to factor the GCF, $-16t$*, for the factoring step and obtain* $-16t\left(t - \frac{3}{4}\right) = 0$*. They should still arrive at the same conclusion.)*

Lesson 7: Creating and Solving Quadratic Equations in One Variable

Student Outcomes

- Students interpret word problems to create equations in one variable and solve them (i.e., determine the solution set) using factoring and the *zero product property*.

Throughout this lesson, students are presented with a verbal description where a relationship can be modeled algebraically. Students must make sense of the quantities presented and decontextualize the verbal description to solve the basic one-variable quadratic equations. Then, they contextualize their solutions to interpret results and answer the questions posed by the examples.

Lesson Notes

Students wrote and solved algebraic expressions and equations based on verbal statements for linear and exponential equations in Modules 1 and 3. In this lesson, students apply the same ideas to solving quadratic equations by writing expressions to represent side lengths and solving equations when given the area of a rectangle or other polygons. Students create equations from a context and solve using the techniques they have developed in the early lessons of this module.

Classwork

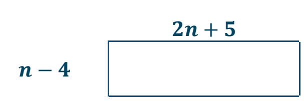
Opening Exercise (5 minutes)

Read the following prompt out loud and have students take notes. Then, have them work with a partner or small group to find the unknown number.

Opening Exercise

The length of a rectangle is 5 in. more than twice a number. The width is 4 in. less than the same number. The perimeter of the rectangle is 44 in. Sketch a diagram of this situation, and find the unknown number.

$$2l + 2w = P$$
$$2(2n+5) + 2(n-4) = 44$$
$$6n + 2 = 44$$
$$n = 7$$



Example 1 (5 minutes)

Review the Opening Exercise with students. Then, ask students: What if, instead of the perimeter, we knew the area?

Example 1

The length of a rectangle is 5 in. more than twice a number. The width is 4 in. less than the same number. If the area of the rectangle is 15 in^2, find the unknown number.

$$\begin{aligned} lw &= A \\ (2n+5)(n-4) &= 15 \\ 2n^2-3n-20 &= 15 \\ 2n^2-3n-35 &= 0 \\ (2n+7)(n-5) &= 0 \\ n &= 5 \text{ or } -\frac{7}{2} \end{aligned}$$

For this context (area), only positive values make sense. So, only $n=5$ is possible.

Give students a few minutes to find a solution. Then, the teacher can either ask a student to demonstrate the solution on the board or present it himself.

Example 2 (5 minutes)

Another way to relate expressions to the area of a rectangle is through proportion. Show students the following example:

Example 2

A picture has a height that is $\frac{4}{3}$ its width. It is to be enlarged so that the ratio of height to width remains the same, but the area is 192 in^2. What are the dimensions of the enlargement?

Let $4x$ to $3x$ represent the ratio of height to width. $A=(h)(w)$, so we have

$$\begin{aligned} (4x)(3x) &= 192 \\ 12x^2 &= 192 \\ x &= 4 \text{ or } -4, \end{aligned}$$

which means that $h=16$ and $w=12$ because only positive values make sense in the context of area. Therefore, the dimensions of the enlargement are 16 inches and 12 inches.

Give students a few minutes to find a solution. Then, the teacher can either have a student demonstrate the solution on the board or present it to the class herself.

Exercises (20 minutes)

The exercises below are scaffolded, beginning with the most basic that follow directly from those above, and culminating with exercises that require deeper reasoning and interpretation from students. If time does not permit assigning all of these exercises in class, select the number and level of difficulty that fit the needs of students.

Exercises

Solve the following problems. Be sure to indicate if a solution is to be rejected based on the contextual situation.

1. **The length of a rectangle is 4 cm more than 3 times its width. If the area of the rectangle is 15 cm^2, find the width.**

$$(4+3w)(w) = 15$$
$$3w^2 + 4w - 15 = 0$$
$$(w+3)(3w-5) = 0$$
$$w = \frac{5}{3} \text{ or } -3$$

However, in this context only the positive value makes sense. Therefore, the width of the rectangle is $\frac{5}{3}$ cm.

2. **The ratio of length to width in a rectangle is $2:3$. Find the length of the rectangle when the area is 150 in^2.**

$$(2x)(3x) = 150$$
$$6x^2 - 150 = 0$$
$$6(x^2 - 25) = 0$$
$$6(x+5)(x-5) = 0$$
$$x = 5 \text{ or } -5$$

In this context, only positive values make sense, which means $x = 5$. Therefore, the length of the rectangle is 10 inches.

3. **One base of a trapezoid is 4 in. more than twice the length of the second base. The height of the trapezoid is 2 in. less than the second base. If the area of the trapezoid is 4 in^2, find the dimensions of the trapezoid.**
(Note: The area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$.)

$$A = \frac{1}{2}(b_1 + b_2)h$$
$$4 = \frac{1}{2}(2b_2 + 4 + b_2)(b_2 - 2)$$
$$4 = \left(\frac{3}{2}b_2 + 2\right)(b_2 - 2)$$
$$\frac{3}{2}{b_2}^2 - b_2 - 8 = 0$$
$$\left(\frac{3}{2}b_2 - 4\right)(b_2 + 2) = 0$$
$$b_2 = \frac{8}{3} \text{ or } -2$$

However, only positive values make sense in this context. The first base is $\frac{28}{3}$ in.; the second base is $\frac{8}{3}$ in.; and the height is $\frac{2}{3}$ in.

4. **A garden measuring 12 m by 16 m is to have a pedestrian pathway that is w meters wide installed all the way around it, increasing the total area to 285 m^2. What is the width, w, of the pathway?**

$$(12+2w)(16+2w)=285$$
$$4w^2+56w-93=0$$
$$(2w+31)(2w-3)=0$$
$$w=\frac{3}{2} \text{ or } -\frac{31}{2}$$

However, only the positive value makes sense in this context, so the width of the pathway is $\frac{3}{2}$ m.

5. **Karen wants to plant a garden and surround it with decorative stones. She has enough stones to enclose a rectangular garden with a perimeter of 68 ft., and she wants the garden to cover 240 ft^2. What is the length and width of her garden?**

$$68=2l+2w$$
$$w=34-l$$
$$240=(l)(34-l)$$
$$l^2-34l+240=0$$
$$(l-10)(l-24)=0$$
$$l=10 \text{ or } 24$$

Important to notice here is that both solutions are positive and could represent the length. Because length and width are arbitrary distinctions here, the garden measures 24 ft. $\times$ 10 ft., with either quantity representing the width and the other representing the length.

For Exercise 6, a discussion on how to identify algebraically an unknown odd number may be necessary. If students have not worked on problems using consecutive integers, this one might be tricky for some. It is likely that they can come up with at least one set of numbers to fit the description without using algebra. If time permits, let them explore the possibilities for an algebraic method of solving this problem. A discussion of the general expressions used to represent odd integers is important for all students.

Have students read Exercise 6 and then ask the following:

- How do you name the odd integers?
  - *If we call the first one n, then the next one would be $n+2$. Or, if we call the first one $2n-1$, the next one would be $2n+1$.*

6. **Find two consecutive odd integers whose product is 99. (Note: There are two different pairs of consecutive odd integers and only an algebraic solution will be accepted.)**

Let n represent the first odd integer and $n+2$ represent the subsequent odd integer. The product is $n(n+2)$, which must equal 99.

$$n(n+2) = 99$$
$$n^2 + 2n - 99 = 0$$
$$(n-9)(n+11) = 0$$
$$n = 9 \text{ or } n = -11$$

If $n = 9$, then $n+2 = 11$, so the numbers could be 9 and 11. Or if $n = -11$, then $n+2 = -9$, so the numbers could be -11 and -9.

OR

Let $2n-1$ represent the first odd integer and $2n+1$ represent the subsequent odd integer. The product is $4n^2-1$, which must equal 99.

$$4n^2 - 1 = 99$$
$$4n^2 = 100$$
$$n^2 = 25$$
$$n = \pm 5$$

The two consecutive pairs of integers would be

$2(5) - 1 = 9; 2(5) + 1 = 11$

AND

$2(-5) - 1 = -11; 2(5) + 1 = -9$

7. **Challenge: You have a 500-foot roll of chain link fencing and a large field. You want to fence in a rectangular playground area. What are the dimensions of the largest such playground area you can enclose? What is the area of the playground?**

$2w + 2l = 500$, so $w + l = 250$, and $l = 250 - w$. $A = (l)(w)$, so $(250 - w)(w) = 0$ gives us roots at $w = 0$ and $w = 250$. This means the vertex of the equation is $w = 125 \rightarrow l = 250 - 125 = 125$. The area of the playground will be $125\text{ ft.} \times 125\text{ ft.}$, or $15,625\text{ ft}^2$.

Scaffolding:

For students who enjoy a challenge, let them try Exercise 7 as a preview of coming attractions. They may use tables or graphs to find the solution. These concepts are addressed in a later lesson.

Closing (5 minutes)

Choose two exercises from this lesson that students struggled the most with and review them quickly on the board. Demonstrate the best strategies for solving.

Lesson Summary

When provided with a verbal description of a problem, represent the scenario algebraically. Start by identifying the unknown quantities in the problem and assigning variables. For example, write expressions that represent the length and width of an object.

Solve the equation using techniques previously learned, such as factoring and using the zero product property. The final answer should be clearly stated and should be reasonable in terms of the context of the problem.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 7: Creating and Solving Quadratic Equations in One Variable

Exit Ticket

1. The perimeter of a rectangle is 54 cm. If the length is 2 cm more than a number, and the width is 5 cm less than twice the same number, what is the number?

2. A plot of land for sale has a width of x ft. and a length that is 8 ft. less than its width. A farmer will only purchase the land if it measures 240 ft^2. What value for x causes the farmer to purchase the land?

Exit Ticket Sample Solutions

1. **The perimeter of a rectangle is 54 cm. If the length is 2 cm more than a number, and the width is 5 cm less than twice the same number, what is the number?**

$$2l + 2w = P$$
$$2(n+2) + 2(2n-5) = 54$$
$$6n - 6 = 54$$
$$n = 10$$

2. **A plot of land for sale has a width of x ft. and a length that is 8 ft. less than its width. A farmer will only purchase the land if it measures 240 ft^2. What value for x causes the farmer to purchase the land?**

$$(x)(x-8) = 240$$
$$x^2 - 8x - 240 = 0$$
$$(x-20)(x+12) = 0$$
$$x = 20 \text{ or } x = -12$$

Since the answer cannot be negative, the answer is $x = 20$. The farmer will purchase the land if the width is 20 ft.

Problem Set Sample Solutions

Solve the following problems.

1. **The length of a rectangle is 2 cm less than its width. If the area of the rectangle is 35 cm^2, find the width.**

$$(w-2)(w) = 35$$
$$w^2 - 2w - 35 = 0$$
$$(w+5)(w-7) = 0$$
$$w = 7 \text{ or } -5$$

However, since the measurement can only be positive, the width is 7 cm.

2. **The ratio of length to width (measured in inches) in a rectangle is $4:7$. Find the length of the rectangle if the area is known to be 700 in^2.**

$$(4x)(7x) = 700$$
$$28x^2 - 700 = 0$$
$$28(x^2 - 25) = 0$$
$$28(x+5)(x-5) = 0$$
$$x = 5 \text{ or } -5$$

However, the measure can only be positive, which means $x = 5$, and the length is 20 inches.

3. **One base of a trapezoid is three times the length of the second base. The height of the trapezoid is 2 in. smaller than the second base. If the area of the trapezoid is 30 in^2, find the lengths of the bases and the height of the trapezoid.**

$$A = \frac{1}{2}(b_1 + b_2)h$$
$$30 = \frac{1}{2}(3b_2 + b_2)(b_2 - 2)$$
$$30 = (2b_2)(b_2 - 2)$$
$$2{b_2}^2 - 4b_2 - 30 = 0$$
$$2(b_2 - 5)(b_2 + 3) = 0$$
$$b_2 = 5 \text{ or } -3$$

However, only the positive value makes sense. The first base is 15 in.; the second base is 5 in.; and the height is 3 in.

4. **A student is painting an accent wall in his room where the length of the wall is 3 ft. more than its width. The wall has an area of 130 ft^2. What are the length and the width, in feet?**

$$(w + 3)(w) = 130$$
$$w^2 + 3w - 130 = 0$$
$$(w + 13)(w - 10) = 0$$
$$w = 10 \text{ or } -13$$

However, since the measure must be positive, the width is 10 ft., and the length is 13 ft.

5. **Find two consecutive even integers whose product is 80. (There are two pairs, and only an algebraic solution will be accepted.)**

$$(w)(w + 2) = 80$$
$$w^2 + 2w - 80 = 0$$
$$(w + 10)(w - 8) = 0$$
$$w = 8 \text{ or } -10$$

So, the consecutive even integers are 8 and 10 or -10 and -8.

Lesson 8: Exploring the Symmetry in Graphs of Quadratic Functions

Student Outcomes

- Students examine quadratic equations in two variables represented graphically on a coordinate plane and recognize the symmetry of the graph. They explore key features of graphs of quadratic functions: y-intercept and x-intercept, the vertex, the axis of symmetry, increasing and decreasing intervals, negative and positive intervals, and end behavior. They sketch graphs of quadratic functions as a symmetric curve with a highest or lowest point corresponding to its vertex and an axis of symmetry passing through the vertex.

Lesson Notes

In this lesson, students recognize the symmetry of the graph of a quadratic equation and sketch their graphs as a symmetric curve with a highest or lowest point corresponding to its vertex (a maximum or minimum value of the function) and the axis of symmetry passing through the vertex. This lesson is an exploration focusing on the symmetric nature of quadratic functions and their graphs and the interpretation of the key features.

Classwork

Opening (4 minutes): Graph Vocabulary

Introduce the following terms, and discuss their meanings. Demonstrate these features on a projection of one of the photographs or on a graph that is either sketched or projected onto the screen. If students keep a journal, have them put these terms in their vocabulary section.

Graph Vocabulary

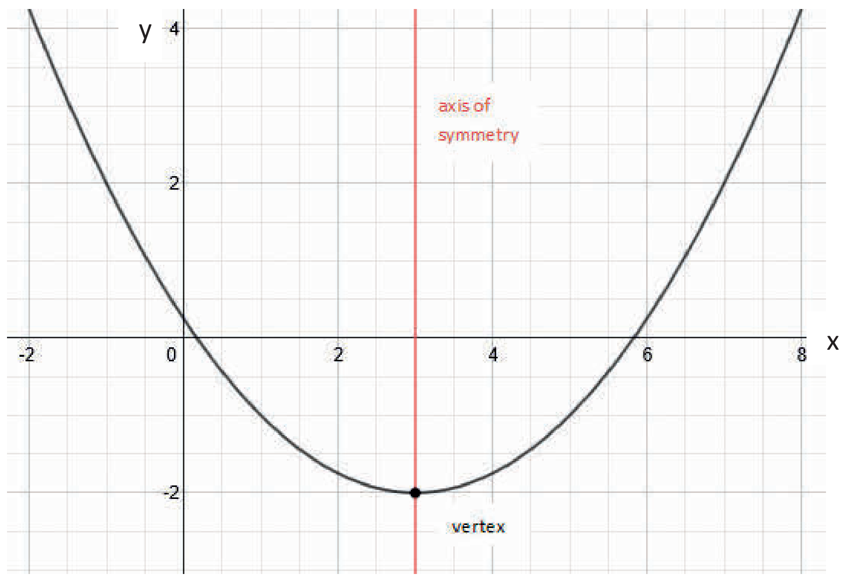
AXIS OF SYMMETRY: Given a quadratic function in standard form, $f(x) = ax^2 + bx + c$, the vertical line given by the graph of the equation $x = -\frac{b}{2a}$ is called the *axis of symmetry* of the graph of the quadratic function.

VERTEX: The point where the graph of a quadratic function and its axis of symmetry intersect is called the *vertex*.

END BEHAVIOR OF A GRAPH: Given a quadratic function in the form $f(x) = ax^2 + bx + c$ (or $f(x) = a(x-h)^2 + k$), the quadratic function is said to *open up* if $a > 0$ and *open down* if $a < 0$.

- **If $a > 0$, then f has a minimum at the x-coordinate of the vertex; that is, f is decreasing for x-values less than (or to the left of) the vertex, and f is increasing for x-values greater than (or to the right of) the vertex.**
- **If $a < 0$, then f has a maximum at the x-coordinate of the vertex; that is, f is increasing for x-values less than (or to the left of) the vertex, and f is decreasing for x-values greater than (or to the right of) the vertex.**

A quadratic function is a polynomial function of degree 2. A quadratic function is in *standard form* if it is written in the form, $f(x) = ax^2 + bx + c$, for constants a, b, c with $a \neq 0$ and for x any real number. Given a quadratic function in standard form, the vertical line given by the graph of the equation, $x = -\frac{b}{2a}$, is called the *axis of symmetry* of the graph of the quadratic function.



End behavior: This graph of a quadratic function opens up. As the values of x approach $+\infty$ and $-\infty$, the values of y approach $+\infty$.

Exploratory Challenge 1 (5 minutes)

Either project the photographs below on the board or print them so that the class can view them together. Use the arched features of the architecture and the two questions beneath the photographs to help students describe the overall shape of the graph of a quadratic function in their own words. Consider finding other photographs of similar curves in nature or in architecture to add to or use in place of the ones presented in this exercise.

IMPORTANT: In the interest of full disclosure, many of the photographs in Exploratory Challenge 1 cannot actually be modeled with a quadratic function but rather are *catenary curves*. These are “quadratic-like” and can be used for our teaching purposes as they display many of the same features, including the symmetry we are exploring in this lesson.

For more information, see the following links for discussions on the difference:

Is the Gateway Arch a Parabola?

http://www.intmath.com/blog/is-the-gateway-arch-a-parabola/4306

Catenary and Parabola Comparison

http://mathforum.org/library/drmath/view/65729.html

Exploratory Challenge 1

Below are some examples of curves found in architecture around the world. Some of these might be represented by graphs of quadratic functions. What are the key features these curves have in common with a graph of a quadratic function?

St. Louis Arch

Bellos Falls Arch Bridge

Arch of Constantine

Roman Aqueduct

The photographs of architectural features above MIGHT be closely represented by graphs of quadratic functions. Answer the following questions based on the pictures.

a. **How would you describe the overall shape of a graph of a quadratic function?**

Student answers will vary. They may use the letter U or a physical object (cup, bowl) to describe the shape. Pose a question for students to think about during the lesson and discuss at the end of the lesson: "Does the letter U (or a cup or a bowl) give an accurate description of a quadratic function? Why, or why not?" Discuss the symmetric nature of the pictures and draw the line of symmetry on each photograph. (The letter U and the physical objects are not infinite and usually do not continue to widen as they open up or down.)

b. **What is similar or different about the overall shape of the above curves?**

All have either a highest or lowest point and spread out from there. Some are wider or narrower at the base.

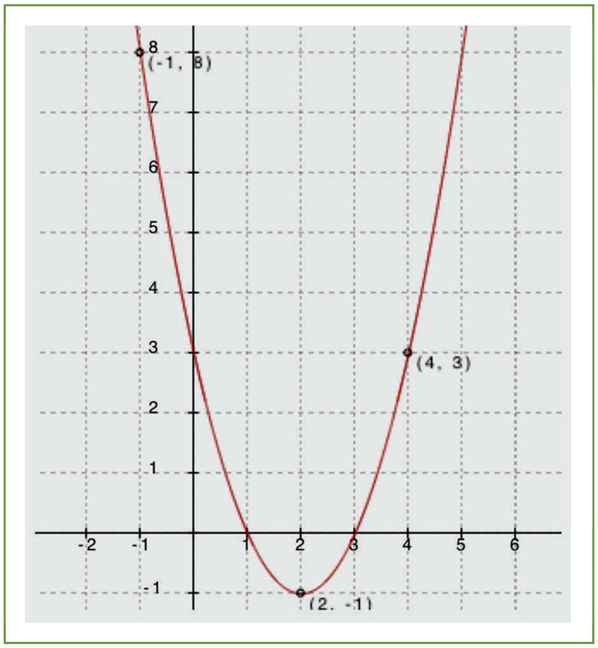
IMPORTANT: Many of the photographs in this activity cannot actually be modeled with a quadratic function but rather are catenary curves. These are "quadratic-like" and can be used for our exploration purposes as they display many of the same features, including the symmetry we are exploring in this lesson.

Exploratory Challenge 2 (20 minutes)

Exploratory Challenge 2

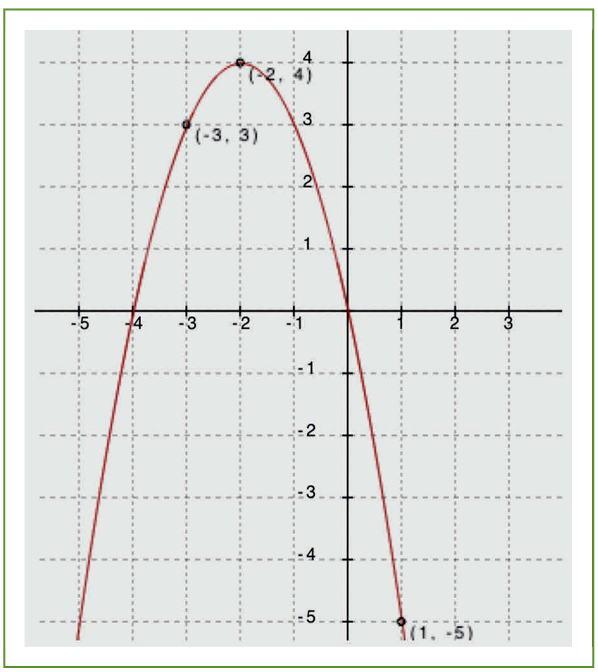
Use the graphs of quadratic functions (Graph A and Graph B) to fill in the table and answer the questions on the following page.

Graph A



x	$f(x)$
-1	8
0	3
1	0
2	-1
3	0
4	3
5	8

Graph B



x	$f(x)$
-5	-5
-4	0
-3	3
-2	4
-1	3
0	0
1	-5

Solution Note: Students may choose different x-values but are likely to choose the ones they can see on the provided graphs.

Use your graphs and tables of values from the previous page to fill in the blanks or answer the questions for each below.

		Graph A		Graph B	
1	x-Intercepts	$(1, 0)$	$(3, 0)$	$(-4, 0)$	$(0, 0)$
2	Vertex	$(2, -1)$		$(-2, 4)$	
3	Sign of the Leading Coefficient	*positive*		*negative*	
4	Vertex Represents a Minimum or Maximum?	*minimum*		*maximum*	
5	Points of Symmetry	Find $f(-1)$ and $f(5)$. $f(-1) = 8$ *and* $f(5) = 8$. Is $f(7)$ greater than or less than 8? Explain. $f(7) > 8$. *Since* $f(5) = 8$ *and the graph is increasing on that side of the vertex, all values beyond* $f(5)$ *will be greater than* $f(5)$.		Find $f(-1)$ and $f(-3)$. $f(-1) = 3$ *and* $f(-3) = 3$. $f(2) = -12$. Predict the value for $f(-6)$ and explain your answer. $f(-6) = -12$. *This is the point of symmetry for* $f(2)$.	
6	Increasing and Decreasing Intervals	On what intervals of the domain is the function depicted by the graph increasing? $(2, +\infty)$ On what intervals of the domain is the function depicted by the graph decreasing? $(-\infty, 2)$		On what intervals of the domain is the function depicted by the graph increasing? $(-\infty, -2)$ On what intervals of the domain is the function depicted by the graph decreasing? $(-2, +\infty)$	
7	Average Rate of Change on an Interval	What is the average rate of change for the following intervals? $[-1, 0]$: -5 $[0, 1]$: -3 $[0, 3]$: -1 $[1, 3]$: 0		What is the average rate of change for the following intervals? $[-5, -4]$: 5 $[-4, -3]$: 3 $[-4, -1]$: 1 $[-3, -1]$: 0	

After students complete the table, have them look at the coordinates of the zeros and those of the vertex in their tables. Then, use the questions below to have a class discussion. Have students refer to the information in the table to help them answer.

Understanding the symmetry of quadratic functions and their graphs (Look at the tables and row 5 in the chart.)

a. What patterns do you see in the tables of values you made next to Graph A and Graph B?

In both tables, the x-values increase by one, and the y-values either increase and then decrease or vice versa. The y-values move toward the vertex value (either increasing or decreasing) and then begin to retrace earlier y-values symmetrically after passing the vertex and moving away from its y-value again.

Finding the vertex and axis of symmetry (Look at rows 1 and 2 of the chart.)

b. How can we know the x-coordinate of the vertex by looking at the x-coordinates of the zeros (or any pair of symmetric points)?

The x-value of the vertex is halfway between any two symmetric points (the average of the x-coordinates). (Developing this concept is crucial for graphing quadratic functions. Build fluency in finding the equation of the axis of symmetry by having students give the equation of the axis of symmetry for several graphs as they work through this lesson.)

Understanding end behavior (Look at rows 3 and 4 of the chart.)

c. What happens to the y-values of the functions as the x-values increase to very large numbers? What about as the x-values decrease to very small numbers (in the negative direction)?

At both ends, the y-values increase toward $+\infty$ for Graph A (opens up) and toward $-\infty$ for Graph B (opens down).

Show students the equations for Graph A, $f(x) = (x-2)^2 - 1$, and Graph B, $f(x) = -(x+2)^2 + 4$, before asking the following question:

d. How can we know whether a graph of a quadratic function opens up or down?

When looking at the function's graph, you can physically see it. When looking at the function's equation, for those that open down, the leading coefficient is any negative number. (Note: These two functions have 1 and -1 for leading coefficients. If students do not see this connection right away, project the graphs of—or have them graph—several equations in their graphing calculator to see the connection.)

Identifying intervals on which the function is increasing or decreasing (Look at row 6 in the chart.)

e. Is it possible to determine the exact intervals that a quadratic function is increasing or decreasing just by looking at a graph of the function?

We can only be certain if enough points of the graph are labeled for us to determine the vertex with certainty.

Computing average rate of change on an interval (Look at row 7 in the chart.)

f. Explain why the average rate of change over the interval $[1, 3]$ for Graph A was zero.

The function values at the two endpoints of the interval were the same.

g. How are finding the slope of a line and finding the average rate of change on an interval of a quadratic function similar? How are they different?

They can both be found using the formula $\frac{f(a)-f(b)}{a-b}$. However, the average rate of change is the SAME for every interval of a linear function and is typically DIFFERENT for most intervals of a quadratic function.

Extension Question:

- One has to look very hard to find two intervals on the graph of a quadratic function with the same average rate of range. Can you find some?
  - *For Graph A, an example is* $[1, 3]$ *and* $[0, 4]$; *both have an average rate of change of* 0. *For Graph B, an example is* $[-2, 1]$ *and* $[-1, 0]$; *both have an average rate of change of* -3.

Point out that intervals having infinity for an end-point need a parenthesis (not a closed bracket) on that side. Remind students of the closed and open holes they used for one-variable graphs in Module 1. Have them think about how you could close the hole on infinity when it is not a distinct value but always just out of reach. Also, point out that the rate of change at $x = 2$ is neither negative nor positive but is, in fact, zero.

Finding a unique quadratic function:

h. Can you graph a quadratic function if you don't know the vertex? Can you graph a quadratic function if you only know the x-intercepts?

Students are likely to intuit that they need more than just a vertex but may not see that the x-intercepts are not quite enough. If they do not, move to the next question.

i. Remember that we need to know at least two points to define a unique line. Can you identify a unique quadratic function with just two points? Explain.

No, there are many different quadratic functions whose graphs pass through any two given points. (If students think they CAN define a unique quadratic function, try giving the class two symmetry points, for example the x-intercepts, and have each student draw a graph of the quadratic function that passes through those zeros. When they compare with each other, they should find that almost every student in the class found a different graph.)

j. What is the minimum number of points needed to identify a unique quadratic function? Explain why.

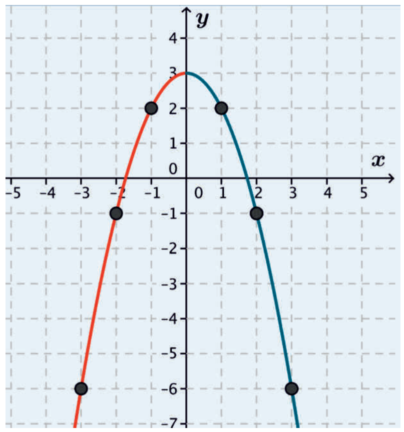
With three distinct points that are not collinear, you can identify the unique quadratic function that passes through all three. (At this stage, students might have trouble writing the equation from just any three points.)

Exploratory Challenge 3 (10 minutes)

Present to students the following graph, table, and questions in their student materials. Have them work with a partner or small group to complete the graph, fill in some values in the table, and then answer the questions. After students have completed the questions, ask the questions aloud to see if students agree about the answers.

Exploratory Challenge 3

Below you see only one side of the graph of a quadratic function. Complete the graph by plotting three additional points of the quadratic function. Explain how you found these points, and then fill in the table on the right.



x	$f(x)$
-3	-6
-2	-1
-1	2
0	3
1	2
2	-1
3	-6

I found the three additional points by using the symmetry of the graph.

a. **What are the coordinates of the x-intercepts?**

$(-1.7, 0)$ *and* $(1.7, 0)$ *(These are estimations but should be close to $\pm\sqrt{3}$.)*

b. **What are the coordinates of the y-intercept?**

$(0, 3)$ *(Point out that this is also the vertex in this case.)*

c. **What are the coordinates of the vertex? Is it a minimum or a maximum?**

The coordinate $(0, 3)$ is a maximum.

d. **If we knew the equation for this curve, what would the sign of the leading coefficient be?**

The leading coefficient would be negative since the graph opens down.

e. **Verify that the average rate of change for the interval $-3 \le x \le -2$, or $[-3, -2]$, is 5. Show your steps.**

Using the formula for average rate of change: $\frac{[f(-2) - f(-3)]}{(-2-(-3))} = \frac{(-1-(-6))}{1} = \frac{5}{1}$.

f. **Based on your answer for row 6 in the table for Exploratory Challenge 2, what interval would have an average rate of change of -5? Explain.**

$2 \le x \le 3$ *The quadratic equation is symmetric.*

If the graph of the function increases at an average rate of 5, then it should decrease at an average rate of -5.

Closing (2 minutes)

- The graphs of quadratic functions have a unique symmetrical nature with a maximum or minimum function value corresponding to the vertex.
- When the leading coefficient of the quadratic expression representing the function is negative, the graph opens down, and when positive, it opens up.

> **Lesson Summary**
>
> **Quadratic functions create a symmetrical curve with its highest (maximum) or lowest (minimum) point corresponding to its vertex and an axis of symmetry passing through the vertex when graphed. The x-coordinate of the vertex is the average of the x-coordinates of the zeros or any two symmetric points on the graph.**
>
> **When the leading coefficient is a negative number, the graph *opens down,* and its end behavior is that both ends move towards negative infinity. If the leading coefficient is positive, the graph *opens up,* and both ends move towards positive infinity.**

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 8: Exploring the Symmetry in Graphs of Quadratic Functions

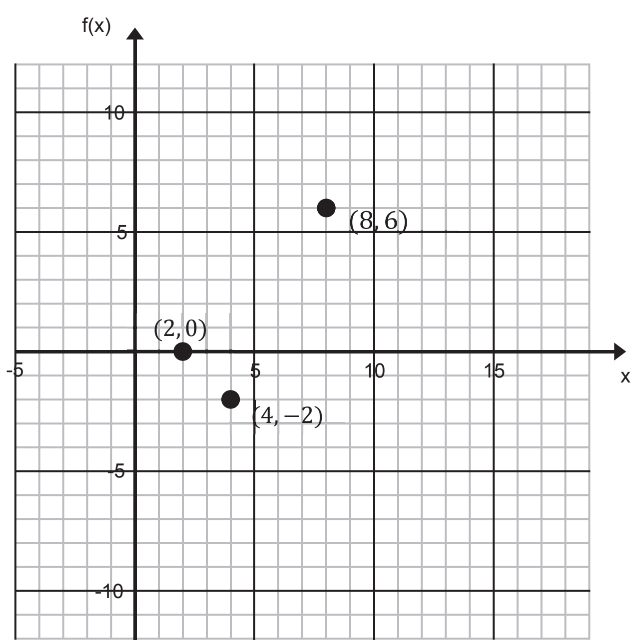
Exit Ticket

1. If possible, find the equation for the axis of symmetry for the graph of a quadratic function with the given pair of coordinates. If not possible, explain why.

   a. $(3, 10)$ $(15, 10)$

   b. $(-2, 6)$ $(6,4)$

2. The point $(4, -2)$ is the vertex of the graph of a quadratic function. The points $(8, 6)$ and $(2, 0)$ also fall on the graph of the function. Complete the graph of this quadratic function by first finding two additional points on the graph. (If needed, make a table of values on your own paper.) Then, answer the questions on the right.



   a. Find the y-intercept.

   b. Find the x-intercept(s).

   c. Find the interval on which the rate of change is always positive.

   d. What is the sign of the leading coefficient for this quadratic function? Explain how you know.

Exit Ticket Sample Solutions

1. **If possible, find the equation for the axis of symmetry for the graph of a quadratic function with the given pair of coordinates. If not possible, explain why.**

   a. $(3, 10)$ $(15, 10)$

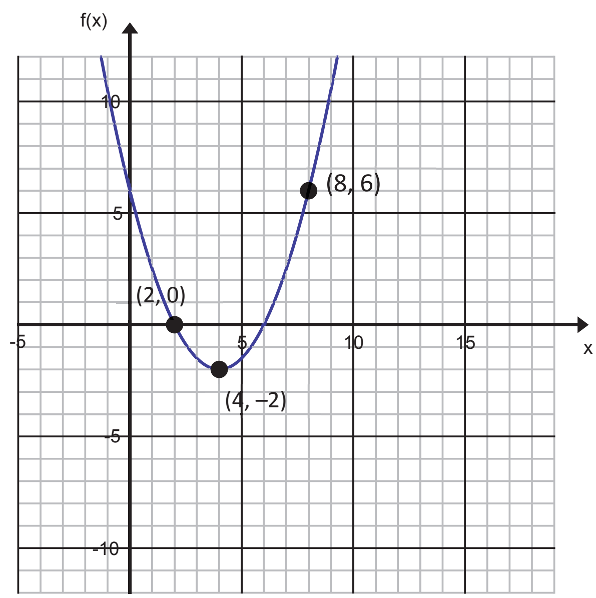
   Even though the coordinate values are not x-intercepts, students can use the symmetric nature of the function to find that the equation of the axis of symmetry goes through the midpoint joining the two x-coordinates of the symmetric points, or the average of the values of the two coordinates:

   $x = \frac{3+15}{2} = \frac{18}{2} = 9$. ***So, $x = 9$ is the equation for the axis of symmetry.***

   b. $(-2, 6)$ $(6, 4)$

   The given coordinates are not symmetric values; therefore, we cannot find the equation of the axis of symmetry.

2. **The point $(4, -2)$ is the vertex of the graph of a quadratic function. The points $(8, 6)$ and $(2, 0)$ also fall on the graph of the function. Complete the graph of this quadratic function by first finding two additional points on the graph. (If needed, make a table of values on your own paper.) Then, answer the questions on the right.**



   a. **Find the y-intercept.**

   $(0, 6)$

   b. **Find the x-intercept(s).**

   $(2, 0)$ $(6, 0)$

   c. **Find the interval on which the rate of change is always positive.**

   $(4, \infty)$ ***or*** $x > 4$

   d. **What is the sign of the leading coefficient for this quadratic function? Explain how you know.**

   Positive. Since the vertex is the minimum value of the graph (opens upward), the leading coefficient is positive.

Problem Set Sample Solutions

1. Khaya stated that every y-value of the graph of a quadratic function has two different x-values. Do you agree or disagree with Khaya? Explain your answer.

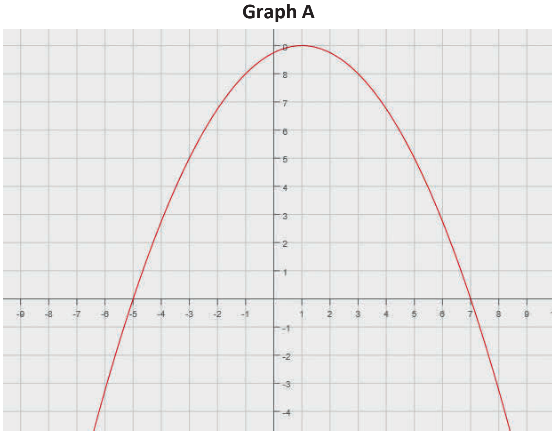
   The graph of a quadratic function has two different x-values for each y-value except at the vertex where there is only one.

2. Is it possible for the graphs of two *different* quadratic functions to each have $x = -3$ as its line of symmetry and both have a maximum at $y = 5$? Explain and support your answer with a sketch of the graphs.

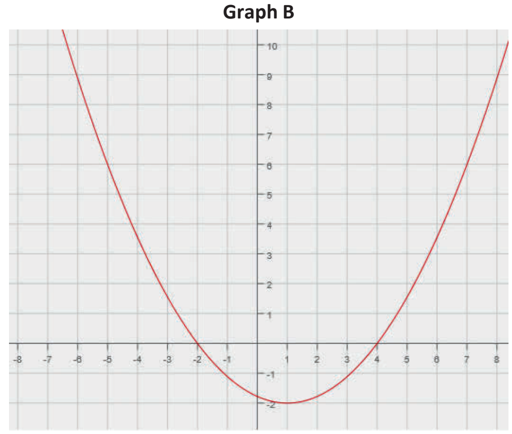
   Students should sketch two graphs with vertex at $(-3, 5)$ and different x-intercepts.

3. Consider the following key features discussed in this lesson for the four graphs of quadratic functions below: x-intercepts, y-intercept, line of symmetry, vertex, and end behavior.

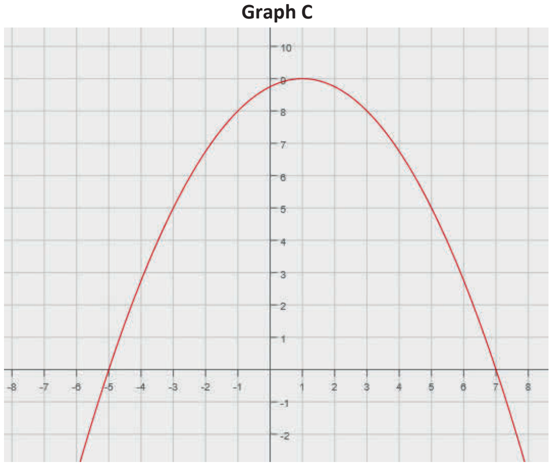
Graph A



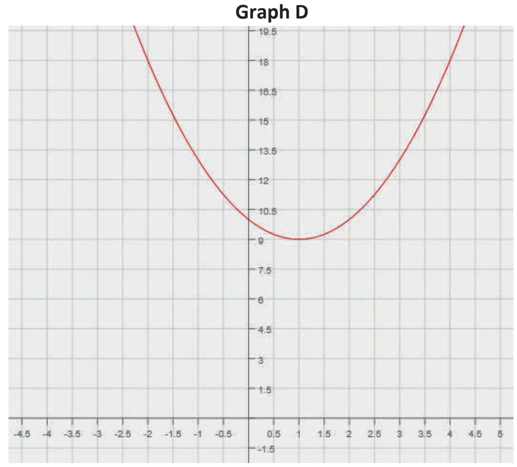
Graph B



Graph C



Graph D



   a. Which key features of a quadratic function do graphs A and B have in common? Which features are not shared?

   Same—line of symmetry.

   Different—vertex, y-intercept, x-intercepts, and end behavior.

b. **Compare graphs A and C, and explain the differences and similarities between their key features.**

Same—line of symmetry, vertex, x-intercepts, y-intercept, and end behavior.

Different—none, Graphs A and C are the same. The images of the graphs are cropped differently, so it may appear that the two are different.

c. **Compare graphs A and D, and explain the differences and similarities between their key features.**

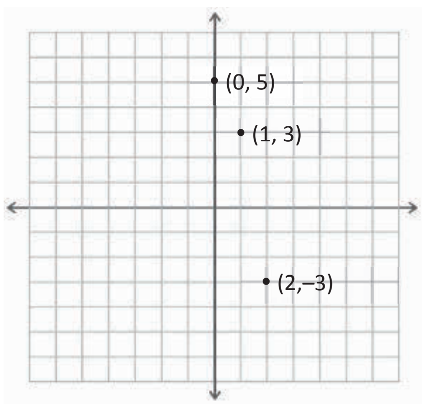
Same—line of symmetry and vertex

Different—x-intercepts (point out that Graph D does not have x-intercepts), y-intercept, and end behavior.

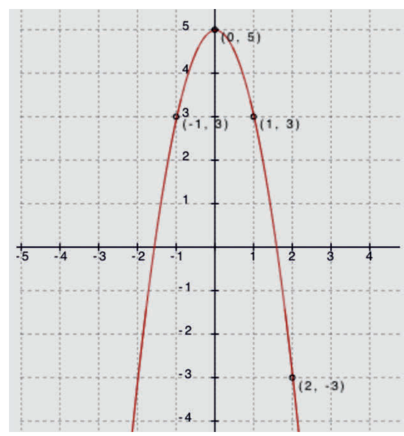
d. **What do all four of the graphs have in common?**

All four graphs have the same line of symmetry.

4. **Use the symmetric properties of quadratic functions to sketch the graph of the function below, given these points and given that the vertex of the graph is the point $(0, 5)$.**



Solution:



Lesson 9: Graphing Quadratic Functions from Factored Form, $f(x) = a(x - m)(x - n)$

Student Outcomes

- Students use the factored form of a quadratic equation to construct a rough graph, use the graph of a quadratic equation to construct a quadratic equation in factored form, and relate the solutions of a quadratic equation in one variable to the zeros of the function it defines.
- Students understand that the number of zeros in a polynomial function corresponds to the number of linear factors of the related expression and that different functions may have the same zeros but different maxima or minima.

Lesson Notes

Throughout this lesson, students apply mathematics to solve problems that arise in the physical world, specifically for objects in motion. They identify the important quantities of the situation and map the relationships between those quantities using graphs.

In this lesson, students relate the solutions of a quadratic equation in one variable to the zeros of the function it defines. They sketch graphs of quadratic functions from tables, expressions, and verbal descriptions of relationships in real-world contexts, identifying key features of the quadratic functions from their graphs. Students are required to graph and show the intercepts and minimum or maximum point. If students do not have time in class to complete Exercise 4, consider adding it to the problem set.

Classwork

Opening Exercise (3 minutes)

Have students complete the opening exercise individually. This review of solving a quadratic equation by factoring leads into the concept development in Example 1.

Opening Exercise

Solve the following equation.

$$x^2 + 6x - 40 = 0$$

The factored form is $(x+10)(x-4)=0$***, so*** $x = -10$ ***or*** 4***.***

Scaffolding:

Remind students of the product-sum rule for factoring quadratic expressions when the leading coefficient is 1: What two factors of the constant term can be added to give the coefficient of the linear term?

Or remind them that they can use the method of splitting the linear term.

Example 1 (7 minutes)

Display the equation $y = x^2 + 6x - 40$ on the board or screen. Make sure students have graph paper before the lesson begins. Have students work with a partner or in small groups to answer the following questions based on the equation.

Example 1

Consider the equation $y = x^2 + 6x - 40$.

a. **Given this quadratic equation, can you find the point(s) where the graph crosses the x-axis?**

(If students stall here, offer a hint. Ask: What is the y-value when the graph crosses the x-axis?)

The factors for $x^2 + 6x - 40$ are $(x-4)(x+10)$, so the solutions for the equation with $y = 0$ are $x = 4$ or $x = -10$.

Give students about two minutes to work with a partner to find the solution. Students should have a head start in figuring out how to proceed based on their results from Example 1. Have students record and label the two x-intercepts. Point out that the ordered pairs are called the x-intercepts of the graph and that the x-values alone, when the equation is equal to zero, are called the *zeros* or *roots* of the equation. Students should be able to generalize that for any quadratic equation, the roots are the solution(s), where $y = 0$, and these solutions correspond to the points where the graph of the equation crosses the x-axis.

b. **In the last lesson, we learned about the symmetrical nature of the graph of a quadratic function. How can we use that information to find the vertex for the graph?**

Since the x-value of the vertex is halfway between the two roots, we just need to find the midpoint of the two roots' x-values: $\frac{4+(-10)}{2} = -3$. Once students know the x-value of the vertex (which also tells us the equation for the axis of symmetry), they can substitute that value back into the equation: $y = (x-4)(x+10)$. Thus, $y = (-3-4)(-3+10) = (-7)(7) = -49$, and the vertex is $(-3, -49)$.

c. **How could we find the y-intercept (where the graph crosses the y-axis and where $x = 0$)?**

If we set x equal to 0, we can find where the graph crosses the y-axis.
$y = (x-4)(x+10) = (0-4)(0+10) = (-4)(10) = -40$

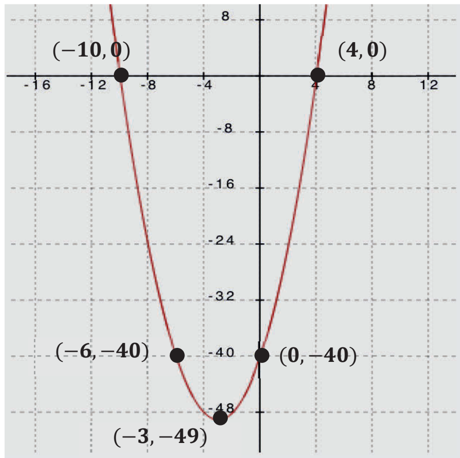
The y-intercept is $(0, -40)$.

d. **What else can we say about the graph based on our knowledge of the symmetrical nature of the graph of a quadratic function? Can we determine the coordinates of any other points?**

We know that the axis of symmetry is at $x = -3$ and that 0 is 3 units to the right of -3. Because the graph of a quadratic function is symmetrical, there exists another point with an x-coordinate 3 units to the left of -3, which would be $x = -6$. The points with x-coordinates of 0 and -6 will have the same y-coordinate, which is -40. Therefore, another point on this graph would be $(-6, -40)$.

Have students plot the five points on graph paper and connect them, making the following graph of a quadratic function:

e. Plot the points you know for this equation on graph paper, and connect them to show the graph of the equation.



Exercise 1 (10 minutes)

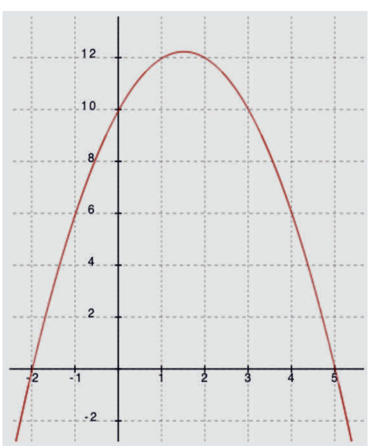
Have students work with a partner or in small groups to graph the following quadratics. Discuss as a class what the key features of a quadratic graph are.

Exercise 1

Graph the following functions, and identify key features of the graph.

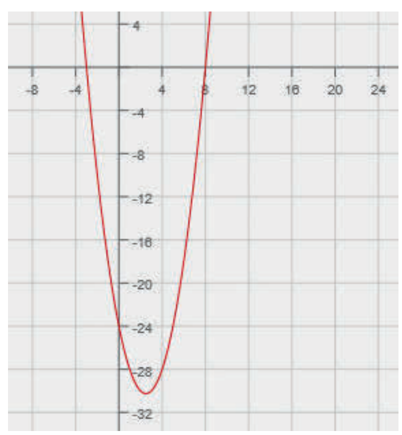
a. $f(x) = -(x + 2)(x - 5)$

Key features: x-intercepts $(-2, 0)$ $(5, 0)$; vertex at $x = 1.5$ $(1.5, 12.25)$; y-intercept $(0, 10)$; end behavior: this graph opens down (as x approaches $\pm\infty$, y approaches $-\infty$)



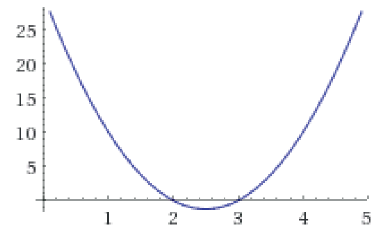
b. $g(x) = x^2 - 5x - 24$

Key features: x-intercepts $(-3, 0)$ $(8, 0)$; vertex at $x = 2.5$ $(2.5, -30.25)$; y-intercept $(0, -24)$; end-behavior: this graph opens up (as x approaches $\pm\infty$, y approaches ∞)



c. $f(x) = 5(x-2)(x-3)$

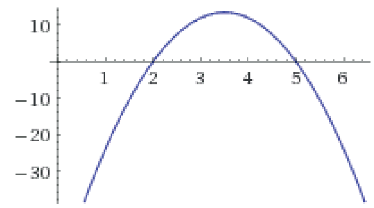
Key features: x-intercepts are $(2, 0)$ and $(3, 0)$; vertex is where $x = 2.5$: $(2.5, -1.25)$; and the y-intercept is $(0, 30)$; end behavior: this graph opens up (as x approaches $\pm\infty$, y approaches ∞).



(x from 0 to 5)

d. $p(x) = -6x^2 + 42x - 60$

Factored form: $p(x) = -6(x-5)(x-2)$
Key features: x-intercepts are $(5, 0)$ and $(2, 0)$; the y-intercept is $(0, -60)$; the axis of symmetry is at $x = 3.5$; the vertex is $(3.5, 13.5)$; end behavior: this graph opens down (as x approaches $\pm\infty$, y approaches $-\infty$).



(x from 0 to 6)

Debrief as a class sharing students' work before moving on to Example 2.

Example 2 (5 minutes)

Display the graph on the board, and discuss the example as a class.

- How can we use the x-intercepts to write a corresponding quadratic function?
  - *When the function is set equal to zero, the solutions to the equation must be $x = -4$ and $x = 2$. From this, we can determine that the factors must have been $(x+4)$ and $(x-2)$.*
- Why are the x-intercepts not enough information to determine a unique quadratic function?
  - *We don't know the leading coefficient, a. There are infinitely many quadratic functions that have these same x-intercepts.*

After completing the example, verify on the board that the function found does indeed have zeros of $x = -4$ and $x = 2$ and therefore the graph will have x-intercepts of -4 and 2.

$$f(x) = 2(x+4)(x-2) = 0$$

$$x + 4 = 0 \text{ or } x - 2 = 0$$

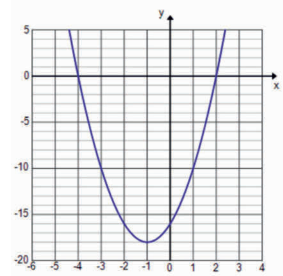
$$x = -4 \text{ or } x = 2$$

Example 2

Consider the graph of the quadratic function shown below with x-intercepts -4 and 2.

a. **Write a formula for a possible quadratic function, in factored form, that the graph represents using a as a constant factor.**

$f(x) = a(x+4)(x-2)$



b. **The y-intercept of the graph is -16.**

Use the y-intercept to adjust your function by finding the constant factor a.

$f(0) = a(0+4)(0-2) = -16 \rightarrow a = 2$

$f(x) = 2(x+4)(x-2)$

Exercises 2–3 (5 minutes)

Allow students to continue working in pairs or groups. Then, debrief as a class.

Exercise 2

Given the x-intercepts for the graph of a quadratic function, write a possible formula for the quadratic function, in factored form.

a. **x-intercepts: 0 and 3**

$f(x) = ax(x-3)$

b. **x-intercepts: -1 and 1**

$f(x) = a(x+1)(x-1)$

c. **x-intercepts: -5 and 10**

$f(x) = a(x+5)(x-10)$

d. **x-intercepts: $\frac{1}{2}$ and 4**

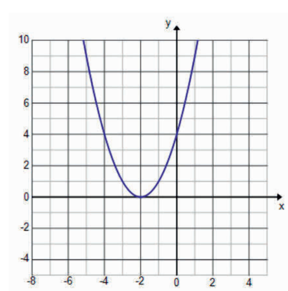
$f(x) = a(x-\frac{1}{2})(x-4)$

Exercise 3

Consider the graph of the quadratic function shown below with x-intercept -2.

a. **Write a formula for a possible quadratic function, in factored form, that the graph represents using a as a constant factor.**

$f(x) = a(x+2)(x+2)$ *or* $f(x) = a(x+2)^2$



b. **The y-intercept of the graph is 4.**

Use the y-intercept to adjust your function by finding the constant factor a.

$f(0) = a(0+2)(0+2) = 4 \rightarrow a = 1$

$f(x) = (x+2)(x+2)$ *or* $f(x) = (x+2)^2$

- In Exercise 2, how many linear factors did each of your functions have?
  - *Each function had two linear factors.*

- Why do you think that is?
  - *These are quadratic functions. When written in standard form, the degree (or highest exponent) of all of the functions will be* 2.
- In Exercise 3, why were the two linear factors the same?
  - *There was only one* x*-intercept, so both factors needed to give the same solution of* $x = -2$. *There couldn't be a different linear factor, or the graph would have a second* x*-intercept. There couldn't be a single factor of* $(x + 2)$ *or the graph would be linear.*

In the example below, students must make sense of the quantities presented in the problem. They are given the problem in its context and must decontextualize to solve the problem and then recontextualize to interpret their solution.

Example 3 (5 minutes)

Present the following problem, and use the questions that follow to guide discussion to a path to the solutions. Students may use their graphing calculators to see the graph. However, some class time may be needed to provide instruction in using the graphing calculator effectively.

Example 3

A science class designed a ball launcher and tested it by shooting a tennis ball straight up from the top of a 15-story building. They determined that the motion of the ball could be described by the function:

$$h(t) = -16t^2 + 144t + 160,$$

where t represents the time the ball is in the air in seconds and $h(t)$ represents the height, in feet, of the ball above the ground at time t. What is the maximum height of the ball? At what time will the ball hit the ground?

a. **With a graph, we can see the number of seconds it takes for the ball to reach its peak and how long it takes to hit the ground. How can factoring the expression help us graph this function?**

   Change the expression to factored form. First, factor out the -16 (GCF): $-16(t^2 - 9t - 10)$. ***Then, we can see that the quadratic expression remaining is factorable***: $-16(t + 1)(t - 10)$.

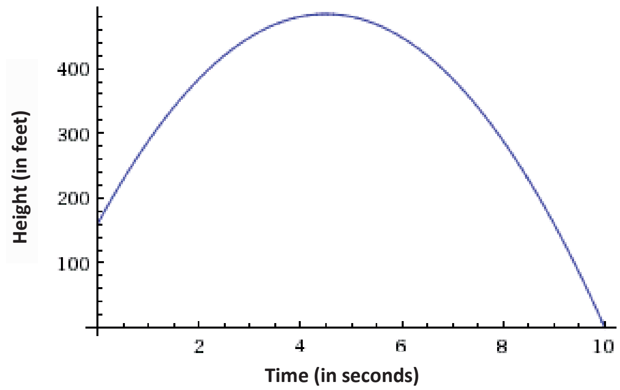
b. **Once we have the function in its factored form, what do we need to know in order to graph it? Now graph the function.**

   We can find the t***-intercepts,*** y***-intercept, axis of symmetry, and the vertex and then sketch the graph of the function.*** t***-intercepts are*** $(10, 0)$ ***and*** $(-1, 0)$; y***-intercept is*** $(0, 160)$; ***the axis of symmetry is*** $t = 4.5$; ***and the vertex is*** $(4.5, 484)$. ***(We find the*** y***-coordinate of the vertex by substituting*** 4.5 ***into either form of the equation.)***

Students determine the key features and graph the function, and the teacher puts the following graph on the board.

- Why is the domain of this function $[0, 10]$?
  - *Because negative time values do not make sense and neither do time values after the ball hits the ground*
- Why is the leading coefficient always negative for functions representing falling objects?
  - *Functions with negative leading coefficients have maximums, while functions with positive leading coefficients have minimums. A launched object rises and then falls and, therefore, has a maximum.*

- Within the context of this problem, what does the y-intercept of the graph represent?
  - *The initial height of the ball*
- Within the context of this problem, what does the vertex of the graph represent?
  - *The time when the ball reaches the maximum height and the maximum height of the ball*



c. **Using the graph, at what time does the ball hit the ground?**

10 seconds

d. **Over what domain is the ball rising? Over what domain is the ball falling?**

The ball is rising from 0 to 4.5 seconds $(0, 4.5)$. It is falling from 4.5 seconds to 10 seconds $(4.5, 10)$. 4.5 is the t-value of the vertex of the graph, and for this context it represents the time that the ball reaches its highest point, and then it begins to fall toward the ground. At $t = 10$, the graph has an h-value of 0, so in this context, it represents the time that the ball hits the ground and stops descending.

e. **Using the graph, what is the maximum height the ball reaches?**

484 ft. *(See work above.)*

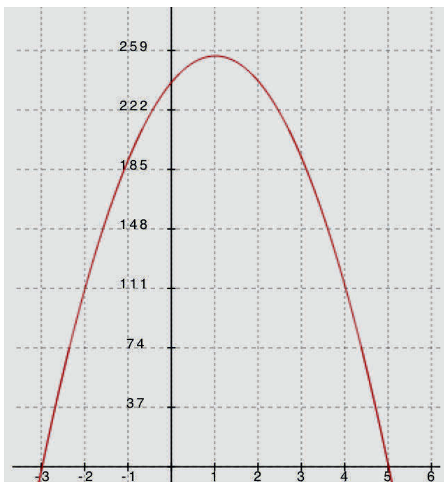
Exercise 4 (5 minutes)

Have students work with a partner or in small groups. If available, they could use technology to confirm their answers.

Exercise 4

The science class in Example 3 adjusted their ball launcher so that it could accommodate a heavier ball. They moved the launcher to the roof of a 23-story building and launched an 8.8-pound shot put straight up into the air. (Note: Olympic and high school women use the 8.8-pound shot put in track and field competitions.) The motion is described by the function $h(t) = -16t^2 + 32t + 240$, where $h(t)$ represents the height, in feet, of the shot put above the ground with respect to time t in seconds. (Important: No one was harmed during this experiment!)

a. **Graph the function, and identify the key features of the graph.**



Key features: Vertex: $(1, 256)$; t-intercepts: $(5, 0)$ and $(-3, 0)$; y-intercept: $(0, 240)$

b. **After how many seconds does the shot put hit the ground?**

The factored form of the function: $h(t) = -16(t - 5)(t + 3)$, so the positive zero of the function is $(5, 0)$, and the shot put hits the ground at 5 seconds.

c. **What is the maximum height of the shot put?**

The vertex is found where $t = 1$, so the vertex is $(1, 256)$, and the shot put reaches a maximum height at 256 ft.

d. **What is the value of $h(0)$, and what does it mean for this problem?**

$h(0) = 240$. This means that the shot put's initial height was 240 ft above the ground.

Closing (2 minute)

Have students write responses to these questions or share responses with a partner before sharing as a class.

- When graphing a quadratic function, why might it be convenient to write the function in factored form?
  - *When the quadratic is in factored form, the x-intercepts of the graph can be easily found. Then, the axis of symmetry can be found by averaging the x-intercepts.*
- If given the x-intercepts of the graph of a quadratic, can you write the equation of the corresponding function?
  - *Given the x-intercepts, you can only write a general formula for the quadratic function. If the x-intercepts are m and n, the formula for the function would be $f(x) = a(x - m)(x - n)$. You would need to know another point on the graph in order to find the leading coefficient, a.*

Lesson Summary

- **When we have a quadratic function in factored form, we can find its x-intercepts, y-intercept, axis of symmetry, and vertex.**
- **For any quadratic equation, the roots are the solution(s) where $y = 0$, and these solutions correspond to the points where the graph of the equation crosses the x-axis.**
- **A quadratic equation can be written in the form $y = a(x - m)(x - n)$, where m and n are the roots of the function. Since the x-value of the vertex is the average of the x-values of the two roots, we can substitute that value back into equation to find the y-value of the vertex. If we set $x = 0$, we can find the y-intercept.**

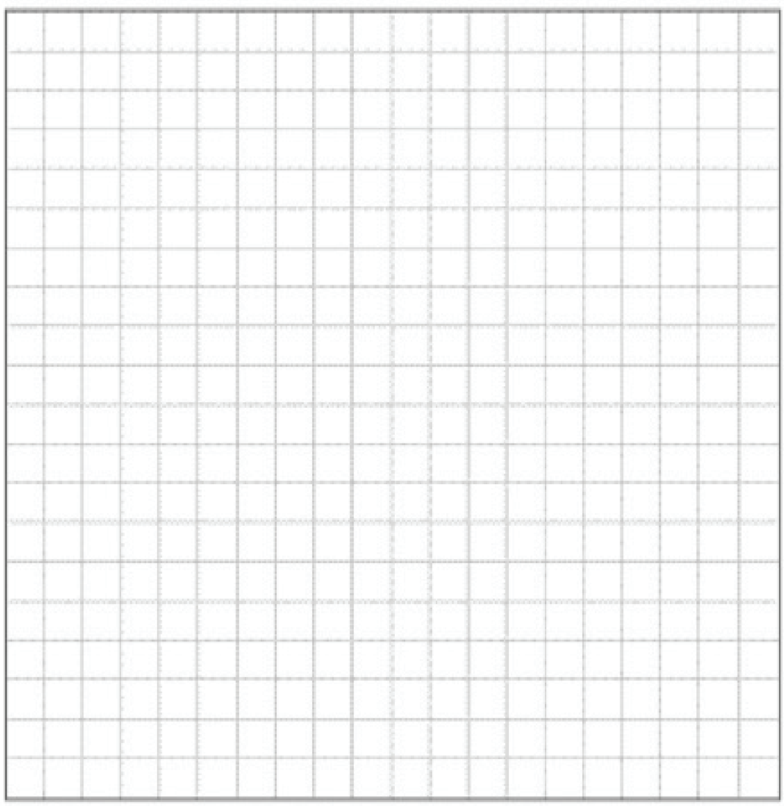
Exit Ticket (3 minutes)

Name ______________________________ Date ________________

Lesson 9: Graphing Quadratic Functions from Factored Form, $f(x) = a(x - m)(x - n)$

Exit Ticket

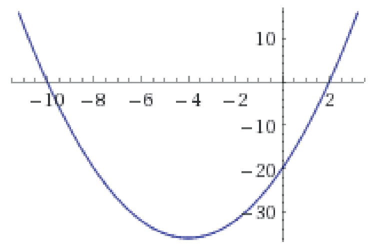
Graph the following function, and identify the key features of the graph: $t(x) = x^2 + 8x - 20$.



Exit Ticket Sample Solutions

Graph the following function, and identify the key features of the graph: $t(x) = x^2 + 8x - 20$.

Factored form: $t(x) = (x + 10)(x - 2)$, so the x-intercepts are $(-10, 0)$ and $(2, 0)$; the y-intercept is $(0, -20)$; and the vertex is where $x = -4$: $(-4, -36)$; end behavior: this graph opens up (as x approaches $\pm\infty$, the y-value approaches ∞).



(x from -10 to 2)

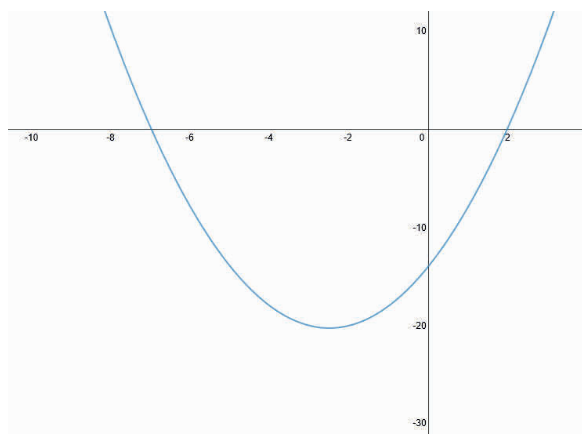
Problem Set Sample Solutions

The first problem in this set offers a variety of quadratic functions to graph, including some in factored form, some in standard form, some that open up, some that open down, one that factors as the difference of squares, one that is a perfect square, and one that requires two steps to complete the factoring (GCF).

1. **Graph the following on your own graph paper, and identify the key features of the graph.**

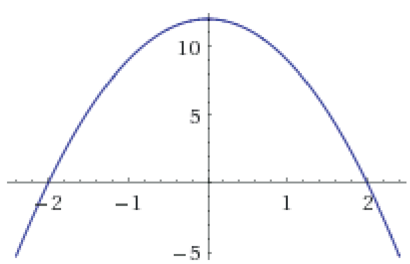
   a. $f(x) = (x - 2)(x + 7)$

   Key features: x-intercepts $(2, 0)$ and $(-7, 0)$; y-intercept $(0, -14)$; vertex at $x = -2.5$ $(-2.5, -20.25)$.



   b. $h(x) = -3(x - 2)(x + 2)$

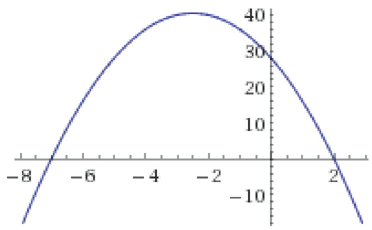
   Key features: vertex $(0, 12)$; axis of symmetry at $x = 0$; y-intercept $(0, 12)$; x-intercepts $(-2, 0)$ and $(2, 0)$.



(x from -2 to 2)

c. $g(x) = -2(x-2)(x+7)$

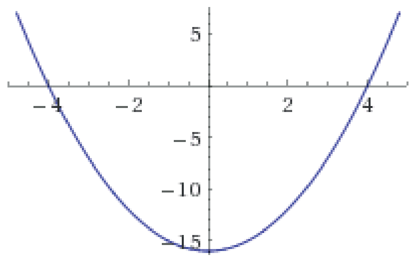
Key features: x-intercepts $(2, 0)$ and $(-7, 0)$; vertex is where $x = -2.5$ $(-2.5, 40.5)$; y-intercept is $(0, 28)$.



(x from −8 to 2)

d. $h(x) = x^2 - 16$

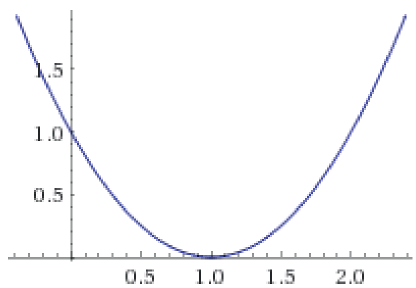
Key features: x-intercepts $(4, 0)$ and $(-4, 0)$; y-intercept and vertex are both $(0, -16)$.



(x from −4 to 4)

e. $p(x) = x^2 - 2x + 1$

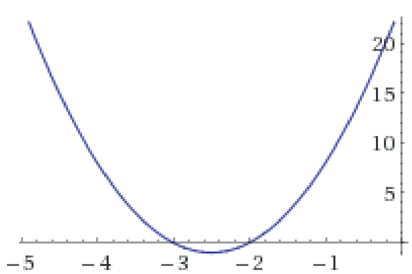
Key features: x-intercept is a double root and is also the vertex $(1, 0)$; the y-intercept is $(0, 1)$.



(x from 0 to 2)

f. $q(x) = 4x^2 + 20x + 24$

Factored form: $q(x) = 4(x^2 + 5x + 6) = 4(x+2)(x+3)$; x-intercepts $(-2, 0)$ and $(-3, 0)$; y-intercept $(0, 24)$; vertex is where $x = -2.5$ at the point $(-2.5, -1)$.



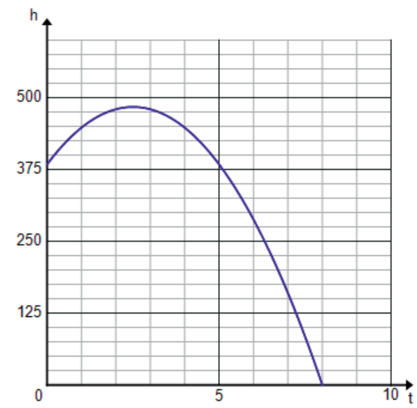
(x from −5 to 0)

2. A rocket is launched from a cliff. The relationship between the height of the rocket, h, in feet, and the time since its launch, t, in seconds, can be represented by the following function:

$$h(t) = -16t^2 + 80t + 384.$$

a. Sketch the graph of the motion of the rocket.

Key features: factored form is $h(t) = -16(t^2 - 5t - 24) = -16(t-8)(t+3)$; t-intercepts $(8, 0)$ and $(-3, 0)$; y-intercept $(0, 384)$; vertex at $t = 2.5$ $(2.5, 484)$.



b. When does the rocket hit the ground?

The rocket will hit the ground after 8 seconds. The zeros of the function are at $t = -3$ and 8. Since the rocket was in the air from $t = 0$, it would be airborne for 8 seconds.

c. When does the rocket reach its maximum height?

The rocket will reach its maximum height after 2.5 seconds. The t-coordinate of the vertex, the highest point on the graph, is at $t = 2.5$.

d. What is the maximum height the rocket reaches?

The maximum height is 484 ft. The vertex, which is the maximum point, is at $(2.5, 484)$.

e. At what height was the rocket launched?

The rocket was launched from a height of 384 ft. This is the height of the rocket at time $t = 0$.

3. Given the x-intercepts for the graph of a quadratic function, write a possible formula for the quadratic function, in factored form.

a. x-intercepts: -1 and -6

$f(x) = a(x+1)(x+6)$

b. x-intercepts: -2 and $\frac{2}{3}$

$f(x) = a(x+2)\left(x - \frac{2}{3}\right)$

c. x-intercepts: -3 and 0

$f(x) = ax(x+3)$

d. x-intercept: 7

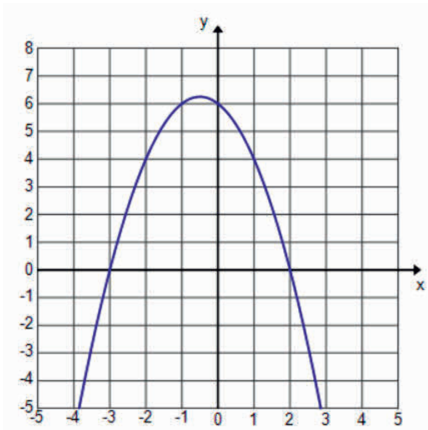
$f(x) = a(x-7)(x-7)$ or $f(x) = a(x-7)^2$

4. Suppose a quadratic function is such that its graph has x-intercepts of -3 and 2 and a y-intercept of 6.

   a. Write a formula for the quadratic function.

   $f(x) = -(x+3)(x-2)$

   b. Sketch the graph of the function.



Lesson 10: Interpreting Quadratic Functions from Graphs and Tables

Student Outcomes

- Students interpret quadratic functions from graphs and tables: zeros (x-intercepts), y-intercept, the minimum or maximum value (vertex), the graph's axis of symmetry, positive and negative values for the function, increasing and decreasing intervals, and the graph's end behavior.
- Students determine an appropriate domain and range for a function's graph and when given a quadratic function in a context, recognize restrictions on the domain.

Throughout this lesson, students make sense of quantities, their units, and their relationships in problem situations.

Lesson Notes

In this lesson, students interpret the key features of graphs and estimate and interpret average rates of change from a graph. They continue to use graphs, tables, and equations to interpret and compare quadratic functions.

Classwork

Opening Exercise (5 minutes): Dolphins Jumping In and Out of the Water

Find a video of a dolphin jumping in and out of the water. (An example is provided below.) This clip is short enough to show more than once or back up and repeat some segments. If the teacher is able to slow or pause at several places in a jump, let students estimate the height of the dolphin at various times (in seconds). Some video players show the time in seconds. (This example is a stock video on YouTube and is about 1.5 minutes longer than the video described in the problem. The video is in slow motion and takes longer to run than the real time lapse.)
http://www.youtube.com/watch?v=g8RoFdyYY3s

After watching the video clip of the dolphins jumping in and out of the ocean as an introduction, ask students what the graph of time vs. the height of the dolphin above and below sea level may look like. Then, project the graph for the problem onto the white board or screen. Note that this same graph and context is used in the End-Of-Module Assessment for Module 5.

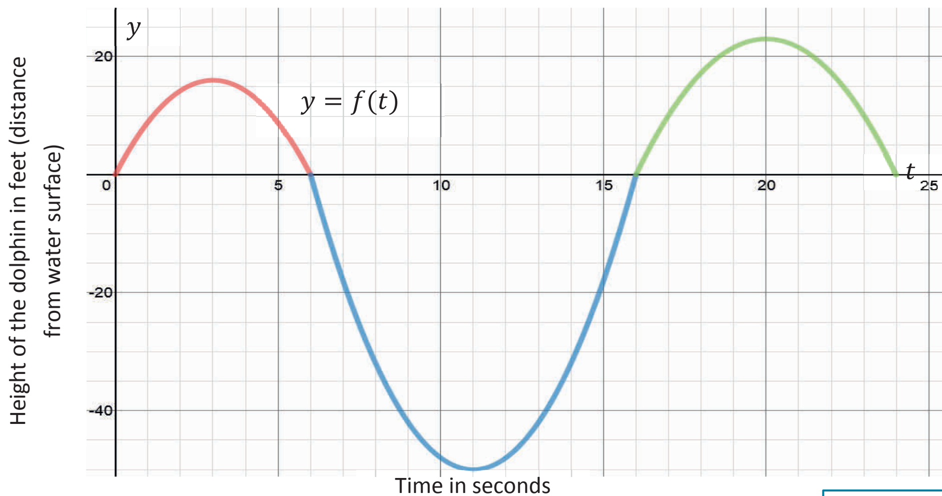
It is important to discuss the fact that there is no measure of horizontal distance represented in the graph below (the graph does NOT trace the path of the dolphin's motion). In fact, the dolphin might be jumping straight up and straight down, exiting and entering the water in exactly the same spot, and the graph would not look different than it does. This is because the height of the dolphin is related to the number of seconds that have passed, not the distance it moves forward or backward. So, the graph and function represent the TIME that is moving forward, not necessarily the dolphin.

Example 1 (10 minutes)

Show or project the graph on the board or screen. Read the prompt below aloud and have students take notes individually. Then, working in groups or with a partner, read the questions below aloud and give students time with their partner or group to work out the answers using their notes.

Example 1

In a study of the activities of dolphins, a marine biologist made a 24-second video of a dolphin swimming and jumping in the ocean with a specially equipped camera that recorded one dolphin's position with respect to time. This graph represents a piecewise function, $y = f(t)$, that is defined by quadratic functions on each interval. It relates the dolphin's vertical distance from the surface of the water, in feet, to the time from the start of the video, in seconds. Use the graph to answer the questions below.



a. **Describe what you know for sure about the actions of the dolphin in the time interval from 0–6 sec. Can you determine the horizontal distance the dolphin traveled in that time interval? Explain why or why not.**

The dolphin jumped out of the water at $t = 0$ and back into the water at $t = 6$. We cannot determine the horizontal distance because the function models the vertical distance to time, not the horizontal distance to time.

Some students may interpret the graph as the path (or trajectory) of the dolphin jumping in and out of the water. It is important to point out that the graph does not indicate the forward motion of the dolphin. In fact, the dolphin can jump straight up and straight down, and the relationship of height to time still has the same graph.

> *Scaffolding:*
>
> For students who are struggling with the concept of horizontal movement not being represented in the graph, remind them of the Module 1 video of the man who jumped straight up and down; the graph was still quadratic.

b. **Where do you find the values for which $f(t) = 0$? Explain what they mean in the context of this problem.**

$f(0) = 0$, $f(6) = 0$, $f(16) = 0$, $f(24) = 0$

$f(t) = 0$ represents the time when the dolphin enters the water or jumps out of the water. It is when the dolphin is at the water's surface.

c. **How long was the dolphin swimming under water in the recorded time period? Explain your answer or show your work.**

10 seconds. Between $t = 6$ and $t = 16$, $f(t) < 0$. This means that the dolphin is below the surface of the water.

d. **Estimate the maximum height, in feet, that the dolphin jumped in the recorded 24-second time period? Explain how you determined your answer.**

The vertex that is in the highest position is estimated to be $(20, 23)$. Students may indicate the vertex during the first jump. However, it is not the maximum of the entire function (24-second time period).

e. **Locate the point on the graph where $f(t) = -50$, and explain what information the coordinates of that point give you in the context of this problem.**

$f(11) = -50$. This means after 11 seconds have passed, the dolphin is 50 feet below the water surface.

Example 2 (15 minutes)

For this example, we interpret a function from a table of values. Project the table below onto the board, and have students study the data and perhaps even make an informal plot. Read the prompt, and have students take notes. Then, have students work with partners or in small groups to answer the questions below as the teacher reads them aloud. Stop for discussion whenever it seems appropriate.

Example 2

The table below represents the value of Andrew's stock portfolio, where V represents the value of the portfolio in hundreds of dollars and t is the time in months since he started investing. Answer the questions that follow based on the table of values.

t (months)	$V(t)$ (hundreds of dollars)
2	325
4	385
6	405
8	385
10	325
12	225
14	85
16	-95
18	-315

a. **What kind of function could model the data in this table? How can you support your conclusion?**

Students can make the conjecture that it might be quadratic based on the shape suggested by plotting the points or by noticing the suggested symmetry of the data. However, they should not make a claim that all U-shaped curves can be well modeled by a quadratic function. A more robust support would be to notice that the sequence of $V(t)$ values has constant second differences over equally spaced intervals of t, which is the characteristic of sequences defined by a quadratic expression.

b. **Assuming this data is in fact quadratic, how much did Andrew invest in his stock initially? Explain how you arrived at this answer.**

Andrew initially invested $\$225$. I used the symmetric value of the quadratic function to find that $V(0) = V(12) = 225$.

c. **What is the maximum value of his stock, and how long did it take to reach the maximum value?**

$V(6) = 405$. It took Andrew 6 months to reach the maximum value of $\$405$.

d. **If the pattern continues to follow the quadratic trend shown above, do you advise Andrew to sell or keep his stock portfolio? Explain why.**

Andrew should sell. The stock initially increased, reaching a maximum value of $\$405$, then decreased. Since this is a quadratic function, it will not increase again. Rather, it continuously decreases after reaching the maximum.

e. **How fast is Andrew's stock value decreasing between $[10, 12]$? Find another two-month interval where the average rate of change is faster than $[10, 12]$ and explain why.**

$$\frac{225-325}{12-10} = -\frac{100}{2} = -50$$

The average rate of change of Andrew's stock portfolio between $[10, 12]$ is $-\$50$. The average rate of change is faster than 50 for any two-month interval after 12 months. Students may calculate the actual rate of change to show that it decreases faster or explain that the quadratic model decreases at a faster rate.

f. **Are there other two-month intervals where the rate of change is the same as $[10, 12]$? Explain your answer.**

The rate of change for a quadratic function is not constant and changes from positive to zero to negative. It is not possible for another two-month interval to have the same rate of change as $[10, 12]$.

Closing (5 minutes)

- Give an example of what the rate of change for an interval of the graph of a quadratic function can tell you.
  - *Answers will vary. For example, the rate of change over an interval can tell us the average rate of increase in profit or the average rate of speed of an object during a given time period.*

Lesson Summary

When interpreting quadratic functions and their graphs, it is important to note that the graph does not necessarily depict the path of an object. In the case of free-falling objects, for example, it is height with respect to time.

The y-intercept can represent the initial value of the function given the context, and the vertex represents the highest (if a maximum) or the lowest (if a minimum) value.

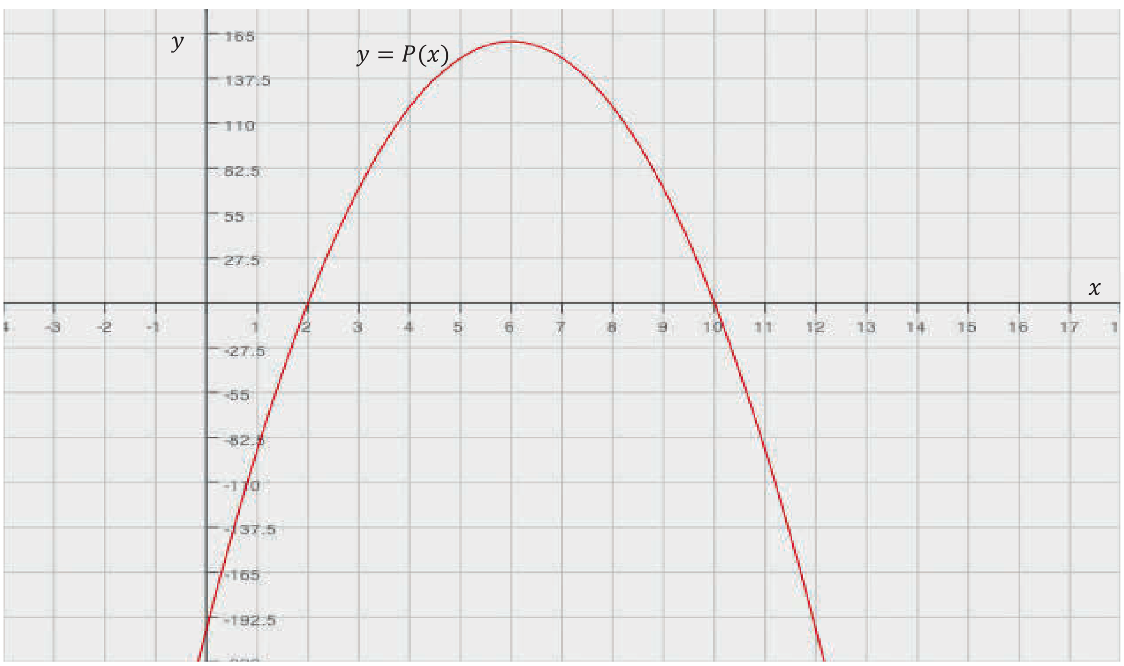
Exit Ticket (10 minutes)

Name ______________________________ Date ____________

Lesson 10: Interpreting Quadratic Functions from Graphs and Tables

Exit Ticket

A toy company is manufacturing a new toy and trying to decide on a price that maximizes profit. The graph below represents profit (P) generated by each price of a toy (x). Answer the questions based on the graph of the quadratic function model.



a. If the company wants to make a maximum profit, what should the price of a new toy be?

b. What is the minimum price of a toy that produces profit for the company? Explain your answer.

c. Estimate the value of $P(0)$, and explain what the value means in the problem and how this may be possible.

d. If the company wants to make a profit of $\$137$, for how much should the toy be sold?

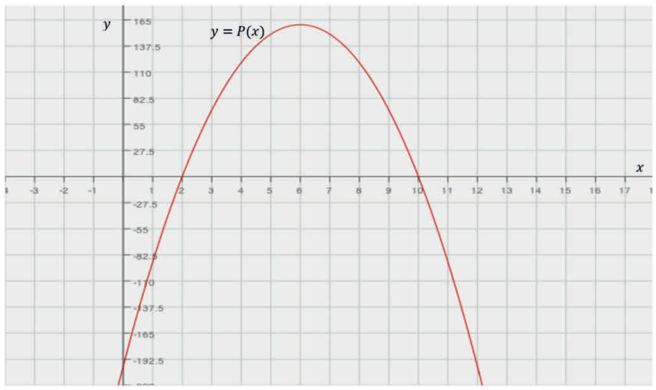
e. Find the domain that only results in a profit for the company, and find its corresponding range of profit.

f. Choose the interval where the profit is increasing the fastest: $[2, 3]$, $[4, 5]$, $[5.5, 6.5]$, $[6, 7]$. Explain your reasoning.

g. The company owner believes that selling the toy at a higher price results in a greater profit. Explain to the owner how selling the toy at a higher price affects the profit.

Exit Ticket Sample Solutions

A toy company is manufacturing a new toy and trying to decide the price that maximizes profit. The graph below represents profit (P) generated by each price of a toy (x). Answer the questions based on the graph of the quadratic function model.



a. **If the company wants to make a maximum profit, what should the price of a new toy be?**

$\$6$

b. **What is the minimum price of a toy that produces profit for the company? Explain your answer.**

The price of a toy must be more than $\$2$ to generate a profit. $\$2$ causes the company to break even and not make any profit.

c. **Estimate the value of $P(0)$, and explain what the value means in the problem and how this may be possible.**

$P(0)$ is approximately $-\$192.50$. Students should interpret an x-value of zero as the toy being given away for free. There is no way to say with certainty what the value $P(0) = -192.50$ represents in this context because we do not know what assumptions were made about how many of the toys would be produced at each price point. We only know what the company concluded about their profit at each price point. It is quite likely that the model is only useful for the domain $(2, 10)$ in this context.

d. **If the company wants to make a profit of $\$137$, for how much should the toy be sold?**

Approximately $\$4.50$ or $\$7.50$

e. **Find the domain that only results in a profit for the company, and find its corresponding range of profit.**

Domain: $(2, 10)$

Range: $(0, 160)$

f. **Choose the interval where the profit is increasing the fastest: $[2, 3]$, $[4, 5]$, $[5.5, 6.5]$, $[6, 7]$. Explain your reasoning.**

The profit is increasing the fastest on the interval $[2, 3]$. The function's rate increases fastest during the interval $[2, 3]$. It should be noted that the function increases and then decreases. However, the rate of change decreases reaching the rate of change of 0 at $x = 6$ and then increases.

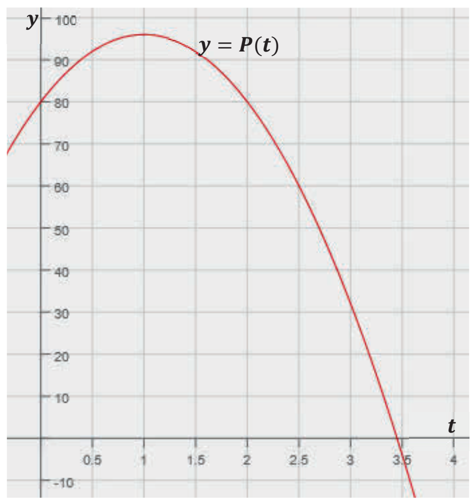
g. **The company owner believes that selling the toy at a higher price results in a greater profit. Explain to the owner how selling the toy at a higher price affects the profit.**

A higher priced toy does not necessarily make for a greater profit. The highest profit is produced when the toy is sold at $\$6$, and then decreases if it is sold at a higher price than $\$6$. Since this is a quadratic function, the profit only decreases after it reaches its maximum.

Problem Set Sample Solutions

Pettitte and Ryu each threw a baseball into the air.

The vertical height of Pettitte's baseball is represented by the graph $y = P(t)$ below. P represents the vertical distance of the baseball from the ground in feet, and t represents time in seconds.



The vertical height of Ryu's baseball is represented by the table values $R(t)$ below. R represents the vertical distance of the baseball from the ground in feet, and t represents time in seconds.

t	$R(t)$
0	86
0.5	98
1	102
1.5	98
2	86
2.5	66
3	38
3.52	0

Use the functions on the previous page to answer the following questions.

a. **Whose baseball reached the greatest height? Explain your answer.**

Ryu's baseball reached a maximum height of 102 feet, and Pettitte's baseball reached a maximum height of 96 feet. Students compare the maximum heights represented in the graph and table and interpret the vertex values in context.

b. **Whose ball reached the ground fastest? Explain your answer.**

Pettitte's ball took less than 3.5 seconds, and Ryu's ball took more than 3.5 seconds. Students interpret x-intercepts in context.

c. **Pettitte claims that his ball reached its maximum faster than Ryu's. Is his claim correct or incorrect? Explain your answer.**

Pettitte's claim is incorrect. It took both balls 1 second to reach their maximum heights. Students recognize that even though the vertex is different, both functions have the same axis of symmetry.

d. **Find $P(0)$ and $R(0)$ values and explain what they mean in the problem. What conclusion can you make based on these values? Did Ryu and Pettitte throw their baseballs from the same height? Explain your answer.**

$P(0) = 80$ and $R(0) = 86$. Students interpret the y-intercept in context. Pettitte and Ryu threw their baseballs from 80 ft. and 86 ft. above the ground, respectively. They were throwing their baseballs from different initial heights.

e. **Ryu claims that he can throw the ball higher than Pettitte. Is his claim correct or incorrect? Explain your answer.**

Ryu's claim is incorrect. Even though Ryu's ball reached maximum height of 102 ft., he started from higher above the ground. He was initially at 86 ft. above the ground. Ryu and Pettitte both threw the ball 16 ft. vertically from their initial positions. Students need to interpret the vertex value in relation to the y-intercept.

Name ______________________________ Date ____________

1. A rectangle with positive area has length represented by the expression $3x^2 + 5x - 8$ and width by $2x^2 + 6x$. Write expressions in terms of x for the perimeter and area of the rectangle. Give your answers in standard polynomial form and show your work.

   a. Perimeter:

   b. Area:

   c. Are both your answers polynomials? Explain why or why not for each.

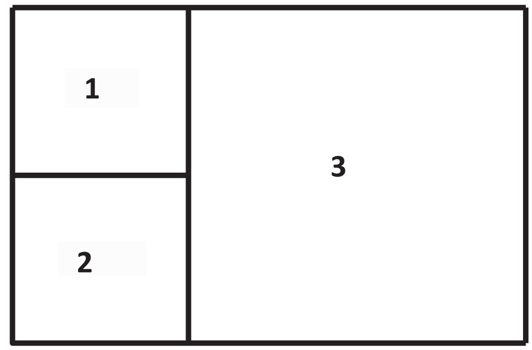
   d. Is it possible for the perimeter of the rectangle to be 16 units? If so, what value(s) of x will work? Use mathematical reasoning to explain how you know you are correct.

e. For what value(s) of the domain will the area equal zero?

f. The problem states that the area of the rectangle is positive. Find and check two positive domain values that produce a positive area.

g. Is it possible that negative domain values could produce a positive function value (area)? Explain why or why not in the context of the problem.

2. A father divided his land so that he could give each of his two sons a plot of his own and keep a larger plot for himself. The sons' plots are represented by squares 1 and 2 in the figure below. All three shapes are squares. The area of square 1 equals that of square 2, and each can be represented by the expression $4x^2 - 8x + 4$.



   a. Find the side length of the father's plot, which is square 3, and show or explain how you found it.

   b. Find the area of the father's plot, and show or explain how you found it.

   c. Find the total area of all three plots by adding the three areas, and verify your answer by multiplying the outside dimensions. Show your work.

3. The baseball team pitcher was asked to participate in a demonstration for his math class. He took a baseball to the edge of the roof of the school building and threw it up into the air at a slight angle so that the ball eventually fell all the way to the ground. The class determined that the motion of the ball from the time it was thrown could be modeled closely by the function

$$h(t) = -16t^2 + 64t + 80,$$

where h represents the height of the ball in feet after t seconds.

   a. Determine whether the function has a maximum value or a minimum value. Explain your answer mathematically.

   b. Find the maximum or minimum value of the function. After how many seconds did the ball reach this value? Show how you found your answers.

   c. For what interval of the domain is the function increasing (i.e., ball going up)? For what interval of the domain is the function decreasing (i.e., ball going down)? Explain how you know.

   d. Evaluate $h(0)$. What does this value tell you? Explain in the context of the problem.

e. How long is the ball in the air? Explain your answer.

f. State the domain of the function, and explain the restrictions on the domain based on the context of the problem.

g. Graph the function indicating the vertex, axis of symmetry, intercepts, and the point representing the ball's maximum or minimum height. Label your axes using appropriate scales. Explain how your answer to part (d) is demonstrated in your graph.

h. Does your graph illustrate the actual trajectory of the ball through the air as we see it?

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a–b**	Student shows little evidence of understanding the properties of polynomial operations. OR Student shows little or no attempt to complete the problems, or the attempt is aborted before calculations are completed.	Student shows some evidence of understanding operations with polynomials, but there are errors in the calculations (e.g., side lengths are not doubled for perimeter, or the product is missing terms). The final answers are not given in standard polynomial form.	Student provides expressions that are treated accurately and appropriately with operations relating to perimeter and area that are carried out correctly. However, the final answers are not given in standard polynomial form.	Student provides expressions that are treated accurately and appropriately with operations relating to perimeter and area that are carried out correctly. Final answers are in simplest and standard polynomial form.
	c	Student makes no attempt to answer the question.	Student makes an attempt to explain, but the explanation shows little understanding of the definition of a polynomial and of the concept of closure for polynomial operations.	Student provides an explanation that shows some understanding of the definition of a polynomial or of the concept of closure for polynomial operations.	Student provides an explanation that is correct and includes understanding of the definition of a polynomial and of closure for polynomial operations.

	d	Student provides an answer (e.g., yes or no) with no supporting explanation. OR Student makes no attempt to answer the question.	Student creates and solves an equation and provides a logical explanation for the process and solution. However, the equation may be incorrect, or there are errors in calculation that lead to an incorrect final answer and incorrect values for x.	Student creates and solves a correct equation and provides a logical explanation for the process and solution. However, both values of x are given as the final solution with some explanation offered as justification.	Student creates and solves a correct equation and provides a logical explanation for the process and solution, which includes only the correct value of x that works in the equation, with the extraneous solution noted in the explanation.
	e	Student makes an attempt to answer this question, but the original equation representing the area is not factored correctly, and no correct results are found. OR Student makes no attempt to answer the question.	Student makes an attempt to factor the original form of the equation representing the area and sets it equal to zero. There is one (or no) correct result given.	Student makes an attempt to factor the original form of the equation representing the area and sets it equal to zero. However, only two correct results are given.	Student accurately factors the equation into its four linear factors and sets them equal to zero. AND Student correctly solves for the four values of x that make the product equal to zero.
	f	Student makes no attempt to find the two values, or the attempt is aborted before a conclusion is reached.	Student only provides one correct value and checks it effectively. OR Student provides two values, but only one is checked effectively. OR Student provides two logically selected values, but the checks attempted are ineffective for both.	Student correctly selects two values and substitutes them into the equation. There are calculation errors in the check that do not affect the final outcome.	Student correctly selects two values and substitutes them into the equation to check whether the x-value produces a positive area. (Note: The zeros found in part (e) might be used as boundaries for the correct values in this part.)

	g	Student makes little or no attempt to answer the question.	Student attempts to answer the question; however, the explanation is missing important parts. For example, there are no references to the dimensions being positive or to the requirement that there must be an even number of negative factors for area. There might be specific examples of negative values that produce a positive area given, but they are without explanation.	Student correctly answers the question but only provides a partially correct explanation. For example, the explanation does not mention the need for an even number of negative factors in the area expression or that both dimensions must be positive (i.e., if two factors are negative, they must both represent the same dimension).	Student correctly answers the question, including the following requirements: there are references in the explanation to the need for positive dimensions and that if an x-value makes any of the factors negative, there must be an even number of negative factors. This means that both negative factors must be for the same dimension.
2	a–b	Student provides an answer with no evidence to indicate a connection is made between the information given in the prompt and the side lengths of squares 1 and 2.	Student provides an answer with evidence to indicate a connection is made between the information in the prompt and the side lengths of squares 1 and 2. However, there is no evidence that a connection is made to the side length of square 3 and the operations needed to answer the questions. Calculations contain errors, and the explanation is missing or inadequate.	Student provides an answer with evidence of understanding the connection between the information given in the prompt and the side length and area of square 3. Calculations are completed accurately, but the explanations are incomplete. OR Student calculations contain errors, but the explanation is adequate and is not dependent on errors in the calculations.	Student provides an answer with evidence of understanding the connection between the information given in the prompt and the side length and area of square 3. Calculations are completed accurately, and the explanations are complete. (Note: Equivalent forms of the solution are acceptable, e.g., $2(2x-2)=4x-4$; $(4x-4)^2=16x^2-32x+16$.)
	c	Student makes little or no attempt to find the area using either method.	Student makes an attempt to find the total area by adding the three smaller areas, but there are errors and verification is impossible. Work is shown.	Student correctly determines the total area by adding the three smaller areas, but there is either no attempt to check by multiplying or there are errors in the attempt to check by multiplying. Work is shown and supports the correct results.	Student correctly determines the total area by adding the three smaller areas and correctly verifies the solution by multiplying the total length by total width. All work is shown and supports the results.

3	**a**	Student makes no attempt to determine the function's value.	Student attempts to find the maximum value for the function but does not make a connection between the sign of the leading coefficient and the direction that the graph opens. Student also makes errors in the calculations. OR Student makes a connection between the sign of the leading coefficient and the direction the graph opens, but the graph is said to have a minimum because the leading coefficient is negative.	Student attempts to find the maximum value for the function, and makes a connection between the sign of the leading coefficient and the direction that the graph opens, but the explanation does not make it clear that the negative leading coefficient indicates that the graph opens down.	Student correctly finds the maximum value for the function and makes a clear connection between the sign of the leading coefficient and the direction that the graph opens. Student provides a clear and logical explanation.
	b–e	Student does not provide evidence of understanding the properties of the key features of the quadratic function. Calculations are ineffective or incorrect. Explanations are missing or ineffective. OR Student makes no attempt to answer the question.	Student provides some evidence of understanding the properties of the key features of the quadratic function. However, calculations are incorrect and explanations are missing or inadequate.	Student provides accurate interpretations of the key features of the quadratic function, but some calculations are incorrectly performed. Complete, logical explanations are supported by calculations.	Student provides accurate interpretations of the key features of the quadratic function, and all calculations are performed correctly and supported by complete, logical explanations.
	f	Student makes no attempt to state a domain of the function.	Student gives an incorrect domain as all real numbers (i.e., the domain of the function with no consideration of the context).	Student describes the domain only as positive or greater than zero, or as less than 5, with no consideration given to the context (i.e., partial consideration of the context).	Student provides an accurate description of domain or gives it as a set. Consideration is given to the beginning of the experiment (0 seconds) and to the end (5 seconds).

	g	Student provides little indication of understanding the graphic representation of the function. The graph is incorrectly drawn, and the key features are missing or incorrectly identified. OR Student makes little or no attempt to graph the function.	Student attempts to graph the function, but key features are not indicated on the graph. The axes are not labeled clearly, but the scale fits the graph or allows for visual verification of the key features (i.e., y-intercept $(0, 80)$, vertex $(2, 144)$, and x-intercept $(5, 0)$).	Student graphs the function clearly and correctly but does not indicate key features on the graph. The axes are labeled clearly with a scale that fits the graph and allows for visual verification of the key features (even though they are not marked).	Student graphs the function clearly and correctly with the y-intercept $(0, 80)$, the vertex $(2, 144)$, and the x-intercept $(5, 0)$ identified correctly. The axes are labeled clearly with a scale that fits the graph.
	h	Student provides an answer, but there is no explanation provided. OR Student makes no attempt to answer the question.	Student attempts to use the laws of physics in the explanation for this question (i.e., the horizontal axis represents the change in time rather than forward motion). However, the answer to the question is given incorrectly.	Student answers the question correctly and provides an explanation. However, the explanation is based only partially on the physics addressed in this problem (i.e., the horizontal axis represents the change in time rather than forward motion).	Student answers the question correctly and provides an explanation that shows an understanding of the physics addressed in this problem (i.e., the horizontal axis represents the change in time rather than forward motion).

Name ______________________________ Date ________________

1. A rectangle with positive area has length represented by the expression $3x^2 + 5x - 8$ and width by $2x^2 + 6x$. Write expressions in terms of x for the perimeter and area of the rectangle. Give your answers in standard polynomial form and show your work.

   a. Perimeter:

   $2(3x^2 + 5x - 8) + 2(2x^2 + 6x)$
   $= 6x^2 + 10x - 16 + 4x^2 + 12x$
   $= 10x^2 + 22x - 16$

   b. Area:

   $(3x^2 + 5x - 8)(2x^2 + 6x)$
   $= 6x^4 + 18x^3 + 10x^3 + 30x^2 - 16x^2 - 48x$
   $= 6x^4 + 28x^3 + 14x^2 - 48x$

   c. Are both your answers polynomials? Explain why or why not for each.

   Yes, both have terms with only whole number exponents (greater than or equal to 0), coefficients that are real numbers, and a leading coefficient that is not 0.

   d. Is it possible for the perimeter of the rectangle to be 16 units? If so, what value(s) of x will work? Use mathematical reasoning to explain how you know you are correct.

   $10x^2 + 22x - 16 = 16$
   $10x^2 + 22x - 32 = 0$
   $2(5x^2 + 11x - 16) = 0$
   $2(5x + 16)(x - 1) = 0$
   So $x = -\frac{16}{5}$ or 1 OR -3.2 or 1

   Check: $2(\text{length}) + 2(\text{width}) = 2(6.72) + 2(1.28) = 13.44 + 2.56 = 16$

   Yes, the perimeter could be 16 units with length 6.72 and width 1.28.

   If $x = 1$, the length would be $3(1)^2 + 5(1) - 8 = 0$; therefore, $x \neq 1$.

   If $x = -3.2$, the length would be $3(-3.2)^2 + 5(-3.2) - 8 = 3(10.24) - 16 - 8 = 30.72 - 24 = 6.72$,

   and the width would be $2(-3.2)^2 + 6(-3.2) = 20.48 - 19.2 = 1.28$.

e. For what value(s) of the domain will the area equal zero?

In factored form: $(3x^2 + 5x - 8)(2x^2 + 6x) = (3x + 8)(x - 1)(2x)(x + 3) = 0$

The Area = 0 when $x = -\frac{8}{3}$, 1, 0, or −3.

f. The problem states that the area of the rectangle is positive. Find and check two positive domain values that produce a positive area.

Check the values around those we found in part (e), since on either side of the zeros there is likely to be either positive or negative values.

Try substituting $x = 2$ into the factored form. The factors will then be $(+)(+)(+)(+)>0$.

So, all numbers greater than 1 will give positive results ($x = 3$, etc.).

Note: If there are any, there must be an even number of negative factors, and any pair of negative factors must be for the same dimension.

g. Is it possible that negative domain values could produce a positive function value (area)? Explain why or why not in the context of the problem.

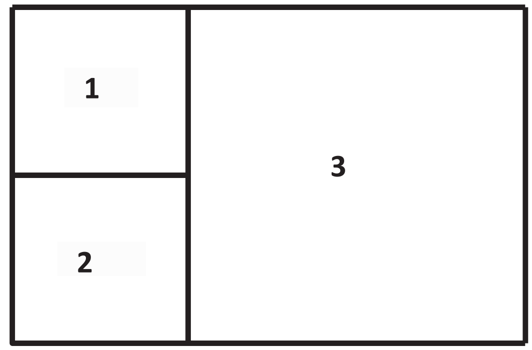
As long as the dimensions are positive, it is possible that the value of x is negative. That means that either two of the four factors must be negative, and the negative factors must both be from the same dimension (length or width), or all four of the factors must be negative. Using the logic in part (f), it is possible that numbers less than $-\frac{8}{3}$ or possibly between 0 and $-\frac{8}{3}$ might work.

Let's try $x = -4$: The factors would be $(-)(-)(-)(-)$. This one works since both dimensions will be positive.

Let's try $x = -1$: The factors would be $(+)(-)$ …. I can stop now because the length is negative, which is impossible in the context of the problem.

So, the answer is YES. There are negative values for x that produce positive area. They are less than $-\frac{8}{3}$, and they result in both positive dimensions and positive area.

2. A father divided his land so that he could give each of his two sons a plot of his own and keep a larger plot for himself. The sons' plots are represented by squares 1 and 2 in the figure below. All three shapes are squares. The area of square 1 equals that of square 2, and each can be represented by the expression $4x^2 - 8x + 4$.



   a. Find the side length of the father's plot, which is square 3, and show or explain how you found it.

      $4x^2 - 8x + 4$ is a perfect square that factors to $(2x - 2)^2$.

      The side length is the square root of $(2x - 2)^2$, which is $(2x - 2)$.

      The father's plot is twice the length of one of the smaller squares, or the sum of the two.

      The side length for plot 3 is $2(2x - 2) = 4x - 4$.

   b. Find the area of the father's plot, and show or explain how you found it.

      The area of the father's plot is the square of the side length:

      $(4x - 4)^2 = 16x^2 - 32x + 16$.

   c. Find the total area of all three plots by adding the three areas, and verify your answer by multiplying the outside dimensions. Show your work.

      By adding the areas of the three squares:

      $(4x^2 - 8x + 4) + (4x^2 - 8x + 4) + (16x^2 - 32x + 16) = 24x^2 - 48x + 24$.

      By multiplying total length by total width:

      Total length $= (2x - 2) + (4x - 4) = 6x - 6$;

      Total width $= (2x - 2) + (2x - 2) = 4x - 4$;

      Area $= (6x - 6)(4x - 4) = 24x^2 - 48x + 24$.

3. The baseball team pitcher was asked to participate in a demonstration for his math class. He took a baseball to the edge of the roof of the school building and threw it up into the air at a slight angle so that the ball eventually fell all the way to the ground. The class determined that the motion of the ball from the time it was thrown could be modeled closely by the function

$$h(t) = -16t^2 + 64t + 80,$$

where h represents the height of the ball in feet after t seconds.

a. Determine whether the function has a maximum value or a minimum value. Explain your answer mathematically.

The function has a maximum because the leading coefficient is negative, making the graph of the function open down.

b. Find the maximum or minimum value of the function. After how many seconds did the ball reach this value? Show how you found your answers.

To find the zeros of the function, we factor as follows:

$$-16(t^2 - 4t - 5) = -16(t - 5)(t + 1) = 0.$$

So, $t = -1$ *or* 5. *Therefore, the* t*-coordinate of the vertex is* $t = \frac{-1+5}{2} = 2$.

If we substitute 2 for t *into the original function, we find that the vertex is at (2, 144); this tells us that the maximum height is 144 ft., which occurs at 2 seconds.*

c. For what interval of the domain is the function increasing (i.e., ball going up)? For what interval of the domain is the function decreasing (i.e., ball going down)? Explain how you know.

The function is increasing from 0 to 2 seconds and decreasing from 2 to 5 seconds. The rate of change over [0, 2] is positive. The rate of change over [2, 5] is negative. For an answer based on the graph: The graph has positive slope from 0 to 2 seconds and negative slope from 2 to 5 seconds.

d. Evaluate $h(0)$. What does this value tell you? Explain in the context of the problem.

$h(0) = 80$. This is the initial height, the height at which the ball was when it was thrown upward. The roof was 80 ft. high.

e. How long is the ball in the air? Explain your answer.

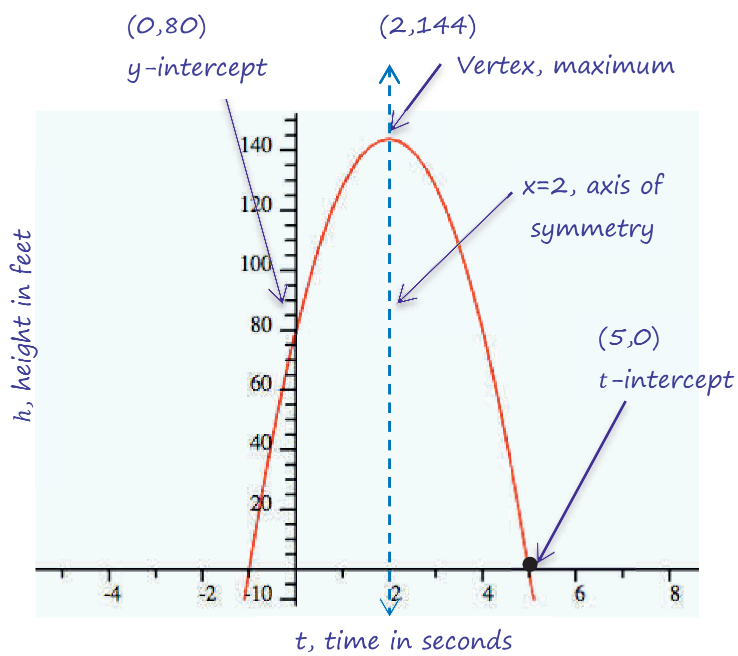
The ball is in the air for 5 seconds. When $t = 0$, the ball is released. When $t = 5$, the height is 0, which means the ball hits the ground 5 seconds after it is thrown.

f. State the domain of the function, and explain the restrictions on the domain based on the context of the problem.

We consider the experiment over at the time the ball reaches the ground, so it must be less than or equal to 5. Additionally, the values for t as described in this context must be greater than or equal to 0 because time began when the ball was thrown.
t: $\{0 \le t \le 5\}$

g. Graph the function indicating the vertex, axis of symmetry, intercepts, and the point representing the ball's maximum or minimum height. Label your axes using appropriate scales. Explain how your answer to part (d) is demonstrated in your graph.

The graph shows the function crossing the y-axis at (0, 80), the height at which the ball was thrown. Then, it travels to a height of 144 ft. at 2 seconds, and hits the ground at 5 seconds.



h. Does your graph illustrate the actual trajectory of the ball through the air as we see it?

No, the graph does not illustrate the actual trajectory of the ball through the air because movement along the horizontal axis represents changes in time, not horizontal distance. The ball could be going straight up and then straight down with very little change in horizontal position, and the graph would be the same.

Topic B

Using Different Forms for Quadratic Functions

Focus Standards:	■ Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. ■ Interpret expressions that represent a quantity in terms of its context.★ □ Interpret parts of an expression, such as terms, factors, and coefficients. □ Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of* P *and a factor not depending on* P. ■ Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$. ■ Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ □ Factor a quadratic expression to reveal the zeros of the function it defines. □ Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. ■ Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial. ■ Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

- Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
- Solve quadratic equations in one variable.
  - Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
  - Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
- For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
- Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
- Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
  - Graph linear and quadratic functions and show intercepts, maxima, and minima.
- Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
  - Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Instructional Days: 7

Lessons 11–12: Completing the Square (E, P)[1]

Lesson 13: Solving Quadratic Equations by Completing the Square (P)

Lesson 14: Deriving the Quadratic Formula (P)

Lesson 15: Using the Quadratic Formula (P)

Lesson 16: Graphing Quadratic Equations From the Vertex Form, $y = a(x-h)^2 + k$ (E)

Lesson 17: Graphing Quadratic Functions From the Standard Form, $f(x) = ax^2 + bx + c$ (P)

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

In Topic A, students expanded their fluency with manipulating polynomials and deepened their understanding of the nature of quadratic functions. They rewrote polynomial expressions by factoring and used the factors to solve quadratic equations in one variable, using rectangular area as a context. They also sketched quadratic functions and learned about the key features of their graphs, with particular emphasis on relating the factors of a quadratic expression to the zeros of the function it defines.

In Lessons 11 and 12 of Topic B, students learn to manipulate quadratic expressions by completing the square. They use this knowledge to solve quadratic equations in one variable in Lesson 13 for situations where factoring is either impossible or inefficient. There is particular emphasis on quadratic functions with irrational solutions in this topic, and students use these solutions as an opportunity to explore the property of closure for rational and irrational numbers. In Lesson 14, students derive the quadratic formula by completing the square for the standard form of a quadratic equation, $ax^2 + bx + c = 0$, and use it to solve quadratic equations that cannot be easily factored. They discover that some quadratic equations do not have real solutions. Students use the discriminant in Lesson 15 to determine whether a quadratic equation has one, two, or no real solutions. In Lesson 16, students learn that the $f(x) = a(x - h)^2 + k$ form of a function reveals the vertex of its graph. They sketch the graph of a quadratic function from its equation in vertex form and construct a quadratic equation in vertex form from its graph.

As students begin to work in two variables, they are introduced to business applications, which can be modeled with quadratic functions, including profit, loss, revenue, cost, etc. Then, students use all of the tools at their disposal in Lesson 17 to interpret functions and their graphs when prepared in the standard form, $f(x) = ax^2 + bx + c$. They explore the relationship between the coefficients and constants in both standard and vertex forms of the quadratic equation, and they identify the key features of their graphs.

Lesson 11: Completing the Square

Student Outcomes

- Students rewrite quadratic expressions given in standard form, $ax^2 + bx + c$ (with $a = 1$), in the equivalent completed-square form, $a(x - h)^2 + k$, and recognize cases for which factored or completed-square form is most efficient to use.

Throughout this lesson, students use the structure of quadratic equations to rewrite them in completed-square form.

Lesson Notes

In the Opening Exercise, students look for patterns in the structure of perfect square quadratic expressions. They recognize that when the leading coefficient is 1, the coefficient of the linear term, b, from the standard form, $x^2 + bx + c$, is always twice the constant term of the perfect square binomial $\left(x + \frac{1}{2}b\right)^2$ and that the constant term, c, must equal $\left(\frac{1}{2}b\right)^2$ for the quadratic expression to be rewritten as a perfect square. This leads directly to the method of completing the square. When given a quadratic expression that is *not* factorable as a perfect square, students can write an equivalent expression that includes a perfect square of a binomial. The importance and elegance of efficient methods is then reinforced. When is factoring the most expedient or useful method for rewriting an expression? When is it more efficient or more useful to complete the square? The examples and exercises support these ideas and introduce students to relevant business applications. Note that all quadratic expressions in this lesson have a leading coefficient of $a = 1$; other leading coefficients are addressed in Lesson 12. It may be helpful to hint to students that in later lessons we use this method of rewriting expressions to solve quadratic equations that are not factorable.

Classwork

Opening Exercise (5 minutes)

Project or draw the table on the board or screen. Demonstrate with the whole class by filling in the first row. Then, have students work in pairs to continue filling in the blanks in the next four rows and writing down their observations.

Opening Exercise

Rewrite the following perfect square quadratic expressions in standard form. Describe patterns in the coefficients for the factored form, $(x + A)^2$, and the standard form, $x^2 + bx + c$.

FACTORED FORM	WRITE THE FACTORS	DISTRIBUTE	STANDARD FORM
Example: $(x+1)^2$	$(x+1)(x+1)$	$x \cdot x + 1x + 1x + 1 \cdot 1$	$x^2 + 2x + 1$
$(x+2)^2$	$(x+2)(x+2)$	$x \cdot x + 2x + 2x + 2 \cdot 2$	$x^2 + 4x + 4$
$(x+3)^2$	$(x+3)(x+3)$	$x \cdot x + 3x + 3x + 3 \cdot 3$	$x^2 + 6x + 9$
$(x+4)^2$	$(x+4)(x+4)$	$x \cdot x + 4x + 4x + 4 \cdot 4$	$x^2 + 8x + 16$
$(x+5)^2$	$(x+5)(x+5)$	$x \cdot x + 5x + 5x + 5 \cdot 5$	$x^2 + 10x + 25$
$(x+20)^2$	$(x+20)(x+20)$	$x \cdot x + 20x + 20x + 20 \cdot 20$	$x^2 + 40x + 400$

For each row, the factored form and standard form are equivalent expressions, so $(x + A)^2 = x^2 + bx + c$. *A, the constant in factored form of the equation, is always half of b, the coefficient of the linear term in the standard form. c, the constant term in the standard form of the quadratic equation, is always the square of the constant in the factored form, A.*

- Can you generalize this pattern so that you can square *any* binomial in the form $(x + A)^2$?
  - $(x + A)^2 = x^2 + 2Ax + A^2$. *The coefficient of the linear term of the equation in standard form, b, is always twice the constant in the binomial, A. The constant term in the standard form of the equation, c, is always the square of the constant in the binomial, A.*

Example (5 minutes)

Have students continue to work with their partner to complete this table. Encourage them to use the patterns discussed above to find the factored form efficiently. Ideally, there should be no need to guess-and-check to factor these expressions. Students should progress through the first five examples but will likely get stuck on the last expression.

Example

Now try working backward. Rewrite the following standard form quadratic expressions as perfect squares.

STANDARD FORM	FACTORED FORM
$x^2 + 12x + 36$	$(x + 6)^2$
$x^2 - 12x + 36$	$(x - 6)^2$
$x^2 + 20x + 100$	$(x + 10)^2$
$x^2 - 3x + \frac{9}{4}$	$\left(x - \frac{3}{2}\right)^2$
$x^2 + 100x + 2500$	$(x + 50)^2$
$x^2 + 8x + 3$	*n/a*

- What is different about $x^2 + 8x + 3$? Why is it impossible to factor this expression as a perfect square binomial?
  - *It is not a perfect square.*
- If you could change something about the last expression to make it a perfect square, what would you change?
  - *If the constant term were a 16, it would be a perfect square (4^2 as the constant term and $4(2)$ as the linear term coefficient).*

Exploratory Challenge (8 minutes)

Exploratory Challenge

Find an expression equivalent to $x^2 + 8x + 3$ that includes a perfect square binomial.

$(x + 4)^2 - 13$

Show students that when an expression is not a perfect square, they can use the tabular method learned in Lesson 2 to rewrite this expression as an equivalent perfect square binomial. Write the expression $x^2 + 8x + 3$ on the board or screen, and lead them through the following process.

Under the quadratic expression, draw a 2×2 table to use as a tool. First, put the x^2 into the upper-left box; next, write one x above the box and one x to the left of the box. Then, follow the steps below.

	x	
x	x^2	

- We are looking for a perfect square binomial that matches our quadratic expression as closely as possible. How do we know there must an x-term in our binomial?
  - *We know this because $x \cdot x$ is the only way to get x^2 (given that we are looking for polynomials which require whole number exponents).*
- The quadratic expression in standard form has a linear term of $+8x$. What constant term must the perfect square binomial have if the linear term coefficient is positive 8? Fill in the missing cells both outside and inside the square.
  - *We want the same two numbers to add to $+8x$ so that would be $+4x$ and $+4x$. Therefore, each binomial must have $+4$ as its constant term since the b coefficient in this example is 8.*

If students have not already come to this conclusion, point out that the constant term in a perfect square binomial is always half of the b coefficient.

	x	$+4$
x	x^2	$+4x$
$+4$	$+4x$	

- If the binomial to be squared is $(x + 4)$, what must the constant term be when this perfect square is expanded? Fill in the final lower right box.
  - *$(x + 4)^2$ has a constant term of 16 when expanded.*

	x	$+4$
x	x^2	$+4x$
$+4$	$+4x$	$+16$

- So, we know that to factor this binomial as the perfect square $(x + 4)^2$, we would need $x^2 + 8x + 16$ instead of the $x^2 + 8x + 3$ that we actually have. Looks like a dead end… but wait!

  $$x^2 + 8x + 16 = x^2 + 8x + 3 + 13$$

  So, all we have to do is add $+13$ to our expression, right?
  - *No, that would not be an equivalent expression.*

Give students a chance to catch the mistake here. They should see that adding 13 to the expression changes its value. Encourage them to find a way to balance the expression; ultimately lead them to write the following:

$$x^2 + 8x + 3 \rightarrow x^2 + 8x + 3 + 13 - 13 \rightarrow x^2 + 8x + (3 + 13) - 13 \rightarrow (x^2 + 8x + 16) - 13 \rightarrow (x + 4)^2 - 13.$$

To verify the result, you may want to undo completing the square by multiplying and combining like terms to prove that the expressions are now equivalent.

Students notice repetition and recognize a pattern through the example above and exercises below. They use this repeated reasoning to generalize the pattern in Exercise 10.

Exercises (20 minutes)

Exercises

Rewrite each expression by completing the square.

1. $a^2 - 4a + 15$

$(a - 2)^2 + 11$ ***(Note: Since the constant term required to complete the square is less than the constant term, $+15$, students may notice that they just need to split the $+15$ strategically.)***

2. $n^2 - 2n - 15$

$(n - 1)^2 - 16$

3. $c^2 + 20c - 40$

$(c + 10)^2 - 140$

4. $x^2 - 1000x + 60\,000$

$(x - 500)^2 - 190\,000$

5. $y^2 - 3y + 10$

$\left(y - \frac{3}{2}\right)^2 + \frac{31}{4}$

6. $k^2 + 7k + 6$

$\left(k + \frac{7}{2}\right)^2 - \frac{25}{4}$

7. $z^2 - 0.2z + 1.5$

$(z - 0.1)^2 + 1.49$

8. $p^2 + 0.5p + 0.1$

$(p + 0.25)^2 + 0.0375$

9. $j^2 - \frac{3}{4}j + \frac{3}{4}$

$\left(j - \frac{3}{8}\right)^2 + \frac{39}{64}$

10. $x^2 - bx + c$

$\left(x - \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$

Closing (2 minutes)

Have students take a look at the expressions in Exercises 2 and 6.

- Is there anything you notice about these two expressions? Although we can (and did) complete the square for each, how else might they be rewritten?
  - *Both of these expressions are easy to write in factored form.*
- Note that in some circumstances, the easiest form may not be the most useful form. Even if an expression is easy to factor, we may still want to write it as a completed square.

Lesson Summary

Just as factoring a quadratic expression can be useful for solving a quadratic equation, completing the square also provides a form that facilitates solving a quadratic equation.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 11: Completing the Square

Exit Ticket

Rewrite the expression $r^2 + 4r + 3$, first by factoring and then by completing the square. Which way is easier? Explain why you think so.

Exit Ticket Sample Solutions

Rewrite the expression $r^2 + 4r + 3$, first by factoring and then by completing the square. Which way is easier? Explain why you think so.

By factoring: $(r+3)(r+1)$

By completing the square: $(r+2)^2 - 1$

Both options are fairly simple, and students may select either for their preference. The important thing is that they are thinking about efficiency in their methods and the various options available for rewriting quadratic expressions.

Problem Set Sample Solutions

Scaffolding:

If students need more practice with completing the square, use these problems or some like them.

Rewrite each expression by completing the square.

1. $q^2 + 12q + 32$

   $(q+6)^2 - 4$

2. $m^2 - 4m - 5$

   $(m-2)^2 - 9$

3. $x^2 - 7x + 6.5$

   $\left(x - \frac{7}{2}\right)^2 - 5.75$

4. $a^2 + 70a + 1225$

   $(a+35)^2$

5. $z^2 - 0.3z + 0.1$

   $(z - 0.15)^2 + 0.0775$

6. $y^2 - 6by + 20$

   $(y - 3b)^2 + 20 - 9b^2$

7. **Which of these expressions would be most easily rewritten by factoring? Justify your answer.**

   Students may respond with either Problem 1, 2, or 4, and justifications may range from a demonstration of the factoring process to a written explanation where students show that the product-sum rule can be applied to either of these expressions.

Lesson 12: Completing the Square

Student Outcomes

- Students rewrite quadratic expressions given in standard form, $ax^2 + bx + c$ (with $a \neq 1$), as equivalent expressions in completed-square form, $a(x-h)^2 + k$. They build quadratic expressions in basic business application contexts and rewrite them in equivalent forms.

Lesson Notes

Lesson 12 is similar in style and sequence to Lesson 11, except that leading coefficients for the quadratic expressions are not 1. Because so much of the groundwork was laid in the preceding lesson, there is more time in this lesson for covering business applications relevant to quadratic expressions.

Classwork

Opening Exercise (5 minutes)

> **Opening Exercise**
>
> **Rewrite each expression by completing the square.**
>
> a. $z^2 - 5z + 8$
>
> $\left(z - \frac{5}{2}\right)^2 + \frac{7}{4}$ *OR* $(z - 2.5)^2 + 1.75$
>
> b. $x^2 + 0.6x + 1$
>
> $(x + 0.3)^2 + 0.91$

> *Scaffolding:*
>
> Students who have difficulty seeing the structure of these expressions may benefit from a written procedure or list of steps they can follow for completing the square.

Example 1 (5 minutes)

> **Example 1**
>
> **Now complete the square for $2x^2 + 16x + 3$.**

Now that students are comfortable with rewriting expressions by completing the square, we can introduce expressions with leading coefficients other than 1. Start by writing the quadratic expression below on the board or screen. Then, walk students through the process of completing the square.

$$2x^2 + 16x + 3$$

Since students already know how to complete the square when the leading coefficient is 1, one way to deal with the leading coefficient is to group the x-terms and factor out the leading coefficient. Then, they can proceed exactly as they did in the previous lesson. Students should be careful to pay attention to the multiplier on the outside of the parentheses and also to the signs involved.

$$2(x^2 + 8x \qquad) + 3$$

Now complete the square of the quadratic expression in the parentheses, and offset the addition on the outside of the parentheses.

$$2(x^2 + 8x + 4^2) + 3 - 2(4^2)$$

Make sure all agree that the two operations will reverse each other and that the new expression is equivalent to the old.

$$2(x + 4)^2 + 3 - 32 \rightarrow 2(x + 4)^2 - 29$$

Check:

$$2(x + 4)^2 - 29 = 2(x^2 + 8x + 16) - 29 = 2x^2 + 16x + 32 - 29 = 2x^2 + 16x + 3$$

Yes, this matches our original expression.

Example 2 (15 Minutes)

If students are not familiar with the vocabulary of business, spend some time going through the terms and definitions below, which are also included in the student materials. Spending a few extra minutes on the vocabulary may extend the time needed for this example but is well worth it. After doing so, guide your students through the example that follows. Remind them to refer to the vocabulary reference in their materials when needed. This information is used in future lessons in this module and may need to become part of the students' math journals or notebooks.

- The relationship between the cost of an item and the quantity sold is often linear. Once we determine a relationship between the selling price of an item and the quantity sold, we can think about how to generate the most profit, that is, at what selling price do we make the most money?
- Quadratic expressions are sometimes used to model situations and relationships in business. A common application in business is to maximize profit, that is, the difference between the total revenue (income from sales) and the production cost. It is important to understand the vocabulary used in business applications.
- It is a good idea to put the following information in a math notebook to use as a reference, as business applications are used in future lessons of this module.

Example 2

Business Application Vocabulary

UNIT PRICE (PRICE PER UNIT): The price per item a business sets to sell its product, which is sometimes represented as a linear expression.

QUANTITY: The number of items sold, sometimes represented as a linear expression.

REVENUE: The total income based on sales (but without considering the cost of doing business).

UNIT COST (COST PER UNIT) OR PRODUCTION COST: The cost of producing one item, sometimes represented as a linear expression.

PROFIT: The amount of money a business makes on the sale of its product. Profit is determined by taking the total revenue (the quantity sold multiplied by the price per unit) and subtracting the total cost to produce the items (the quantity sold multiplied by the production cost per unit): $\text{Profit} = \text{Total Revenue} - \text{Total Production Costs}$.

- We can integrate the linear relationship of selling price to quantity and the profit formula to create a quadratic equation, which we can then maximize.

Use the example below to model this.

The following business formulas are used in this and the remaining lessons in the module:

$\text{Total Production Costs} = (\text{cost per unit})(\text{quantity of items sold})$

$\text{Total Revenue} = (\text{price per unit})(\text{quantity of items sold})$

$\text{Profit} = \text{Total Revenue} - \text{Total Production Costs}$

Have students work in pairs or small groups to solve the following problem:

Now solve the following problem:

A certain business is marketing its product and has collected data on sales and prices for the past few years. The company determined that when it raised the selling price of the product, the number of sales went down. The cost of producing a single item is $\$10$.

Selling Price (s)	Quantity Sold (q)
10	$1,000$
15	900
20	800
25	700

a. Using the data the company collected in this table, determine a linear expression to represent the quantity sold, q.

$q = -20s + 1200$

Use any two of the points of the data to find the slope of the linear relationship, as found in the data, to be -20. Substitute the coordinates of any (s, q) ordered pair from the table into the equation $q = -20s + b$, and solve for b to find the y-intercept, which is $1,200$.

Or use a graphic method to find the linear equation. Plot the data on a coordinate plane with s on the horizontal axis and q on the vertical axis. Use any two known points to determine the slope to be -20. The y-intercept will be visible, but students should use an algebraic method to verify rather than just estimate from the graph. Substitute these values into slope-intercept form, $y = mx + b$.

Use student responses from above to derive the following steps as a class. Discuss the algebraic reasoning behind each step, and then follow with the questions on the following page.

b. **Now find an expression to represent the profit function, P.**

Let q represent the quantity sold, s represent the selling price, and P represent the total profit. The expression $P(s)$ represents the total profit with respect to the selling price.

$P = (s)(q) - 10q$ — *The profit formula is $P = \text{Total Revenue} - \text{Production Costs}$, where $\text{Total Revenue} = \text{price} \cdot \text{quantity sold}$ and $\text{Production Costs} = \text{cost per item} \cdot \text{quantity sold}$.*

$P(s) = s(-20s + 1200) - 10(-20s + 1200)$ — *Substitute $-20s + 1200$ for q in the profit formula. (Note that if we factor the common factor from this form of $P(s)$, we get $P(s) = (-20 + 1200)(s - 10)$. This could save time later when we need to factor.)*

$P(s) = -20s^2 + 1200s + 200s - 12\,000$ — *Multiply the expressions, and combine like terms. We now have a quadratic function relating the price per item, s, to profit, P.*

$P(s) = -20s^2 + 1400s - 12\,000$ — *This is the expression that represents the profit function.*

> *Scaffolding:*
> Have students graph the quadratic equation to visually represent the function, $y = P(s)$, and the maximum fit.

- Find the equivalent factored form of the expression. How might we use the factored form?
  - $-20(s^2 - 70s + 600)$
    $-20(s - 60)(s - 10)$
    We can use the factored form of the expression to find the zeros of P by factoring and then setting the expression equal to zero.
- Use the zeros to find the minimum or maximum value for the profit function, P. How might we use the vertex to interpret the profit function? (Remind students of the process used in earlier lessons in this module.)
  - *If we set the expression above equal to 0, we get the following:*
    $-20(s - 60)(s - 10) = 0$, *which leads to $s = 10$ or 60.*
    Halfway between them is the axis of symmetry and the s-value for the vertex, which is $s = 35$.
    Then, $P(35) = 12\,500$.
    By finding the vertex, we can determine the selling price that generates the maximum profit. The x-values (domain) represent selling price, so the value of the x-coordinate at the vertex represents the best price. The P-value at the vertex tells us the amount of profit made at the maximum.
- What is the equivalent completed-square form for the profit expression?
  $-20s^2 + 1400s - 12\,000$
  - $-20(s^2 - 70s + 35^2) - 12\,000 + 20(35^2)$
    $-20(s - 35)^2 - 12\,000 + 24\,500$
    $-20(s - 35)^2 + 12\,500$
- What do you notice about the values in the completed-square form?
  - *The values of 35 and $12{,}500$ in the completed-square form are the same values we found to be the vertex when using the factored form. It appears that writing a quadratic expression in completed-square form reveals the vertex of the graph of the quadratic function.*

Hopefully, students notice that the vertex can be seen in the parameters of the completed-square form. If not, point it out to them: $(x - h)^2 + k$, where (h, k) is the vertex.

Exercises (10 minutes)

Exercises 1–6 include a business application but primarily focus on the procedure for completing the square. If time is short, choose two or three of these to work in class, and assign the others along with the Problem Set.

Exercises

For Exercises 1–5, rewrite each expression by completing the square.

1. $3x^2 + 12x - 8$

   $3(x^2 + 4x) - 8 \rightarrow 3(x+2)^2 - 8 - 12 \rightarrow 3(x+2)^2 - 20$

2. $4p^2 - 12p + 13$

   $4(p^2 - 3p) + 13 \rightarrow 4\left(p - \frac{3}{2}\right)^2 + 13 - 9 \rightarrow 4\left(p - \frac{3}{2}\right)^2 + 4$

3. $\frac{1}{2}y^2 + 3y - 4$

   $\frac{1}{2}(y^2 + 6y) - 4 \rightarrow \frac{1}{2}(y+3)^2 - 4 - \frac{9}{2} \rightarrow \frac{1}{2}(y+3)^2 - \frac{17}{2}$

4. $1.2n^2 - 3n + 6.5$

   $1.2(n^2 - 2.5n) + 6.5 \rightarrow 1.2(n - 1.25)^2 + 6.5 - 1.875 \rightarrow 1.2(n - 1.25)^2 + 4.625$

5. $\frac{1}{3}v^2 - 4v + 10$

   $\frac{1}{3}(v^2 - 12v) + 10 \rightarrow \frac{1}{3}(v-6)^2 + 10 - 12 \rightarrow \frac{1}{3}(v-6)^2 - 2$

6. **A fast food restaurant has determined that its price function is $3 - \frac{x}{20\,000}$, where x represents the number of hamburgers sold.**

   a. **The cost of producing x hamburgers is determined by the expression $5000 + 0.56x$. Write an expression representing the profit for selling x hamburgers.**

      $\text{Profit} = \text{Total Revenue} - \text{Total Production Costs} = (\text{quantity})(\text{price}) - \text{cost}$

      $$= (x)\left(3 - \frac{x}{20\,000}\right) - (5000 + 0.56x)$$

      $$= 3x - \frac{x^2}{20\,000} - 5000 - 0.56x$$

      $$= -\frac{x^2}{20\,000} + 2.44x - 5000$$

b. **Complete the square for your expression in part (a) to determine the number of hamburgers that need to be sold to maximize the profit, given this price function.**

$$-\frac{1}{20\,000}(x^2 - 48\,800x + \quad) - 5000 = -\frac{1}{20\,000}(x^2 - 48\,800x + 24\,400^2) - 5000 + \frac{24\,400^2}{20\,000}$$
$$= -\frac{1}{20\,000}(x - 24\,400)^2 - 5000 + 29\,768$$
$$= -\frac{1}{20\,000}(x - 24\,400)^2 + 24\,768$$

So, $24,400$ hamburgers must be sold to maximize the profit using this price expression. Note: The profit will be $\$24,768$.

Lesson Summary

Here is an example of completing the square of a quadratic expression of the form $ax^2 + bx + c$.

$$3x^2 - 18x - 2$$
$$3(x^2 - 6x) - 2$$
$$3(x^2 - 6x + 9) - 3(9) - 2$$
$$3(x - 3)^2 - 3(9) - 2$$
$$3(x - 3)^2 - 29$$

Exit Ticket (10 minutes)

Name ______________________________ Date ____________________

Lesson 12: Completing the Square

Exit Ticket

1. Complete the square: $ax^2 + x + 3$.

2. Write the expression for the profit, P, in terms of q, the quantity sold, and s, the selling price, based on the data collected below on sales and prices. Use the examples and your notes from class to then determine the function that represents yearly profit, P, in terms of the sales, s, given the production cost per item is $\$30$.

Selling Price, $\$$ (s)	**Quantity Sold (q)**
100	7000
200	6000
500	3000
600	2000
800	0

Exit Ticket Sample Solutions

1. **Complete the square: $ax^2 + x + 3$.**

   $$a\left(x^2 + \frac{1}{a}x\right) + 3 \rightarrow a\left(x + \frac{1}{2a}\right)^2 + 3 - \frac{1}{4a}$$

2. **Write the expression for the profit, P, in terms of q, the quantity sold, and s, the selling price, based on the data collected below on sales and prices. Use the examples and your notes from class to then determine the function that represents yearly profit, P, in terms of the sales, s, given the production cost per item is $\$30$.**

Selling Price, $\$$ (s)	Quantity Sold (q)
100	7000
200	6000
500	3000
600	2000
800	0

   The revenue (total income from sales) is the price per item times number of items, $(s)(q)$, and the cost is $30q$, so the profit is $P = (s)(q) - 30q$.

   The graph showing the relationship between s and q is a line with slope -10 and q-intercept $8,000$, so $q = -10s + 8000$.

   This means the profit function is $P(s) = s(-10s + 8000) - 30(-10s + 8000)$.

   Factoring the common factors out gives us $P(s) = (-10s + 8000)(s - 30) = -10(s - 800)(s - 30)$.

   Or, multiplying and combining like terms gives us $P(s) = -10s^2 + 8300s - 240\,000$.

Solution Notes:

1. Some students may give the answer as $q = -10s + 8000$, which is an important part of finding the profit but is not complete. The revenue (total income from sales) is the product $(s)(q)$, and the cost is $30q$, so the profit is $P = (s)(q) - 30q$. Using the equation for q: $P = s(-10s + 8000) - 30(-10s + 8000)$. Multiplying and combining like terms gives $P = -10s^2 + 8300s - 240\,000$.
2. Students may also come up with $P = (s)(q) - 30q$. While it is true that $P = (s)(q) - 30q$, this expression has two variables. q needs to be substituted for its equivalent in terms of s so that the function P is written in terms of s.
3. The graph showing the relationship between q and s is a line with slope -10 and q-intercept 8,000, so $q = -10s + 8000$. This means $P(s) = s(-10s + 8000) - 30(-10s + 8000)$. Multiplying and simplifying gives $P(s) = -10s^2 + 8300s - 240\,000$.

Problem Set Sample Solutions

Rewrite each expression by completing the square.

1. $-2x^2 + 8x + 5$

   $-2(x^2 - 4x + 4) + 5 + 8 \rightarrow -2(x-2)^2 + 13$

2. $2.5k^2 - 7.5k + 1.25$

   $2.5(k^2 - 3k + 2.25) + 1.25 - 5.625 \rightarrow 2.5(k - 1.5)^2 - 4.375$

3. $\frac{4}{3}b^2 + 6b - 5$

   $\frac{4}{3}\left(b^2 + \frac{9}{2}b + \frac{81}{16}\right) - 5 - \frac{27}{4} \rightarrow \frac{4}{3}\left(b + \frac{9}{4}\right)^2 - \frac{47}{4}$

4. $1000c^2 - 1250c + 695$

   $1000(c^2 - 1.25c + 0.625^2) + 695 - 390.625 \rightarrow 1000(c - 0.625)^2 + 304.375$

5. $8n^2 + 2n + 5$

   $8\left(n^2 + \frac{1}{4}n + \frac{1}{64}\right) + 5 - \frac{1}{8} \rightarrow 8\left(n + \frac{1}{8}\right)^2 + 4\frac{7}{8}$

Lesson 13: Solving Quadratic Equations by Completing the Square

Student Outcomes

- Students solve complex quadratic equations, including those with a leading coefficient other than 1, by completing the square. Some solutions may be irrational. Students draw conclusions about the properties of irrational numbers, including closure for the irrational number system under various operations.

Lesson Notes

Throughout this lesson students look for and make use of the structure of expressions when completing the square. They see algebraic expressions, such as $(x-h)^2$, as single objects and the completed-square form as a single object being composed of several objects. They recognize that the graph has a minimum or maximum point from the standard form of the quadratic equation, and they can use the structure to transform it into its vertex form. Once in that form, they use the structure of the completed-square expression to identify the coordinates of the vertex.

Classwork

Opening Exercise (4 minutes)

Students review two previous concepts relevant to this one: (1) solving a quadratic equation by factoring and (2) rewriting a quadratic expression by completing the square. In the first Opening Exercise, students gather the variable terms on one side of the equation. This is preparation for the first steps in solving an equation by completing the square. The second Opening Exercise is a reminder of the process used in completing the square for a quadratic expression.

Have students work with a partner or small group to solve the following exercises.

Scaffolding:

- Remind students of the multiplicative property of zero (zero product property): If $ab=0$, then either $a=0$ and/or $b=0$.
- Common error: Students may subtract 16 outside of the parentheses to balance the $+16$ inside in Opening Exercise 2. Remind them to consider the leading coefficient $\frac{1}{2}$ when balancing the expression.

Opening Exercise

a. **Solve the equation for b: $2b^2-9b=3b^2-4b-14$.**

To solve by factoring, gather all terms to one side of the equation and combine like terms so that the remaining expression is equal to zero:

$b^2+5b-14=0$*, then factor:* $(b+7)(b-2)=0$*, and solve:* $b=-7$ *or* 2.

b. **Rewrite the expression by completing the square: $\frac{1}{2}b^2-4b+13$.**

Factor $\frac{1}{2}$ from the first two terms: $\frac{1}{2}(b^2-8b \qquad)+13$*, and then complete the square by adding $+16$ inside the parentheses (which is really $+8$ since there is $\frac{1}{2}$ outside the parentheses). Now, to compensate for the $+8$, we need to add -8 outside the parentheses and combine it with the constant term:*

$$\frac{1}{2}(b^2 - 8b + 16) + 13 - 8 = \frac{1}{2}(b - 4)^2 + 5.$$

Example 1 (8 minutes)

Students have learned how to solve equations by factoring. But what if an equation is not conducive to this method? Write the equation below on the board or screen, and ask students to work with a partner or small group to solve it.

> **Example 1**
>
> **Solve for x.**
>
> $$12 = x^2 + 6x$$

Scaffolding:

- For visual learners, show students a graph of $y = x^2 + 6x - 12$. The axis of symmetry is $x = -3$, and the roots are each a distance of $\sqrt{21}$ (or approximately 4.6) to the left and right of this axis.
- Some students may need to be instructed to write the two solutions for the square root as separate numbers so that the $\pm$ is not lost. For example, in the third step of this problem, it would be $x + 3 = +\sqrt{21}$ or $-\sqrt{21}$. Then, when they add -3 to both sides of the equation, they have two numbers.

Let students work at it for a few minutes, but they will find that this equation is not factorable over the integers.

- What strategies did you use to try to solve this equation? Where did you get stuck?
  - *We can subtract 12 from both sides, but then the product-sum method will not work because there are no pairs of factors of -12 that also add to 6.*

Students have learned to complete the square for an expression but not to use the technique for solving an equation. They may try to complete the square to solve this equation but will likely be confused about what to do with the equal sign.

Demonstrate that by completing the square, we can solve our quadratic equation a different way. Walk students through each step in the process. Start by gathering the variable terms on one side of the equation (usually the left side feels more *comfortable*) and the constant(s) on the other. Then, find the constant value that *completes* the square on the left, and balance the operation with the same on the right side of the equation.

$x^2 + 6x + 9 = 12 + 9$	Add 9 to complete the square: $\left[\frac{1}{2}(6)\right]^2$.
$(x + 3)^2 = 21$	Factor the perfect square.
$x + 3 = \pm\sqrt{21}$	Take the square root of both sides. Remind students NOT to forget the $\pm$.
$x = -3 \pm \sqrt{21}$	Add -3 to both sides to solve for x.
$x = -3 + \sqrt{21}$ or $x = -3 - \sqrt{21}$	

- Remember to put the -3 to the left of the $\pm$ square root.
- Is there a simpler way to write this answer?
  - *No, since the value under the radical is not a perfect square, it cannot be written in a simpler form or combined with the -3 by addition or subtraction. The number cannot be expressed exactly or in any simpler form.*
  - *The solutions can also be estimated using decimals, approximately 1.58 or -7.58.*

Discussion (10 minutes): Rational and Irrational Numbers

- In the last example, we said that we couldn't combine a rational number, -3, with irrational numbers, $\sqrt{21}$ and $-\sqrt{21}$. Let's take a moment to examine what happens when we apply operations to rational and irrational numbers. For each column, describe what you observe. What do you notice?

Display the chart below, giving students an opportunity to write or share their observations with a partner and then with the class as a whole.

Column A	Column B	Column C	Column D
$4 + 9.5$	$-5(9.5)$	$-4 + \pi$	$2 \times \pi$
$\frac{4}{7} + \frac{5}{3}$	$5 \times \frac{1}{2}$	$3.5 + \sqrt{11}$	$4\sqrt{11}$
$\sqrt{9} - 2$	$(\sqrt{16})(\sqrt{9})$	$\sqrt{9} - \sqrt{10}$	$\sqrt{25} \times \sqrt{13}$

- What do you notice about Column A?
  - *In Column A, all of the examples are sums or differences of rational numbers. In every case, we can name a rational number that is equal to the sum or difference. For example,* $4 + 9.5 = 13.5$.
- What do you notice about Column B?
  - *In Column B, all of the examples are products of two rational numbers. In every case, we can name a rational number that is equal to the product. For example,* $(\sqrt{16})(\sqrt{9}) = 12$.
- What do you notice about Column C?
  - *In Column C, all of the examples are sums or differences of a rational number and an irrational number. In every case, we cannot name a rational number that is equal to the sum or difference. For example, we can only write the sum* $-4 + \pi$ *as* $-4 + \pi$. *We could determine a decimal approximation, but the sum is irrational. The sums and differences are all equal to irrational numbers in this column.*
- What do you notice about Column D?
  - *In Column D, all of the examples are products of a rational number and an irrational number. In every case, we cannot name a rational number that is equal to the product. For example, we can only write the product* $2 \times \pi$ *as* 2π. *We could also determine a decimal approximation here, but the product is irrational. The products are all equal to irrational numbers in this column.*

Rational and Irrational Numbers

The *sum or product of two rational numbers* is always a *rational number*.

The *sum of a rational number and an irrational number* is always an *irrational number.*

The *product of a rational number and an irrational number* is an *irrational number* as long as the rational number is not zero.

Example 2 (8 minutes)

Give students one more example, this time with a leading coefficient not equal to 1, and ask them to solve it.

> **Example 2**
>
> **Solve for x.**
>
> $$4x^2 - 40x + 94 = 0$$

Have students explore solving by factoring. They should see that 2 is a GCF and find that $2(2x^2 - 20x + 47) = 0$ is the result. Now they may try to factor (by looking for a pair of numbers with a product of 94 and a sum of -20) the remaining quadratic factor but to no avail.

- Would it be more efficient to factor or complete the square?
  - *This equation is more conducive to completing the square because it cannot be factored using the product-sum method.*
- *Now that we have decided to complete the square, do you have any suggestions for how to start?*
  - *Gather the variable terms on the left and the constant on the right of the equation, or complete the square on the left, and then solve.*

There are two ways to approach this process: (1) by gathering all the terms to the left side of the equation and completing the square of the full expression before solving, or (2) by gathering only the variable terms on the left and the constants on the right, and then completing the square on the left before solving. Write the two starting positions on the board or screen.

$$4x^2 - 40x + 94 = 0 \text{ or } 4x^2 - 40x = -94$$

Let students struggle with these two different entry points for just a minute or two and then demonstrate BOTH strategies, allowing each step to settle before moving to the next. You might even work the steps for each side by side on the board or screen. The important thing is that students see and understand the difference in the balancing step.

Gathering everything on the left:

$$4(x^2 - 10x + \_\_\_) + 94 = 0$$
$$4(x^2 - 10x + 25) + 94 - 100 = 0$$
$$4(x - 5)^2 - 6 = 0$$
$$4(x - 5)^2 = 6$$
$$(x - 5)^2 = \frac{6}{4}$$
$$x - 5 = \pm\sqrt{\frac{6}{4}}$$
$$x = 5 \pm \frac{\sqrt{6}}{2}$$
$$x = 5 + \frac{\sqrt{6}}{2} \text{ or } 5 - \frac{\sqrt{6}}{2}$$

Gathering variable terms on the left and the constant on the right:

$$4(x^2 - 10x + \_\_\_) = -94$$
$$4(x^2 - 10x + 25) = -94 + 100$$
$$4(x - 5)^2 = 6$$
$$(x - 5)^2 = \frac{6}{4}$$
$$x - 5 = \pm\sqrt{\frac{6}{4}}$$
$$x = 5 \pm \frac{\sqrt{6}}{2}$$
$$x = 5 + \frac{\sqrt{6}}{2} \text{ or } 5 - \frac{\sqrt{6}}{2}$$

- What is the same about these two strategies for solving by completing the square?
  - *Both complete the square for $x^2 - 10x$, but one subtracts 100 and the other adds 100 to compensate for completing the square.*

Emphasize that in the method used above and on the left, we needed to add -100 to balance the $+4(25)$ because they are both on the same side of the equation. (Adding -100 on the left side is equivalent to adding $+100$ on the right side.) In the method used above and on the right, we added $+4(25)$ on the left side of the equation to complete the square; therefore, we had to also add $+100$ on the right to balance. Make sure that students see the reasoning behind this difference. Take a few minutes to make sure all students see the importance of this difference.

- Why was the fraction $\frac{6}{4}$ not reduced in this example?
  - *Sometimes it is wise to leave the fraction unreduced if one or both parts are perfect squares, and you know that you will later be taking the square root.*

Exercises (20 minutes)

Reinforce the topic by having students individually solve the following equations by completing the square. Students can check their work by comparing their solutions to a classmate's. Some of these have fraction coefficients.

Exercises

Solve each equation by completing the square.

1. $x^2 - 2x = 12$

$$x^2 - 2x + 1 = 12 + 1$$
$$(x-1)^2 = 13$$
$$x = 1 \pm \sqrt{13}$$

2. $\frac{1}{2}r^2 - 6r = 2$

$$\frac{1}{2}(r^2 - 12r + 36) = 2 + 18$$

(Be careful with factoring out the rational leading coefficient.)

$$\frac{1}{2}(r-6)^2 = 20$$
$$(r-6)^2 = 40$$
$$r - 6 = \pm\sqrt{40}$$
$$r = 6 \pm \sqrt{40} = 6 \pm 2\sqrt{10}$$

(The last step should be optional at this point.)

3. $2p^2 + 8p = 7$

$$2(p^2 + 4p + 4) = 7 + 8$$
$$2(p+2)^2 = 15$$
$$(p+2)^2 = \frac{15}{2}$$
$$(p+2) = \pm\sqrt{\frac{15}{2}}$$
$$p = -2 \pm \sqrt{\frac{15}{2}};\ -2 + \sqrt{\frac{15}{2}} \text{ or } -2 - \sqrt{\frac{15}{2}}$$

4. $2y^2 + 3y - 5 = 4$

$$2y^2 + 3y = 4 + 5$$
$$2\left[y^2 + \left(\frac{3}{2}\right)y + \frac{9}{16}\right] = 9 + \frac{9}{8}$$

(Notice the square in the numerator. It is best to leave the fraction as it is in this step since we know we will eventually be taking the square root.)

$$2\left(y + \frac{3}{4}\right)^2 = \frac{81}{8}$$
$$\left(y + \frac{3}{4}\right)^2 = \frac{81}{16}$$
$$\left(y + \frac{3}{4}\right) = \pm\sqrt{\frac{81}{16}}$$
$$y = -\frac{3}{4} \pm \frac{9}{4}$$
$$y = \frac{3}{2} \text{ or } -3$$

- Now that you know the solutions for the last exercise, could you have approached it differently?
  - *Since the solutions are rational, this one might have been easier to solve by factoring.*

Closing (1 minute)

- When a quadratic equation is not conducive to factoring, we can solve by completing the square.
- Completing the square is a good technique to use when the solutions are irrational and difficult to determine by factoring.
- If a simplified solution includes both rational and irrational components (without a perfect square under the radical), it cannot be rewritten equivalently as a single rational or irrational number. We need to learn to appreciate expressions such as $2 + 5\sqrt{3}$ and $\frac{\sqrt{2}}{7}$ as one single number on the real number line, which cannot be simplified any further.

Lesson Summary

When a quadratic equation is not conducive to factoring, we can solve by completing the square.

Completing the square can be used to find solutions that are irrational, something very difficult to do by factoring.

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 13: Solving Quadratic Equations by Completing the Square

Exit Ticket

1. Solve the following quadratic equation both by factoring and by completing the square: $\frac{1}{4}x^2 - x = 3$.

2. Which method do you prefer to solve this equation? Justify your answer using algebraic reasoning.

Exit Ticket Sample Solutions

1. **Solve the following quadratic equation both by factoring and by completing the square: $\frac{1}{4}x^2 - x = 3$.**

   Factoring—you can eliminate the fraction by multiplying both sides by 4 to obtain an equivalent expression.

   $$x^2 - 4x = 12$$
   $$x^2 - 4x - 12 = 0$$
   $$(x-6)(x+2) = 0$$
   $$x = 6 \text{ or } -2$$

   Completing the Square—you can start by either multiplying both sides by 4 or by factoring out $\frac{1}{4}$ as the GCF.

   $$\frac{1}{4}(x^2 - 4x + \_\_\_) = 3$$
   $$\frac{1}{4}(x^2 - 4x + 4) = 3 + 1$$
   $$x^2 - 4x + 4 = 16$$
   $$(x-2)^2 = 16$$
   $$x = 2 \pm 4$$
   $$x = 6 \text{ or } -2$$

2. **Which method do you prefer to solve this equation? Justify your answer using algebraic reasoning.**

   Students may have a personal preference about which way is easier; therefore, either answer is correct, but it should have mathematical reasoning to support it. They may prefer factoring since they are more familiar with it or because having a fraction as the leading coefficient can make completing the square trickier. Notice that in the factoring response, the fraction was eliminated in the first step. Remind students that this is often an option to make an equation look friendlier.

Problem Set Sample Solutions

Notice that Problem 3 has only one solution. Ask students why this happened. They should identify the point where $m = -3$ as the vertex of the graph of the original quadratic equation, making -3 a *double* solution.

Solve each equation by completing the square.

1. $p^2 - 3p = 8$

   $$p^2 - 3p + \frac{9}{4} = 8 + \frac{9}{4}$$
   $$\left(p - \frac{3}{2}\right)^2 = \frac{41}{4}$$
   $$\left(p - \frac{3}{2}\right) = \pm\sqrt{\frac{41}{4}}$$
   $$p = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$$

2. $2q^2 + 8q = 3$

$$2(q^2 + 4q + 4) = 3 + 8$$
$$2(q+2)^2 = 11$$
$$(q+2)^2 = \frac{11}{2}$$
$$(q+2) = \pm\sqrt{\frac{11}{2}}$$
$$q = -2 \pm \sqrt{\frac{11}{2}}$$

3. $\frac{1}{3}m^2 + 2m + 8 = 5$

$$\frac{1}{3}(m^2 + 6m) + 8 - 8 = 5 - 8$$
$$\frac{1}{3}(m^2 + 6m + 9) = -3 + 3$$
$$\frac{1}{3}(m+3)^2 = 0$$
$$(m+3)^2 = 0$$
$$m = -3$$

4. $-4x^2 = 24x + 11$

$$-4x^2 - 24x = 11$$ *Gather variable terms.*
$$-4(x^2 + 6x + 9) = 11 - 36$$ *Factor out -4; complete the square and balance the equality.*
$$-4(x+3)^2 = -25$$ *Factor the perfect square.*
$$(x+3)^2 = +\frac{25}{4}$$ *Divide both sides by -4.*
$$x + 3 = \pm\frac{5}{2}$$
$$x = -3 \pm \frac{5}{2} = -\frac{1}{2} \text{ or } -5\frac{1}{2}$$

Lesson 14: Deriving the Quadratic Formula

Student Outcomes

- Students derive the quadratic formula by completing the square for a general quadratic equation in standard form, $ax^2 + bx + c = 0$, and use it to verify the solutions for equations from the previous lesson for which they have already factored or completed the square.

Lesson Notes

Throughout this lesson, students use the structure of the equation to determine the best strategy for solving. If factoring is not possible, they solve by completing the square. While solving, they continue to notice the structure involved in the expressions in the steps to the solution, such as when fractional results are perfect squares, keeping in mind that the next step in the process will be to take the square root.

Classwork

Opening Exercise (5 minutes)

Students review the previous lesson: how to solve a quadratic equation by completing the square.

Opening Exercise

a. **Solve for x by completing the square: $x^2 + 2x = 8$.**

To solve by completing the square, factor out the leading coefficient, complete the square, and balance the equality.

$$(x^2 + 2x + \quad) = 8 +$$
$$(x^2 + 2x + 1) = 8 + 1$$
$$(x + 1)^2 = 9$$
$$(x + 1) = \pm\sqrt{9}$$
$$x = -1 \pm \sqrt{9}$$
$$x = -1 + 3 \text{ or } -1 - 3$$
$$x = 2 \text{ or } -4$$

Note: This equation can also be solved by factoring.

b. **Solve for p by completing the square: $7p^2 - 12p + 4 = 0$.**

To solve by completing the square, first gather the variable terms to one side of the equation and factor out the leading coefficient.

$$7p^2 - 12p = -4$$
$$7\left(p^2 - \frac{12}{7}p + \left(\frac{6}{7}\right)^2\right) = -4 + \frac{7(36)}{49}$$
$$7\left(p - \frac{6}{7}\right)^2 = -4 + \frac{36}{7}$$
$$7\left(p - \frac{6}{7}\right)^2 = \frac{8}{7}$$
$$\left(p - \frac{6}{7}\right)^2 = \frac{8}{49}$$
$$\left(p - \frac{6}{7}\right) = \pm\frac{\sqrt{8}}{7}$$
$$p = \frac{6}{7} \pm \frac{\sqrt{8}}{7} \text{ OR approximately } 1.26 \text{ or } 0.45$$

- Which of these problems makes more sense to solve by completing the square? Which makes more sense to solve by factoring? How could you tell early in the problem solving process which strategy to use?
  - *The best strategy for solving the second equation is by completing the square, while the first equation can be easily factored using product-sum. You can see this from the solutions since the first set of solutions is rational. To tell right away which of the expressions can be factored easily, test the sum and the product.*

Discussion (12 minutes)

Have students start with solving the general form of a one-variable linear equation, $ax + b = 0$, for x.

- How would you solve this equation for x: $ax + b = 0$, where a and b could be replaced with any numbers?
  - *Isolate the x-term and then divide by the leading coefficient:*
  $$ax = -b \rightarrow x = -\frac{b}{a}.$$
- Can we say that $-\frac{b}{a}$ is a *formula* for solving any equation in the form $ax + b = 0$?
  - *Yes, it will always give us the value for x, based on the values of a and b.*

As students discuss the following questions, guide them to realize that the parameters of a quadratic equation are the key to determining the best entry point for solving an equation. Then, follow up by having students work in pairs or small groups to try to solve the standard form of a one-variable quadratic equation, $ax^2 + bx + c = 0$, for x.

- What factors determine how we solve an equation and what the solutions are? What makes the first Opening Exercise different from the second?
  - *The coefficients of the quadratic and linear terms, a and b, along with the constant, c, of the quadratic equation in standard form are what make the solution unique.*

- What would happen if we tried to come up with a way to use just the values of a, b, and c to solve a quadratic equation? Can we solve the general quadratic equation $ax^2 + bx + c = 0$? Is this even possible? Which method would make more sense to use, factoring or completing the square?
- Encourage students to play around with this idea for a few minutes. Point them in the right direction by asking questions such as the following: Can we factor if we do not know the coefficients—using a, b, and c, rather than numbers? Can we complete the square using only a, b, and c?

Ultimately, show students how to derive the quadratic formula by completing the square and have them record this in the space provided in the student handout.

> **Discussion**
>
> **Solve $ax^2 + bx + c = 0$.**

$ax^2 + bx + c = 0$	The steps:
$a\left(x^2 + \frac{b}{a}x + \_\_\right) = -c$	Gather the variable terms, and factor out the leading coefficient.
$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) = -c + a\left(\frac{b}{2a}\right)^2$	Complete the square inside the parentheses, and balance the equality.
$\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) = -\frac{c}{a} + \left(\frac{b^2}{4a^2}\right)$	Now, simplify the fraction on the right and multiply both sides by $\frac{1}{a}$. Remember that $a \neq 0$ since that would make the original equation not quadratic.
$\left(x + \left(\frac{b}{2a}\right)\right)^2 = \frac{-4a(c)}{4a(a)} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$	Factor the perfect square on the left and clean up the right. Be careful. You need to combine the two fractions on the right by finding the common denominator. Use the commutative property to reverse the two fractions on the right.
$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Now, combine the two right-side fractions, and take the square root of both sides. Notice that the denominator is a perfect square, and do not forget the $\pm$.
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Add $\frac{-b}{2a}$ to both sides to finally solve for x. The final step is combining the two fractions, which have the same denominator since we took the square root of $4a^2$.

Then discuss by asking the following:

- How can we verify that this formula is correct?

Encourage students to review problems from the Opening Exercises and see if they get the same answers using the quadratic formula that they got by completing the square for the standard form of a quadratic equation.

Make sure students put the equation in standard form before solving; otherwise, the quadratic formula will not work.

- Now try solving the equations in the Opening Exercises using the quadratic formula. Check your answers to make sure they are correct.
  - $x^2 + 2x = 8 \to x^2 + 2x - 8 = 0 \to a = 1, b = 2, c = -8$ *[The negative is important here.]*

    $$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)} = \frac{-2 \pm \sqrt{4+32}}{2} = \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2} = 2 \text{ or } -4$$

    Now check: Substitute 2 *and* -4 *into the original equation. Is* $(2)^2 + 2(2) = 8? \to 4 + 4 = 8.$

    Is $(-4)^2 + 2(-4) = 8? \to 16 - 8 = 8.$ *Or, look back at the first example to compare answers.*
  - $7p^2 - 12p + 4 = 0 \to a = 7, b = -12, c = 4$

    $$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(7)(4)}}{2(7)} = \frac{12 \pm \sqrt{144 - 112}}{14} = \frac{12 \pm \sqrt{32}}{14} = \frac{12 \pm 4\sqrt{2}}{14} = \frac{6 \pm 2\sqrt{2}}{7},$$

    which is approximately 1.26 *or* 0.45.
- Have students check whether this matches the answers from the Opening Exercises.
  - *Yes*

Note: The simplification of the square root would not be required to get the same decimal approximations.

- Now, take a minute and have your students look closely at the quadratic formula. Point out that the whole expression can be split into two separate expressions as follows:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Notice that the first part of the expression (in red) represents the axis of symmetry (the x-coordinate for the vertex). Now we step to the right and the left by the amount represented in blue to find the x-intercepts (i.e., zeros or roots). While it is never a good idea to offer memorization tricks as problem solving strategies, do not be shy about using $\frac{-b}{2a}$, as it is presented here to quickly find the axis of symmetry and the vertex for a quadratic function in standard form.

Exercises 1–4 (12 minutes)

Have students work independently and check against their work from the previous lesson. If time is short, you might select two of these exercises and move on to Exercise 5. Some of these are more challenging than others. If you shorten this set, take the needs of your students into account.

After verifying that the Opening Exercises can be solved using the quadratic formula, return to the in-class exercises for Lesson 13, and solve them using the quadratic formula. Check to make sure your answers are the same.

Exercises 1–4

Use the quadratic formula to solve each equation.

1. $x^2 - 2x = 12 \to a = 1, b = -2, c = -12$ **[Watch the negatives.]**

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-12)}}{2(1)} = \frac{2 \pm \sqrt{52}}{2} = \frac{2 \pm \sqrt{4(13)}}{2} = \frac{2 \pm 2\sqrt{13}}{2} = 1 \pm \sqrt{13}$$

2. $\frac{1}{2}r^2 - 6r = 2 \rightarrow a = \frac{1}{2}, b = -6, c = -2$ **[Did you remember the negative?]**

$$r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\left(\frac{1}{2}\right)(-2)}}{2\left(\frac{1}{2}\right)} = \frac{6 \pm \sqrt{36+4}}{1} = 6 \pm \sqrt{4(10)} = 6 \pm 2\sqrt{10}$$

3. $2p^2 + 8p = 7 \rightarrow a = 2, b = 8, c = -7$

$$p = \frac{-8 \pm \sqrt{8^2 - 4(2)(-7)}}{2(2)} = \frac{-8 \pm \sqrt{64+56}}{4} = \frac{-8 \pm \sqrt{4(30)}}{4} = \frac{-8 \pm 2\sqrt{30}}{4} = \frac{-4 \pm \sqrt{30}}{2}$$

Note: In the Lesson 13 problem, the radical in the final answer was $\sqrt{\frac{15}{2}}$*, which is equivalent to* $\frac{\sqrt{30}}{2}$*.*

4. $2y^2 + 3y - 5 = 4 \rightarrow a = 2, b = 3, c = -9$

$$y = \frac{-3 \pm \sqrt{3^2 - 4(2)(-9)}}{2(2)} = \frac{-3 \pm \sqrt{9+72}}{4} = \frac{-3 \pm \sqrt{81}}{4} = \frac{-3 \pm 9}{4} = \frac{3}{2} \text{ or } -3$$

- Based on the original equation, which of the problems in the exercises would be best solved using the quadratic formula?
  - *Exercises 1–3 would need to be solved by using either the quadratic formula or by completing the square. Exercise 4 could be factored.*

Exercise 5 (7 minutes)

Have students work with a partner or in small groups to solve the following equations, using a different method for each: solve by factoring, solve by completing the square, and solve using the quadratic formula. Before they begin, ask them to consider the method they will use.

Exercise 5

Solve these quadratic equations, using a different method for each: solve by factoring, solve by completing the square, and solve using the quadratic formula. Before starting, indicate which method you will use for each.

Method *by factoring* — $2x^2 + 5x - 3 = 0$

Method *by quadratic formula* — $x^2 + 3x - 5 = 0$

Method *by completing the square* — $\frac{1}{2}x^2 - x - 4 = 0$

(Students may choose any method to solve. However, the first and third are factorable, but the middle expression is not.)

$(2x-1)(x+3) = 0$

$x = \frac{1}{2}$ *or* -3

$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-5)}}{2(1)}$

$x = \frac{-3 \pm \sqrt{29}}{2}$

$x = \frac{-3 + \sqrt{29}}{2}$ *or* $\frac{-3 - \sqrt{29}}{2}$

$x \approx 1.19$ *or* -4.19

$\frac{1}{2}(x^2 - 2x \quad) = 4$

$\frac{1}{2}(x^2 - 2x + 1) = 4 + \frac{1}{2}$

$\frac{1}{2}(x-1)^2 = 4.5$

$(x-1)^2 = 9$

$(x-1) = \pm 3$

$x = 1 \pm 3$

$x = 4$ *or* -2

Closing (4 minutes)

- When is completing the square the most efficient method to use for solving a quadratic equation?
  - *When it is not possible (or it is very difficult) to factor the quadratic expression, and when the leading coefficient and linear term coefficient are easy to deal with (i.e., when the leading coefficient is easily factored out resulting in an even linear term coefficient).*
- When is the quadratic formula best?
  - *When it is not possible (or it is very difficult) to factor the quadratic expression, and when the leading coefficient and/or linear term coefficient are fractions that are not easily eliminated.*
- When is factoring the most efficient method to use for solving a quadratic equation?
  - *When the factors of the equation are obvious or fairly easy to find. When factoring out the GCF or eliminating any fractional coefficients is possible.*
- Is using the quadratic formula really just completing the square? Why or why not?
  - *Yes, using the quadratic formula is really solving by completing the square because the formula is derived by completing the square for a quadratic equation in standard form. When we use the formula, we are substituting values into the expression derived by completing the square.*

Lesson Summary

The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is derived by completing the square on the general form of a quadratic equation: $ax^2 + bx + c = 0$, where $a \neq 0$. The formula can be used to solve any quadratic equation and is especially useful for those that are not easily solved by using any other method (i.e., by factoring or completing the square).

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 14: Deriving the Quadratic Formula

Exit Ticket

Solve for R using any method. Show your work.

$$\frac{3}{2}R^2 - 2R = 2$$

Exit Ticket Sample Solutions

Solve for R using any method. Show your work.

$$\frac{3}{2}R^2 - 2R = 2$$

Let's start each method by multiplying both sides of the equation by 2 to eliminate the fraction.

$$3R^2 - 4R = 4$$

By factoring:	*By completing the square:*	*By using the quadratic formula:*
$3R^2 - 4R - 4 = 0$ $(R-2)(3R+2) = 0$ $R = 2 \text{ or } -\frac{2}{3}$	$3\left(R^2 - \frac{4}{3}R \quad\right) = 4$ $3\left(R^2 - \frac{4}{3}R + \left(-\frac{2}{3}\right)^2\right) = 4 + \frac{2^2}{3}$ $R^2 - \frac{4}{3}R + \left(-\frac{2}{3}\right)^2 = \frac{12}{9} + \frac{4}{9}$ $\left(R - \frac{2}{3}\right)^2 = \frac{16}{9}$ $R - \frac{2}{3} = \pm\sqrt{\frac{16}{9}}$ $R = \frac{2}{3} \pm \frac{4}{3}$ $R = 2 \text{ or } -\frac{2}{3}$	$3R^2 - 4R - 4 = 0$ $a = 3, b = -4, \text{and } c = -4$ $R = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-4)}}{2(3)}$ $R = \frac{4 \pm \sqrt{16+48}}{6}$ $R = \frac{4 \pm \sqrt{64}}{6} = \frac{4 \pm 8}{6}$ $R = 2 \text{ or } -\frac{2}{3}$

Problem Set Sample Solutions

Use the quadratic formula to solve each equation.

1. Solve for z: $z^2 - 3z - 8 = 0$.

   $a = 1, b = -3, c = -8$

   $$z = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-8)}}{2(1)} = \frac{3 \pm \sqrt{41}}{2}$$

2. Solve for q: $2q^2 - 8 = 3q$.

   $a = 2, b = -3, c = -8$

   $$q = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-8)}}{2(2)} = \frac{3 \pm \sqrt{73}}{4}$$

3. Solve for m: $\frac{1}{3}m^2 + 2m + 8 = 5$.

   $a = \frac{1}{3}, b = 2, c = 3$

   $$m = \frac{-2 \pm \sqrt{2^2 - 4\left(\frac{1}{3}\right)(3)}}{2\left(\frac{1}{3}\right)} = \frac{-2 \pm \sqrt{0}}{\frac{2}{3}} = -3$$

Lesson 15: Using the Quadratic Formula

Student Outcomes

- Students use the quadratic formula to solve quadratic equations that cannot be easily factored.
- Students understand that the discriminant, $b^2 - 4ac$, can be used to determine whether a quadratic equation has one, two, or no real solutions.

Lesson Notes

In this lesson, students continue to use the quadratic formula strategically (i.e., when other, simpler methods of solution are impossible or too difficult). Students identify the discriminant as $b^2 - 4ac$ and use it to determine the number and nature of the solutions. They understand that the sign of the discriminant can be used to determine the number of real solutions a quadratic equation has, which are defined as follows: a positive discriminant yields two real solutions, a negative discriminant yields no real solutions, and a discriminant equal to zero yields only one real solution. In addition, a discriminant that is a perfect square indicates the solutions are rational; therefore, the quadratic expression is factorable over the integers. Finally, students return to the concept of using the x-intercepts of the graph of a quadratic to write a formula for the corresponding quadratic function which was explored in Lesson 9 of Topic A.

Classwork

Opening Exercise (4 minutes)

Students review how to solve quadratic equations using the quadratic formula from the previous lesson.

Opening Exercise

Solve the following:

a. $4x^2 + 5x + 3 = 2x^2 - 3x$

Students should recognize that this is a difficult quadratic equation to solve. Accordingly, they should set it equal to zero and solve it using the quadratic formula:

$$2x^2 + 8x + 3 = 0$$

$$x = \frac{-8 \pm \sqrt{8^2 - 4(2)(3)}}{2(2)}$$

$$x = \frac{-8 \pm \sqrt{40}}{4}$$

$$x = \frac{-8 \pm \sqrt{4(10)}}{4}$$

$$x = \frac{-8 \pm 2\sqrt{10}}{4}$$

$$x = \frac{-4 \pm \sqrt{10}}{2}$$

$$x = -2 \pm \frac{\sqrt{10}}{2}$$

Scaffolding:

- Discuss the definition of mathematical efficiency.
- Encourage all students to look for ways to make finding the solution for an equation more efficient (e.g., by factoring out the GCF or setting the equation equal to zero).

Check by substituting the approximated decimal value(s) into the original equation or by completing the square:

$$2(x^2+4x)=-3 \rightarrow 2(x^2+4x+4)=-3+8 \rightarrow 2(x+2)^2=5$$
$$(x+2)^2=\frac{5}{2} \rightarrow x+2=\pm\sqrt{\frac{5}{2}} \rightarrow x=-2\pm\sqrt{\frac{5}{2}}$$

(Have students use their calculators to check that these are the same decimal values as the previous solutions.)

b. $c^2-14=5c$

Initially, students may approach this problem by using the quadratic formula. While this approach works, encourage students to look for a more efficient pathway to the solution (in this case, to solve by factoring):

$$c^2-5c-14=0 \rightarrow (c-7)(c+2)=0 \rightarrow c=7 \text{ or } -2$$

Checks:

$$7^2-14=5(7) \rightarrow 49-14=35 \rightarrow 35=35$$
$$(-2)^2-14=5(-2) \rightarrow 4-14=-10 \rightarrow -10=-10$$

Discussion (3 minutes)

Before moving on, review the Opening Exercises.

- What are the differences between these two quadratic equations? Is one easier to solve than the other?
  - *Some students will have used the quadratic formula for both; some may have observed the second exercise to be more efficiently solved by factoring. Both need to be manipulated before a decision can be made.*
- Is one pathway to solution *more correct* than another?
  - *Ultimately, students should be aware that while there are many ways to arrive at a correct solution, some methods are more efficient than others.*

Exercises (10 minutes)

Have students solve the following three standard form quadratic equations independently using the quadratic formula. Ask them to watch for special circumstances in each.

Exercises

Solve Exercises 1–5 using the quadratic formula.

1. $x^2-2x+1=0$

   $a=1, b=-2, c=1$

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(1)}}{2(1)}=\frac{2\pm\sqrt{0}}{2}=1$$

This exercise factors easily as a perfect square, $(x-1)^2=0$, with a solution of $x=1$. However, notice that when we use the quadratic formula, we can see that the number under the radical sign equals zero. This translates as one real solution which is also rational. We also sometimes call this a double solution (or a double root) since the factors of the perfect square give the equation $(x-1)(x-1)=0$, for which we find that there are two identical solutions, both of which are 1.

2. $3b^2 + 4b + 8 = 0$

$a = 3, b = 4, c = 8$

$b = \frac{-4 \pm \sqrt{4^2 - 4(3)(8)}}{2(3)} = \frac{-4 \pm \sqrt{-80}}{6} = ???$

This exercise may be confusing. Point out that substituting into the quadratic formula yields a negative square root, which is not a real number. Have students try finding $\sqrt{-80}$ on their calculators. They should get an error message or a message that says non-real answer. (Note that some calculators will produce an answer in terms of i when in *complex mode*.) Explain that there is no *real* number whose square is a negative number. So, when we find that the number under the radical is negative, we say that the equation has *no real solutions*.

3. $2t^2 + 7t - 4 = 0$

$a = 2, b = 7, c = -4$

$t = \frac{-7 \pm \sqrt{7^2 - 4(2)(-4)}}{2(2)} = \frac{-7 \pm \sqrt{49 + 32}}{4} = \frac{-7 \pm \sqrt{81}}{4} = \frac{-7 \pm 9}{4} = \frac{1}{2} \text{ or } -4$

Point out that this time, the value under the radical, 81, is a perfect square. This translates into *two rational* solutions.

4. $q^2 - 2q - 1 = 0$

$a = 1, b = -2, c = -1$

$q = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

5. $m^2 - 4 = 3$

$a = 1, b = 0, c = -7$

$m = \frac{0 \pm \sqrt{0^2 - 4(-7)}}{2(1)} = \frac{0 \pm \sqrt{28}}{2} = \frac{0 \pm 2\sqrt{7}}{2} = 0 \pm \sqrt{7} = \pm\sqrt{7}$

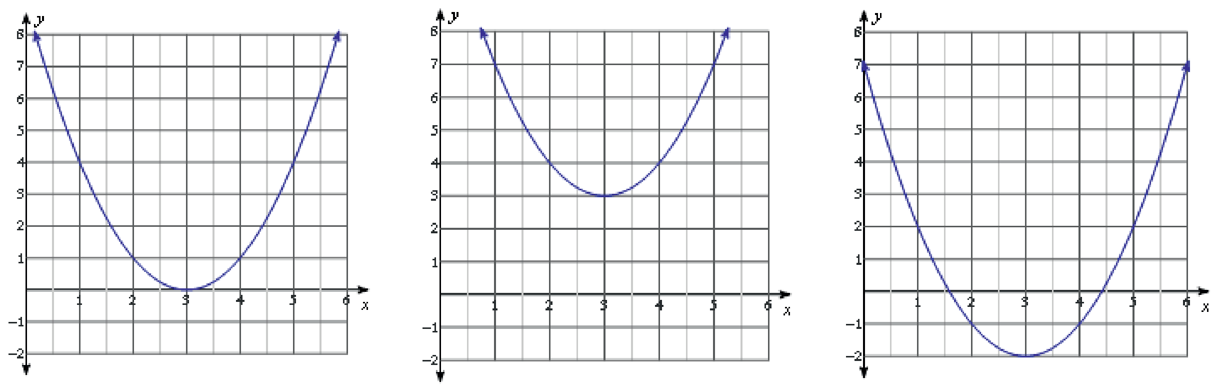
Point out that these last two times, the values under the radicals were not perfect squares but were greater than zero. This translated into *two irrational* solutions.

Discussion (4 minutes)

- Describe the solutions of these quadratic equations in your own words. This is where you can delve into what makes these equations different.
  - *We usually get two solutions, but the first equation has only one solution. We couldn't solve the second problem because we get a negative under the radical sign. The third solution has a perfect square under the radical so that the square root is eliminated from the answer. The last two are not perfect squares under the radical but do have a positive value; therefore, both answers are irrational.*

The expression under the radical is called the *discriminant*: $b^2 - 4ac$. The value of the discriminant determines the number and nature of the solutions for a quadratic equation. As we saw in the Opening Exercises, when the discriminant is positive, then we have $\pm\sqrt{(\text{positive number})}$, which yields two real solutions (two rational solutions if the value is a perfect square). When the discriminant equals zero, as it did in Example 1, then we have $\pm\sqrt{0}$, which yields only one solution, $\frac{-b}{2a}$. When the discriminant is a negative number, then we have $\pm\sqrt{(\text{negative number})}$, which can never lead to a real solution.

Show students the following graphs of quadratic equations. Project or sketch them on the board or screen.



- What are the differences among these three graphs? Which of these graphs belongs to a quadratic equation with a positive discriminant? Which belongs to a quadratic equation with a negative discriminant? Which graph has a discriminant equal to zero?
  - *The first graph touches the x-axis exactly once, corresponding to one real root and a discriminant equal to zero. The second graph lies entirely above the x-axis, so it has no real roots; therefore, its discriminant must be negative. The third graph intersects the x-axis in two points, so it has two real roots; therefore, its discriminant is positive.*

Note that the third graph *looks* like it may have rational solutions since it appears that the intercepts are 1.5 and 4.5. Remind students that graphs are excellent for estimating, but algebra is used to find the exact solutions.

Exercises 6–10 (7 minutes)

For Exercises 6–9, determine the number of real solutions for each quadratic equation without solving.

6. $p^2 + 7p + 33 = 8 - 3p$

$a = 1, b = 10, c = 25 \rightarrow 10^2 - 4(1)(25) = 0 \rightarrow$ ***one real solution***

7. $7x^2 + 2x + 5 = 0$

$a = 7, b = 2, c = 5 \rightarrow 2^2 - 4(7)(5) = -136 \rightarrow$ ***no real solutions***

8. $2y^2 + 10y = y^2 + 4y - 3$

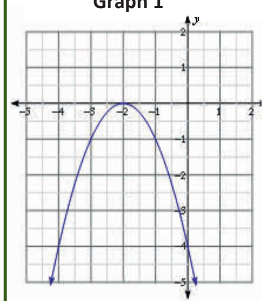
$a = 1, b = 6, c = 3 \rightarrow 6^2 - 4(1)(3) = 24 \rightarrow$ *two real solutions*

9. $4z^2 + 9 = -4z$

$a = 4, b = 4, c = 9 \rightarrow 4^2 - 4(4)(9) = -128 \rightarrow$ *no real solutions*

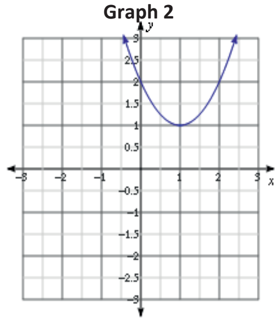
10. **On the line below each graph, state whether the discriminant of each quadratic equation is positive, negative, or equal to zero. Then, identify which graph matches the discriminants below.**

Graph 1



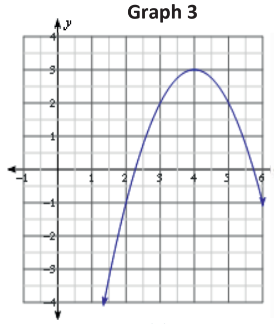
Equal to zero

Graph 2



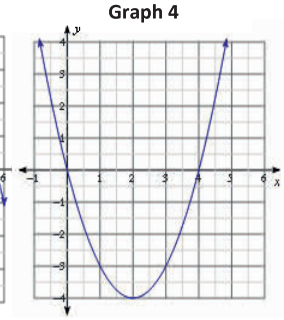
Negative

Graph 3



Positive

Graph 4



Positive (perfect square)

Discriminant A: $(-2)^2 - 4(1)(2)$
Graph: 2

Discriminant B: $(-4)^2 - 4(-1)(-4)$
Graph: 1

Discriminant C: $(-4)^2 - 4(1)(0)$
Graph: 4

Discriminant D: $8^2 - 4(-1)(-13)$
Graph: 3

Discussion (3 minutes)

Before students work on Exercise 11, have the following discussion.

- If a quadratic function is written in factored form, $f(x) = a(x - m)(x - n)$, what key feature of the graph is easily recognizable?
  - *The x-intercepts, $x = m$ and $x = n$. (Students may also note that the direction of opening can be seen by looking at the value of a.)*
- Is the converse of this true? In other words, if we are given the x-intercepts, $x = m$ and $x = n$, of the graph of a quadratic function, can we write a formula for the function?
  - *Students will likely say yes.*

Students did exercises like this in Lesson 9, but we haven't proven that this holds true for all quadratic functions. Give students time to work on Exercise 11, guiding them through parts (b) and (c) as needed. Students should see that even when the x-intercepts are irrational numbers, they can still be used to write the factored form of the corresponding quadratic function. In Algebra II, students extend this idea to polynomials of any degree.

Exercise 11 (7 minutes)

11. Consider the quadratic function $f(x) = x^2 - 2x - 4$.

a. Use the quadratic formula to find the x-intercepts of the graph of the function.

$x = 1 + \sqrt{5}$ and $x = 1 - \sqrt{5}$

b. Use the x-intercepts to write the quadratic function in factored form.

$f(x) = \left(x - (1 + \sqrt{5})\right)\left(x - (1 - \sqrt{5})\right)$

c. Show that the function from part (b) written in factored form is equivalent to the original function.

$f(x) = x^2 - (1 + \sqrt{5})x - (1 - \sqrt{5})x + (1 + \sqrt{5})(1 - \sqrt{5})$

$f(x) = x^2 - x - \sqrt{5}x - x + \sqrt{5}x + 1 - 5$

$f(x) = x^2 - 2x - 4$

This exercise still does not exactly prove that this holds true for all quadratic functions, regardless of the types of solutions. If time permits, work the extension as a class.

Extension: Consider the quadratic equation $ax^2 + bx + c = 0$.

a. Write the equation in factored form, $a(x - m)(x - n) = 0$, where m and n are the solutions to the equation.

The solutions to the equation $ax^2 + bx + c = 0$ are $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$. So in factored form the equation would be

$$a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = 0$$

b. Show that the equation from part (a) is equivalent to the original equation.

$$\begin{aligned} a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) &= a\left(x^2 - \frac{-b + \sqrt{b^2 - 4ac}}{2a}x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}x + \frac{b^2 - (b^2 - 4ac)}{4a^2}\right) \\ &= a(x^2 + \left(\frac{b - \sqrt{b^2 - 4ac}}{2a} + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)x + \frac{c}{a} \\ &= a\left(x^2 + \frac{2b}{2a}x + \frac{c}{a}\right) \\ &= ax^2 + bx + c \end{aligned}$$

Closing (2 minutes)

- The quadratic formula can be used to solve any quadratic equation in standard form.
- The discriminant is the part of the quadratic formula that is under the radical. It can be used to determine the nature and number of solutions for a quadratic equation and whether the quadratic expression can be factored over the integers.

Lesson Summary

You can use the sign of the discriminant, $b^2 - 4ac$, to determine the number of real solutions to a quadratic equation in the form $ax^2 + bx + c = 0$, where $a \neq 0$. If the equation has a positive discriminant, there are two real solutions. A negative discriminant yields no real solutions, and a discriminant equal to zero yields only one real solution.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 15: Using the Quadratic Formula

Exit Ticket

1. Solve the following equation using the quadratic formula: $3x^2 + 6x + 8 = 6$.

2. Is the quadratic formula the most efficient way to solve this equation? Why or why not?

3. How many zeros does the function $f(x) = 3x^2 + 6x + 2$ have? Explain how you know.

Exit Ticket Sample Solutions

1. **Solve the following equation using the quadratic formula: $3x^2 + 6x + 8 = 6$.**

$$3x^2 + 6x + 2 = 0 \rightarrow$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{-6 \pm \sqrt{36 - 24}}{6}$$

$$x = \frac{-6 \pm \sqrt{12}}{6}$$

$$x = \frac{-6 \pm \sqrt{4(3)}}{6}$$

$$x = \frac{-6 \pm 2\sqrt{3}}{6}$$

$$x = \frac{-3 \pm \sqrt{3}}{3} \text{ or } -1 \pm \frac{\sqrt{3}}{3}$$

2. **Is the quadratic formula the most efficient way to solve this equation? Why or why not?**

   This is a personal preference. Some may consider the quadratic formula to be more efficient, while others may prefer completing the square. After the leading coefficient is factored out, the linear term coefficient is still even, making this a good candidate for completing the square.

3. **How many zeros does the function $f(x) = 3x^2 + 6x + 2$ have? Explain how you know.**

   Since the discriminant of the original equation is positive, 12, and yields two real solutions, the function must have two zeros. OR After solving the equation $3x^2 + 6x + 2 = 0$, I found that there were two irrational solutions. This means that the corresponding function has two zeros.

Problem Set Sample Solutions

The Problem Set is identical in scope and style to the exercise set from class. Students are *not* being asked to solve the quadratic equations in each question, only to use the discriminant to find the number of roots or to use the number of roots to discuss the value of the discriminant.

Without solving, determine the number of real solutions for each quadratic equation.

1. $b^2 - 4b + 3 = 0$

   $a = 1, b = -4, c = 3 \rightarrow (-4)^2 - 4(1)(3) = 4 \rightarrow$ *two real solutions*

2. $2n^2 + 7 = -4n + 5$

   $a = 2, b = 4, c = 2 \rightarrow (4)^2 - 4(2)(2) = 0 \rightarrow$ *one real solution*

3. $x - 3x^2 = 5 + 2x - x^2$

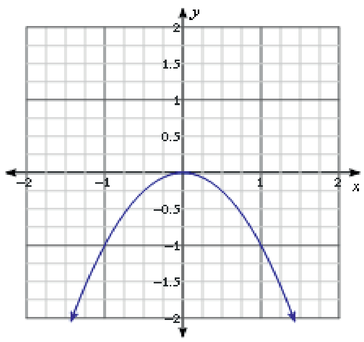
   $a = -2, b = -1, c = -5 \rightarrow (-1)^2 - 4(-2)(-5) = -39 \rightarrow$ *no real solutions*

4. $4q + 7 = q^2 - 5q + 1$

$a = -1, b = 9, c = 6 \rightarrow (9)^2 - 4(-1)(6) = 105 \rightarrow$ *two real solutions*

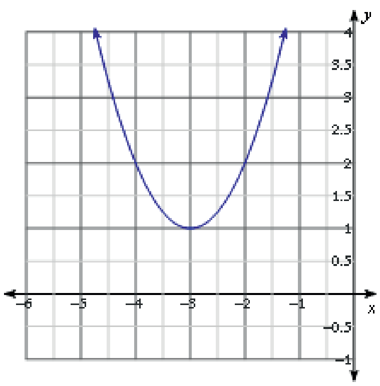
Based on the graph of each quadratic function, $y = f(x)$, determine the number of real solutions for each corresponding quadratic equation, $f(x) = 0$.

5.



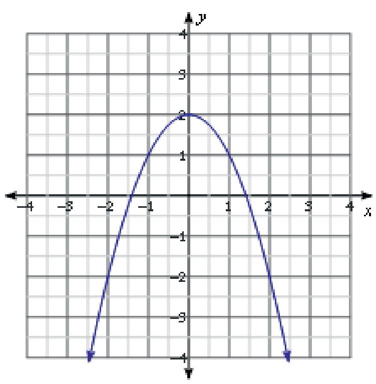
One real solution

6.



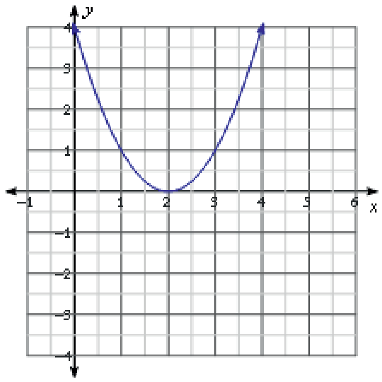
No real solutions

7.



Two real solutions

8.



One real solution

9. Consider the quadratic function $f(x) = x^2 - 7$.

a. Find the x-intercepts of the graph of the function.

$x = \pm\sqrt{7}$

b. Use the x-intercepts to write the quadratic function in factored form.

$f(x) = (x - \sqrt{7})(x + \sqrt{7})$

c. Show that the function from part (b) written in factored form is equivalent to the original function.

$f(x) = (x - \sqrt{7})(x + \sqrt{7}) = x^2 + \sqrt{7}x - \sqrt{7}x - 7 = x^2 - 7$

10. Consider the quadratic function $f(x) = -2x^2 + x + 5$.

a. Find the x-intercepts of the graph of the function.

$x = \dfrac{1 \pm \sqrt{41}}{4}$

b. Use the x-intercepts to write the quadratic function in factored form.

$f(x) = -2\left(x - \dfrac{1+\sqrt{41}}{4}\right)\left(x - \dfrac{1-\sqrt{41}}{4}\right)$

c. Show that the function from part (b) written in factored form is equivalent to the original function.

$f(x) = -2\left(x - \dfrac{1+\sqrt{41}}{4}\right)\left(x - \dfrac{1-\sqrt{41}}{4}\right) = -2\left(x^2 - \dfrac{1+\sqrt{41}}{4}x - \dfrac{1-\sqrt{41}}{4}x - \dfrac{5}{2}\right) = -2\left(x^2 - \dfrac{1}{2}x - \dfrac{5}{2}\right)$
$= -2x^2 + x + 5$

Lesson 16: Graphing Quadratic Equations from the Vertex Form, $y = a(x-h)^2 + k$

Student Outcomes

- Students graph simple quadratic equations of the form $y = a(x-h)^2 + k$ (completed-square or vertex form), recognizing that (h, k) represents the vertex of the graph and use a graph to construct a quadratic equation in vertex form.
- Students understand the relationship between the leading coefficient of a quadratic function and its concavity and slope and recognize that an infinite number of quadratic functions share the same vertex.

Lesson Notes

The Opening Exercise is set up so students get three different quadratic equations, and they are asked to analyze the similarities and differences in the structure of their equations and the corresponding graphs, which leads to a generalization about the graphs of quadratic equations of the form $y = (x-h)^2$.

Throughout this lesson, we use $y =$ notation when we are talking about the equation that represents the function and $f(x) =$ when talking about the function itself. It may be important to review the reason for using both notations and the difference between them. Students will need a graphing calculator and graph paper for this lesson.

Scaffolding:

- Recall: What does the graph of a quadratic equation look like?
- Remind students that when calculating squares in their calculator, they need to watch out for a common error: $(-1)^2 \neq -1^2$.

Classwork

Opening Exercise (5 minutes)

> **Opening Exercise**
>
> **Graph the equations $y = x^2$, $y = (x-2)^2$, and $y = (x+2)^2$ on the interval $-3 \le x \le 3$.**

Have students graph the equations $y = x^2$, $y = (x-2)^2$, and $y = (x+2)^2$ on the interval $-3 \le x \le 3$. Consider having students graph the equations using the graphing calculator or graph paper.

- How are the graphs of these equations similar? How are they different?
  - *The graphs look similar in that they have the same shape. Point out that the graphs are all translations of each other. They have different vertices: $y = x^2$ has its vertex at $(0, 0)$, while $y = (x-2)^2$ has its vertex at $(2, 0)$, and $y = (x+2)^2$ has its vertex at $(-2, 0)$.*

Scaffolding:

- For students who quickly grasp the horizontal shift implied in the Opening Exercise: We already know how to move the vertex left and right. How might we move it up and down?
- The function $f(x) = x^2$ is called the *parent function* for all quadratic functions and their graphs.

- Now consider the graph of $y = (x-5)^2$. Where would you expect this graph to be in relation to the other three?
  - *The graph of $y = (x-5)^2$ is 5 units to the right of the graph of $y = x^2$, 3 units to the right of $y = (x-2)^2$, and 7 units to the right of $y = (x+2)^2$.*

Discussion (10 minutes)

Based on the Opening Exercise and the lessons in Module 3 (e.g., see horizontal translations in Module 3, Lesson 18), students should be able to reason that replacing x with $(x+N)$ moves the vertex N units to the right (for a negative N) or the left (for a positive N) on the x-axis. Push further. Allow students to discuss the following questions with their partner or small group before taking suggested answers from the class:

- Why do you think the graph moves to the *right* when we *subtract* a positive number from x inside the parentheses and to the *left* when we *add*?
  - *As an example, let's start with a quadratic function with vertex location $(0, k)$, giving us the equation: $y = x^2 + k$. After a horizontal translation, $x \rightarrow (x-h)$, the height of the vertex should remain the same, namely $y = k$. That means $y = (x-h)^2 + k = k$ at the vertex. We are curious about where the new vertex is horizontally; that is, what x-value will make the previous equation true. This implies $(x-h)^2 = 0$; therefore, $x = h$. That means, after the translation $x \rightarrow (x-h)$, the vertex $(0, k)$ is translated to (h, k), and the whole graph is translated h units to the right. Since h is positive, the graph shifts to the right; if h were negative, $(x-h)$ would read as x plus a positive number, and the graph shifts to the left.*
- Your teacher is 6 units tall and standing at the position $x = -2$ on a horizontal axis. Is it possible to find a quadratic equation that looks just like $y = x^2$ but that sits directly on top of your teacher's head?

  With a partner, take five minutes and experiment to see if you can find the quadratic equation to represent this situation. Use what we have already learned in earlier lessons and modules to help you get started. Construct tables and draw graphs to verify your results. Remember you performed vertical translations in Module 3, Lesson 17.
  - *Students should be able to discover that, in addition to moving the graph left or right by adding or subtracting within the parentheses (adjusting the horizontal position), they can move up and down by adding or subtracting outside the parentheses (adjusting the vertical position).*

> *Scaffolding:*
> - Visual learners will benefit from experimenting with their graphing calculators to determine the effect of changing the values of h and k in equations of the form, $y = a(x-h)^2 + k$.
> - For now, let the leading coefficient equal 1.

In the activity above, students model the situation using tables and graphs. Then, they conclude that the graphs of the equations can move up or down by adding or subtracting a constant outside the parentheses.

Exercises (8 minutes)

Exercises

1. **Without graphing, state the vertex for each of the following quadratic equations.**
   a. $y = (x-5)^2 + 3$

      $(5, 3)$

b. $y = x^2 - 2.5$

$(0, -2.5)$

c. $y = (x + 4)^2$

$(-4, 0)$

2. **Write a quadratic equation whose graph will have the given vertex.**

a. $(1.9, -4)$

$y = (x - 1.9)^2 - 4$

b. $(0, 100)$

$y = x^2 + 100$

c. $\left(-2, \frac{3}{2}\right)$

$y = (x + 2)^2 + \frac{3}{2}$

Scaffolding:

If students need more examples to reinforce this concept, have them compare the following graphs either on their graphing calculator or graph paper:

$$y = \frac{1}{2}(x - 1)^2;$$
$$y = 2(x - 1)^2;$$
$$y = -2(x - 1)^2.$$

Ask them to comment on how the three graphs are similar and how they are different. Hopefully, they will notice that all three have the same vertex, but some are stretched or shrunk vertically, and some open down rather than up.

Discussion (6 minutes)

Review the problems above, and when discussing solutions for Exercise 2, ask the following:

- Are these the *only* quadratic equations with graphs with these vertices? Is there another way to write two equations that have the same vertex but are different?

Using the equation from Exercise 2(b), ask students to experiment with ways to change the graph without changing the vertex. Encourage them to write equations, evaluate them using a table, and graph the results.

Students come to the conclusion that if we multiply the term $(x - h)^2$ by some other number, we keep the vertex the same, but the graph experiences a vertical stretch or shrink, or in the case of a negative coefficient, the direction that the graph opens is reversed.

Verify that this is true by applying the same rule to the equation from Exercise 2(c).

- Can we generalize and discuss the effect the leading coefficient, a, has on the graph of $f(x) = a(x - h)^2 + k$? Compared to the graph when $a = 1$:
  - *The graph is shrunk vertically when* $-1 < a < 1$.
  - *The graph is stretched vertically when* $a < -1$ *or* $a > 1$.
  - *The graph opens up when* a *is positive.*
  - *The graph opens down when* a *is negative.*

Exploratory Challenge (8 minutes)

This application problem about rectangular area connects vertex form to a real-world context. Consider asking students to create the graph of the function in part (b) to connect the problem's solution explicitly to the graph of a quadratic function. To aid in comprehension of the problem, consider asking students to name some hypothetical lengths and widths of the pen given the 60-foot constraint (e.g., 20 by 10 or 5 by 25).

Exploratory Challenge

Caitlin has 60 feet of material that can be used to make a fence. Using this material, she wants to create a rectangular pen for her dogs to play in. What dimensions will maximize the area of the pen?

a. **Let w be the width of the rectangular pen in feet. Write an expression that represents the length when the width is w feet.**

$\frac{(60-2w)}{2}$ or $30 - w$

b. **Define a function that describes the area, A, in terms of the width, w.**

$A(w) = w(30 - w)$ or $A(w) = -w^2 + 30w$

c. **Rewrite $A(w)$ in vertex form.**

$$A(w) = -w^2 + 30w$$
$$= -(w^2 - 30w)$$
$$= -(w^2 - 30w + 225) + 225$$
$$= -(w - 15)^2 + 225$$

d. **What are the coordinates of the vertex? Interpret the vertex in terms of the problem.**

The vertex is located at $(15, 225)$. Since the leading coefficient is negative, the function has a maximum. The maximum value of the function is 225, which occurs when $w = 15$. For this problem, this means that the maximum area is 225 square feet, which happens when the width is 15 feet.

e. **What dimensions maximize the area of the pen? Do you think this is a surprising answer?**

The pen has the greatest area when the length and width are both 15 feet. Students may or may not be surprised to note that this occurs when the rectangle is a 15×15 square.

Closing (3 minutes)

- How many quadratic equations are there whose graphs share a given vertex?
  - *There are infinitely many graphs that share a given vertex because there are an infinite number of values possible for the leading coefficient.*

Lesson Summary

When graphing a quadratic equation in vertex form, $y = a(x - h)^2 + k$, (h, k) are the coordinates of the vertex.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 16: Graphing Quadratic Equations from the Vertex Form, $y = a(x - h)^2 + k$

Exit Ticket

1. Compare the graphs of the function, $f(x) = -2(x + 3)^2 + 2$ and $g(x) = 5(x + 3)^2 + 2$. What do the graphs have in common? How are they different?

2. Write two different equations representing quadratic functions whose graphs have vertices at $(4.5, -8)$.

Exit Ticket Sample Solutions

1. Compare the graphs of the function, $f(x) = -2(x+3)^2 + 2$ and $g(x) = 5(x+3)^2 + 2$. What do the graphs have in common? How are they different?

   Both quadratic equations have their vertices at $(-3, 2)$. However, the graph of f has less vertical stretch than the graph of g, and the graph of f opens down, whereas the graph of g opens up.

2. Write two different quadratic equations whose graphs have vertices at $(4.5, -8)$.

   $y = (x - 4.5)^2 - 8$ *and* $y = -(x - 4.5)^2 - 8$ *or any similar responses with different leading coefficients.*

Problem Set Sample Solutions

1. Find the vertex of the graphs of the following quadratic equations.

   a. $y = 2(x-5)^2 + 3.5$

   $(5, 3.5)$

   b. $y = -(x+1)^2 - 8$

   $(-1, -8)$

2. Write a quadratic equation to represent a function with the following vertex. Use a leading coefficient other than 1.

   a. $(100, 200)$

   $y = -2(x - 100)^2 + 200$

   b. $\left(-\frac{3}{4}, -6\right)$

   $y = 4\left(x + \frac{3}{4}\right)^2 - 6$

3. Use vocabulary from this lesson (i.e., *stretch, shrink, opens up,* and *opens down*) to compare and contrast the graphs of the quadratic equations $y = x^2 + 1$ and $y = -2x^2 + 1$.

   The quadratic equations share a vertex at $(0, 1)$, but the graph for the equation $y = -2x^2 + 1$ opens down and has a vertical stretch, while the graph of the equation $y = x^2 + 1$ opens up.

Lesson 17: Graphing Quadratic Functions from the Standard Form, $f(x) = ax^2 + bx + c$

Student Outcomes

- Students graph a variety of quadratic functions using the form $f(x) = ax^2 + bx + c$ (standard form).
- Students analyze and draw conclusions about contextual applications using the key features of a function and its graph.

Lesson Notes

Throughout this lesson, students are presented with a verbal description where a relationship can be modeled symbolically and graphically. Students must decontextualize the verbal description and graph the quadratic relationship they describe. They then contextualize their solution to fully answer the questions posed by the examples.

Students may wonder why all physics applications of quadratic functions have the same leading coefficients. Lesson 23 has a teaching moment for students to learn the laws of physical objects in motion. Here is a simplified way to explain it if they have questions now:

Due to Earth's gravity, there is a constant downward force on any object. The force is proportional to the mass of the object. The proportional constant is called the gravitational acceleration. The constant is $-32\ \text{ft/s}^2$, or $-9.8\ \text{m/s}^2$, near Earth's surface.

- When we use a quadratic function to model the height (or vertical position) over time of any falling or projected object, the leading coefficient is calculated to be half of the gravitational acceleration. Therefore, the leading coefficient must either be -16 or -4.9, depending on the unit of choice.
- The coefficient of the linear term in the function represents the initial vertical speed of the object (if in feet, then feet per second, or if in meters, then meters per second).
- The constant of the quadratic function represents the initial height (or vertical position) of the object.

In summary, the following quadratic function h can be used to describe the height as a function of time for any projectile motion under a constant vertical acceleration due to gravity.

$$h(t) = \frac{1}{2}gt^2 + v_0t + h_0$$

t: time (in sec.)

g: gravitational acceleration (in ft/s^2 or m/s^2)

v_0: initial vertical speed (in ft/s or m/s)

h_0: initial height (in ft. or m)

Classwork

Opening Exercise (10 minutes)

Present students with the following problem. Write or project it on the board or screen, and have students work with a partner or in small groups to solve the problem.

Opening Exercise

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function $h(t) = -16t^2 + 96t + 6$, where t represents the time (in seconds) since the ball was thrown, and h represents the height (in feet) of the ball above the ground.

Scaffolding:

You may want to start by substituting values and creating a graph together as a class before getting into the problem analysis. Students can then use that graph as a visual reference during the discussion.

Have students informally consider and discuss the following questions in a small group or with a partner. Some possible student responses are provided after each question.

Opening Exercise

a. **What does the domain of the function represent in this context?**

The time (number of seconds) since the ball was thrown

b. **What does the range of this function represent?**

The height (in feet) of the ball above the ground

c. **At what height does the ball get thrown?**

The initial height of the ball is when t is 0 sec. (i.e., $h(0)$), which is the y-intercept. The initial height is 6 ft.

d. **After how many seconds does the ball hit the ground?**

The ball's height is 0 when $h(t) = 0$. We can solve using any method. Since this does not appear to be easily factorable, and the size of the numbers might be cumbersome in the quadratic formula, let's solve by completing the square.

$$-16t^2 + 96t + 6 = 0$$
$$-16(t^2 - 6t) = -6$$
$$-16(t^2 - 6t + 9) = -6 - 144$$

From here, we see the completed-square form: $h(t) = -16(t-3)^2 + 150$.

$$-16(t-3)^2 = -150$$
$$(t-3)^2 = \frac{150}{16}$$
$$t - 3 = \pm\frac{\sqrt{150}}{4}$$
$$t = 3 \pm \frac{\sqrt{150}}{4}$$
$$t \approx 6.0618 \ \textit{or} \ -0.0618$$

For this context, the ball hits the ground at approximately 6.1 seconds.

e. **What is the maximum height that the ball reaches while in the air? How long will the ball take to reach its maximum height?**

Completing the square (and using the work from the previous question), we get $h(t) = -16(t-3)^2 + 150$, so the vertex is $(3, 150)$, meaning that the maximum height is 150 ft., and it will reach that height in 3 sec.

f. **What feature(s) of this quadratic function are *visible* since it is presented in the standard form, $f(x) = ax^2 + bx + c$?**

We can see the initial position, or height of the ball, or the height when $t = 0$, in the constant term. We can also see the leading coefficient, which tells us about the end behavior and whether the graph is wider or narrower than the graph of $f(x) = x^2$.

g. **What feature(s) of this quadratic function are *visible* when it is rewritten in vertex form, $f(x) = a(x-h)^2 + k$?**

We can only see the coordinates of the vertex and know that $x = h$ is the equation of the axis of symmetry. We can still see the leading coefficient in this form, which tells us about the end behavior and whether the graph is wider or narrower than the graph of $f(x) = x^2$.

To understand and solve a problem presented in a context, point out the importance of graphing the function, along with interpreting the domain and range. Demonstrate how the key features of a quadratic function that are discoverable from the algebraic form can help us create the graph. We use the function from the Opening Exercise as our first example.

Have students contemplate and write a general strategy for graphing a quadratic function from the standard form in their student materials. Discuss or circulate to ensure that what they have written will be helpful to them later. Note that a correct strategy is provided in the Lesson Summary.

A general strategy for graphing a quadratic function from the standard form:

- *Look for hints in the function's equation for general shape, direction, and y-intercept.*
- *Solve $f(x) = 0$ to find the x-intercepts by factoring, completing the square, or using the quadratic formula.*
- *Find the vertex by completing the square or using symmetry. Find the axis of symmetry and the x-coordinate of the vertex using $\frac{-b}{2a}$ and the y-coordinate of the vertex by finding $f\left(\frac{-b}{2a}\right)$.*
- *Plot the points you know (at least three are required for a unique quadratic function), sketch the graph of the curve that connects them, and identify the key features of the graph.*

Example (10 minutes)

Have students use the steps to graph the thrown baseball example in the Opening Exercise: $h(t) = -16t^2 + 96t + 6$. Have students answer the following questions in their student materials with a partner or in small groups.

Example

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function $h(t) = -16t^2 + 96t + 6$, where t represents the time (in seconds) since the ball was thrown, and h represents the height (in feet) of the ball above the ground.

Remind students to look back at the work done in the Opening Exercise.

a. **What do you notice about the equation, just as it is, that will help us in creating our graph?**

The leading coefficient is negative, so we know the graph opens down. $h(0) = 6$, *so the point* $(0, 6)$, *which is the* y*-intercept, is on the graph.*

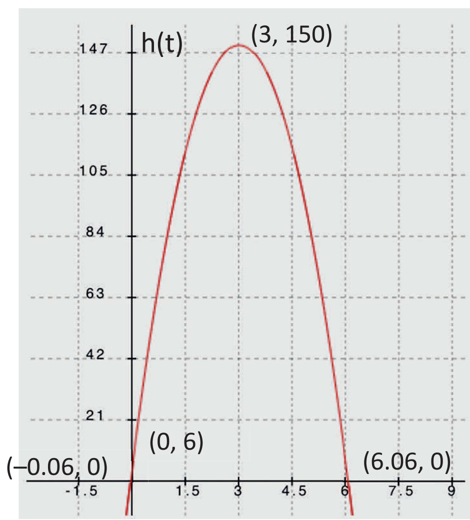
b. **Can we factor to find the zeros of the function? If not, solve** $h(t) = 0$ **by completing the square.**

This function is not factorable, so we complete the square to find the zeros to be $(6.06, 0)$ *and* $(-0.06, 0)$.

c. **What is the vertex of the function? What method did you use to find the vertex?**

Since we already completed the square (and the zeros are irrational and more difficult to work with), we can easily find the vertex using the completed-square form, $h(t) = -16(t-3)^2 + 150$, *which means the vertex is* $(3, 150)$.

d. **Now plot the graph of** $h(t) = -16t^2 + 96t + 6$, **and label the key features on the graph.**



After students have graphed the function, ask the following questions to probe further into their conceptual understanding of the graphic representation:

- What is the appropriate domain for the context of the function?
  - *Since the time must be positive, the domain for the context is* $[0, 6.06]$.
- What do the 3 and the 150 in the vertex tell us?
  - *The* 3 *and the* 150 *tell us that the ball reached its highest point of* 150 ft. *after* 3 sec., *and then it started back down.*

- What do the zeros of the function tell us about the ball's flight?
  - *Since the zeros tell us when the ball was at ground level (height $= 0$), the negative value indicates that the ball was above the ground at the time of the throw*. The 6.06 tells us that it took 6.06 sec. for the ball to complete its flight and hit the ground.*

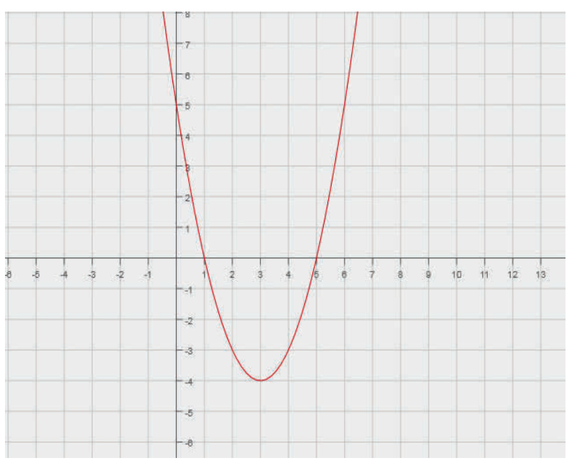
    **The -0.06 does not describe part of the baseball throw. It is a thought experiment using mathematics. Hypothetically, the -0.06 could mean, based on the graph, that if we backtracked in time and asked, "What if, instead of starting at 6 ft. high at time 0, we assume the ball was somehow thrown up from the ground level at an earlier time and reached 6 ft. at time 0?" then, -0.06 would be the time the ball got started.*
- Does this curve represent the path of the ball? Explain.
  - *No, the problem says that the ball went straight up, so it would probably come straight down. The graph does not show forward movement for the ball but only represents the elapsed time as it relates to the ball's height.*

Exercises (20 minutes)

Students use the steps in the examples above to complete the following exercises. The first two are pure mathematical examples. Then, the next two will be in a context.

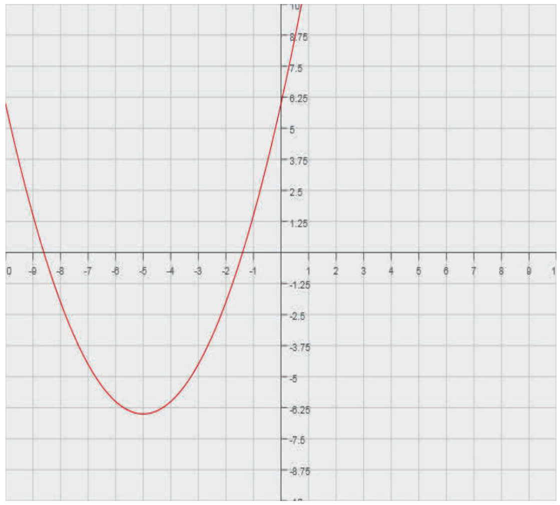
Exercises

1. Graph the function $n(x) = x^2 - 6x + 5$, and identify the key features.



***x*-intercepts:** $(5, 0), (1, 0)$
***y*-intercept:** $(0, 5)$
Vertex: $(3, -4)$

2. Graph the function $f(x) = \frac{1}{2}x^2 + 5x + 6$, and identify the key features.



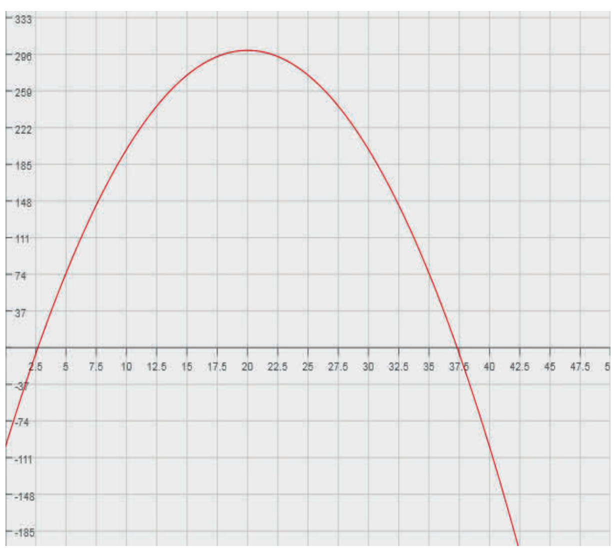
x-intercepts: $(-5+\sqrt{13}, 0), (-5-\sqrt{13}, 0)$
y-intercept: $(0, 6)$
Vertex: $(-5, -6.5)$

3. Paige wants to start a summer lawn-mowing business. She comes up with the following profit function that relates the total profit to the rate she charges for a lawn-mowing job:

$$P(x) = -x^2 + 40x - 100.$$

Both profit and her rate are measured in dollars. Graph the function in order to answer the following questions.

a. Graph P.



x-intercepts: $(20+10\sqrt{3}, 0)$, $(20-10\sqrt{3})$
y-intercept: $(0, -100)$
Vertex: $(20, 300)$

b. **According to the function, what is her initial cost (e.g., maintaining the mower, buying gas, advertising)? Explain your answer in the context of this problem.**

When Paige has not mown any lawns or charged anything to cut grass, her profit would be -100. A negative profit means that Paige is spending $\$100$ to run her business.

c. **Between what two prices does she have to charge to make a profit?**

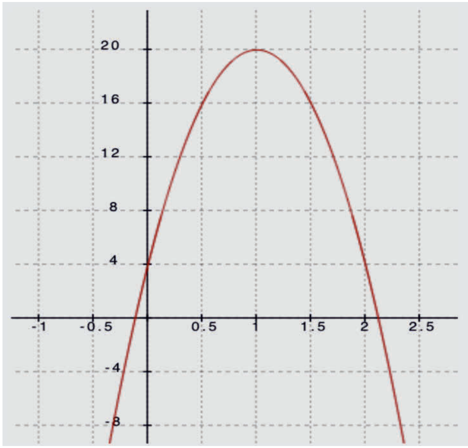
Using completing the square, we find the intercepts at $(20+10\sqrt{3})$ and $(20-10\sqrt{3})$. However, since this question is about money, we approximate and find that her rates should be between $\$2.68$ and $\$37.32$.

d. **If she wants to make a $\$275$ profit this summer, is this the right business choice?**

Yes. Looking at the graph, the vertex, $(20, 300)$, is the maximum profit. So, if Paige charges $\$20$ for each lawn she mows, she can make a $\$300$ profit, which is $\$25$ more than she wants.

4. **A student throws a bag of chips to her friend. Unfortunately, her friend does not catch the chips, and the bag hits the ground. The distance from the ground (height) for the bag of chips is modeled by the function $h(t) = -16t^2 + 32t + 4$, where h is the height (distance from the ground in feet) of the chips, and t is the number of seconds the chips are in the air.**

a. **Graph h.**



t-intercepts: $\left(1+\frac{\sqrt{5}}{2}, 0\right), \left(1-\frac{\sqrt{5}}{2}, 0\right)$
h-intercept: $(0, 4)$
Vertex: $(1, 20)$

b. **From what height are the chips being thrown? Tell how you know.**

4 ft. This is the initial height, or when $t = 0$.

c. **What is the maximum height the bag of chips reaches while airborne? Tell how you know.**

From the graph, the vertex is $(1, 20)$, which means that at 1 second, the bag is 20 ft. above the ground for this problem. Since this is the vertex of the graph, and the leading coefficient of the quadratic function is negative, the graph opens down (as $t \to \pm\infty$, $h(t) \to -\infty$), and the vertex is the maximum of the function. This means 20 ft. is the maximum height of the thrown bag.

d. **How many seconds after the bag was thrown did it hit the ground?**

By completing the square, we find that $t = 1 \pm \frac{\sqrt{5}}{2}$*. Since this is time in seconds, we need a positive value,* $1 + \frac{\sqrt{5}}{2}$*, which is about* 2.12 sec.

e. **What is the average rate of change of height for the interval from 0 to $\frac{1}{2}$ second? What does that number represent in terms of the context?**

$\frac{\left[f\left(\frac{1}{2}\right) - f(0)\right]}{\frac{1}{2} - 0} = \frac{[16 - 4]}{\frac{1}{2}} = 24$. *The rate of change for the interval from* 0 *to* $\frac{1}{2}$ sec. *is* 24 ft/s, *which represents the average speed of the bag of chips from* 0 *to* $\frac{1}{2}$ sec.

f. **Based on your answer to part (e), what is the average rate of change for the interval from 1.5 to 2 sec.?**

The average rate of change for the interval from 1.5 *to* 2 sec. *will be the same as it is from* 0 *to* $\frac{1}{2}$ *except that it will be negative:* -24 ft/s.

5. **Notice how the profit and height functions both have negative leading coefficients. Explain why this is.**

The nature of both of these contexts is that they have continually changing rates, and both require the graph to open down since each would have a maximum. Problems that involve projectile motion have maxima because an object can only go so high before gravity pulls it back down. Profits also tend to increase as prices increase only to a point before sales drop off, and profits begin to fall.

Closing (2 minutes)

- For a profit function in the standard form, $P(x) = ax^2 + bx + c$, what does the constant, c, identify?
  - *Starting cost*
- For a height function in the standard form, $h(t) = at^2 + bt + c$, what does the constant, c, identify?
  - *Starting height*
- Describe a strategy for graphing a function represented in standard form, $f(x) = ax^2 + bx + c$.
  - *Examine the form of the equation for hints.*
  - *Find the zeros by factoring, completing the square, or using the quadratic formula.*
  - *Find the vertex by completing the square or using symmetry.*
  - *Plot the points that you know (at least three), sketch the curve, and identify the key features.*

Lesson Summary

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$. A general strategy for graphing a quadratic function from the standard form:

- **Look for hints in the function's equation for general shape, direction, and y-intercept.**
- **Solve $f(x) = 0$ to find the x-intercepts by factoring, completing the square, or using the quadratic formula.**
- **Find the vertex by completing the square or using symmetry. Find the axis of symmetry and the x-coordinate of the vertex using $\frac{-b}{2a}$ and the y-coordinate of the vertex by finding $f\left(\frac{-b}{2a}\right)$.**
- **Plot the points that you know (at least three are required for a unique quadratic function), sketch the graph of the curve that connects them, and identify the key features of the graph.**

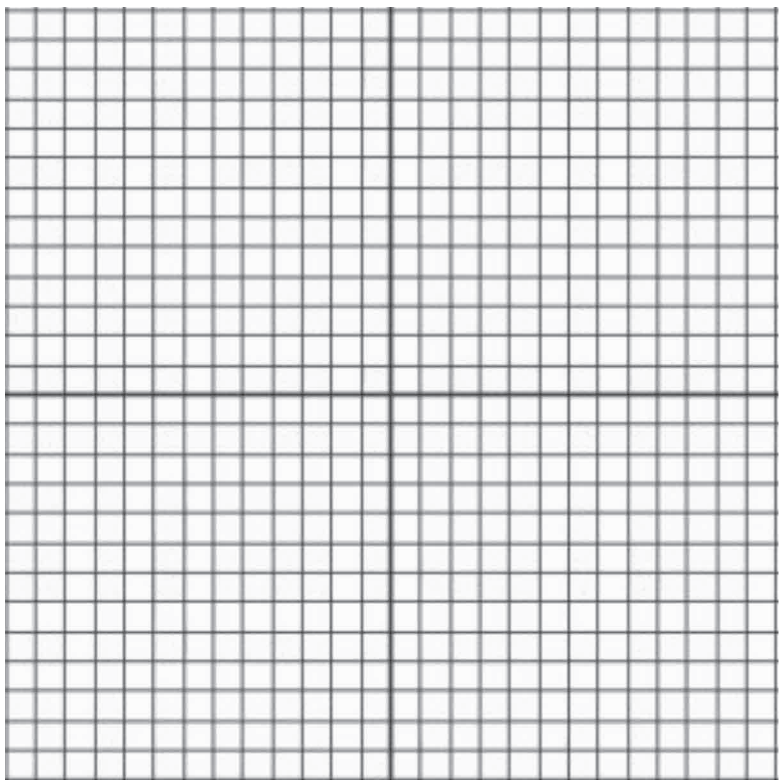
Exit Ticket (3 minutes)

Name ____ Date ____________________

Lesson 17: Graphing Quadratic Functions from the Standard Form, $f(x) = ax^2 + bx + c$

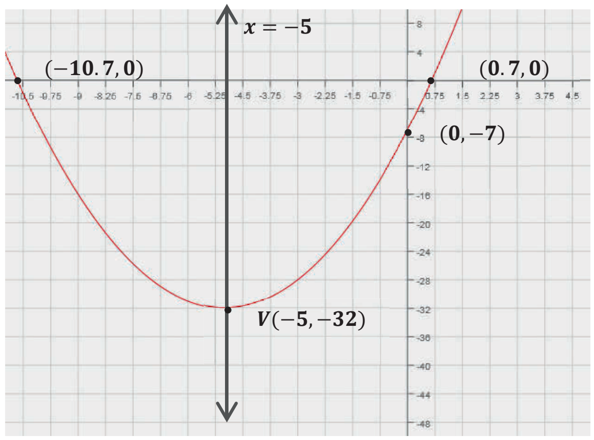
Exit Ticket

Graph $g(x) = x^2 + 10x - 7$, and identify the key features (e.g., vertex, axis of symmetry, x- and y-intercepts).



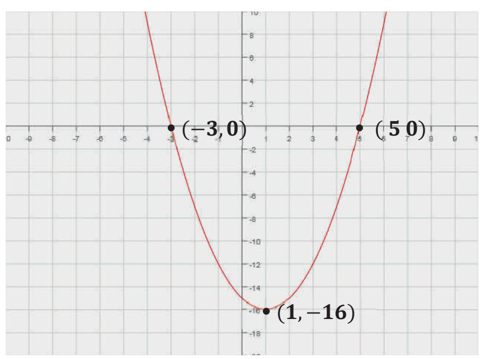
Exit Ticket Solutions

Graph $g(x) = x^2 + 10x - 7$, and identify the key features (e.g., vertex, axis of symmetry, x- and y-intercepts).



Problem Set Solutions

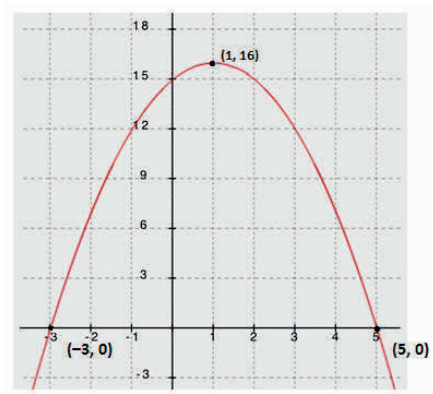
1. **Graph $f(x) = x^2 - 2x - 15$, and identify its key features.**



x-intercepts: $(-3, 0)$ $(5, 0)$
y-intercept: $(0, -15)$
Vertex: $(1, -16)$

End behavior: As $x \to \pm\infty$, $y \to \infty$.

2. Graph $f(x) = -x^2 + 2x + 15$, and identify its key features.



x-intercepts: $(-3, 0)$ $(5, 0)$
y-intercept: $(0, 15)$
Vertex: $(1, 16)$
End behavior: As $x \to \pm\infty$, $y \to -\infty$.

3. Did you recognize the numbers in the first two problems? The equation in the second problem is the product of -1 and the first equation. What effect did multiplying the equation by -1 have on the graph?

The graph gets reflected across the x-axis. The x-intercepts remain the same. The y-intercept becomes the opposite of the original y-intercept. The end behavior of the graph reversed. The vertex became the maximum instead of the minimum.

4. Giselle wants to run a tutoring program over the summer. She comes up with the following profit function:

$$P(x) = -2x^2 + 100x - 25,$$

where x represents the price of the program. Between what two prices should she charge to make a profit? How much should she charge her students if she wants to make the most profit?

Using the quadratic formula, the two roots are $25 \pm \frac{35\sqrt{2}}{2}$, which is about $\$0.25$ and $\$50$. If Giselle charges between $\$0.25$ and $\$50$, she can expect to make a profit. If she charges $\$25$, she will make a maximum profit of $\$1,225$.

5. Doug wants to start a physical therapy practice. His financial advisor comes up with the following profit function for his business:

$$P(x) = -\frac{1}{2}x^2 + 150x - 10000$$

where x represents the amount, in dollars, that he charges his clients. How much will it cost for him to start the business? What should he charge his clients to make the most profit?

The formula suggests it would cost him $\$10,000$ to start his business. He should charge $\$150$ to make the most profit.

Topic C

Function Transformations and Modeling

Focus Standards:

- Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
- Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
- Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
  - Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
- Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
  - Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
- Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*
- Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Instructional Days: 7

Lesson 18: Graphing Cubic, Square Root, and Cube Root Functions (E)[1]

Lesson 19: Translating Graphs of Functions (P)

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

In Lesson 18 of this topic, students build an understanding of the transformational relationship between basic quadratic and square root functions, as well as cubic and cube root functions. (Note: Square and cube roots are not treated as inverse functions in this course but rather as rotations and reflections of quadratic and cubic functions.) The topic builds on students' prior experience of transforming linear, exponential, and absolute value functions in Module 3 to include transforming quadratic, square root, and cube root functions in Lessons 19 and 20. Students create graphs of quadratic, square root, and cube root functions by recognizing in the given functions a parent function along with the transformations to be performed. Students also write the function of the given graph by recognizing the parent function and different transformations being performed. It is crucial that students understand that complex functions can be built from basic parent functions and that this recognition can simplify both graphing functions and creating function equations from graphs. They recognize the application of transformations in the vertex form for the quadratic function and use it to expand their ability to efficiently sketch graphs of square root and cube root functions.

In Lesson 21, students use what they know about transformations of functions to build both graphs and new, related functions from the quadratic parent function. Then, in Lesson 22, they compare key features of three functions (quadratic, square root, or cube root), each represented in a different way, including graphically, algebraically, numerically in tables, or verbally with a description.

In the final two lessons, students create quadratic functions from contextual situations described verbally and from data sets, create graphs of their functions, interpret key features of both the functions and their graphs in terms of the contexts, and answer questions related to the functions and their graphs. They justify their solutions, as well as choose and explain the level of precision they used in reporting their results.

Lesson 18: Graphing Cubic, Square Root, and Cube Root Functions

Student Outcomes

- Students compare the basic quadratic (parent) function, $y = x^2$, to the square root function and do the same with cubic and cube root functions. They then sketch graphs of square root and cube root functions, taking into consideration any constraints on the domain and range.

Lesson Notes

In these exercises, students explore the effects of squaring, taking the square root of, cubing, and taking the cube root of various values. They then use visible patterns to make generalizations about the graphs of square root and cube root functions, as well as cubic functions.

Classwork

Opening Exercise (5 minutes)

Students review evaluating expressions that involve radicals and exponents so that they are prepared to work with quadratic, square root, cubic, and cube root functions.

Scaffolding:

For students who are not as familiar working with radicals, it may be important to spend a few minutes discussing the difference between the solutions for the equations $x^2 = 25$ and $x = \sqrt{25}$. In the first case, there are two possible solutions: 5 and -5. But in the second, there is only one: 5. The square root symbol originates from the geometric application of finding the length of the hypotenuse. The use of that symbol is reserved to mean the positive value that when squared would yield the value inside. Hence, when we solve for x in the equation $x^2 = 25$, we state the solution as $x = \pm\sqrt{25}$.

Opening Exercise

a. **Evaluate x^2 when $x = 7$.**

49

b. **Evaluate $\sqrt{x}$ when $x = 81$.**

9

c. **Evaluate x^3 when $x = 5$.**

125

d. **Evaluate $\sqrt[3]{x}$ when $x = 27$.**

3

Exploratory Challenge 1 (5 minutes)

Exploratory Challenge 1

Use your graphing calculator to create a data table for the functions $y = x^2$ and $y = \sqrt{x}$ for a variety of x-values. Use both negative and positive numbers, and round decimal answers to the nearest hundredth.

x	$y = x^2$	$y = \sqrt{x}$
4	16	2
2	4	1.41
0	0	0
-2	4	*Error*
-4	16	*Error*

Scaffolding:

- Provide struggling students with tables that include several x-values.
- Some students might need to try many more values before recognizing the patterns.

Discussion (5 minutes)

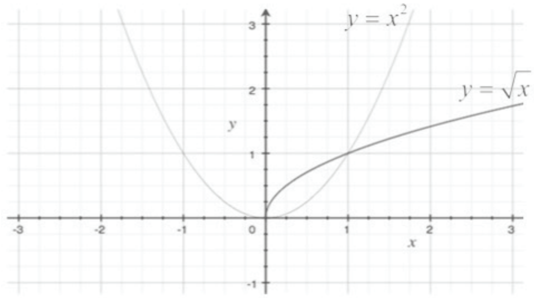
Compare the two y-columns in the data table from Exercise 1. Students can use their calculators to explore additional values of these functions.

- Observe both y-columns from above. What do you notice about the values in the two y-columns?
  - *In the column for $y = x^2$, all y-values are positive.*
  - *In the column for $y = \sqrt{x}$, all negative x-values produce an error.*
- Why are all y-values for $y = x^2$ positive?
  - *All y-values are positive since they are obtained by squaring the x-value.*
- Why do all negative x-values produce an error for $y = \sqrt{x}$?
  - *No real number, when squared, produces a negative result. Therefore, the calculator produces an error.*
- What is the domain of $y = x^2$ and $y = \sqrt{x}$?
  - *The domain of $y = x^2$ is all real numbers.*
  - *The domain of $y = \sqrt{x}$ is $x \geq 0$.*
- What is the range of $y = x^2$ and $y = \sqrt{x}$?
  - *The range of $y = x^2$ is $y \geq 0$.*
  - *The range of $y = \sqrt{x}$ is $y \geq 0$.*
- Compare the domain and range of $y = x^2$ and $y = \sqrt{x}$.
  - *The domain of $y = \sqrt{x}$ is limited to nonnegative values, while the domain of $y = x^2$ includes all real numbers.*
  - *The range of both functions is the same set of all nonnegative real numbers.*
- What is the result if we take the square root of x^2? Have students try making a third column in the chart to see if they can come up with a rule for $\sqrt{x^2}$. This should help them understand the need for the $\pm$ when taking the square root of a variable expression.
  - *If x is a negative number, the result is $(-x)$, and if x is a positive number, the result is x. So, $\sqrt{x^2} = |x|$.*

Exploratory Challenge 2 (10 minutes)

Exploratory Challenge 2

Create the graphs of $y = x^2$ and $y = \sqrt{x}$ on the same set of axes.



Scaffolding:

Provide students with the axes drawn and numbered.

- What additional observations can we make when comparing the graphs of these functions?
  - *They intersect at $(0, 0)$ and $(1, 1)$. The square root function is a reflection of the part of the quadratic function in the first quadrant, about $y = x$ (when x is nonnegative).*
- Why do they intersect at $(0, 0)$ and $(1, 1)$?
  - 0^2 *and* $\sqrt{0}$ *are both equal to* 0.
  - 1^2 *and* $\sqrt{1}$ *are both equal to* 1.

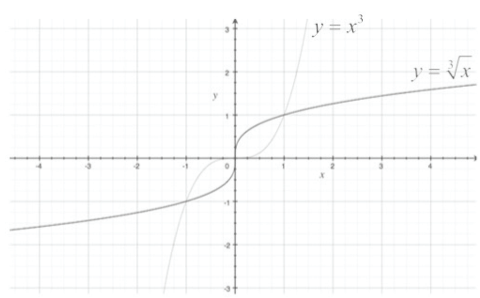
Exploratory Challenge 3 (15 minutes)

- Predict the relationship between $y = x^3$ and $y = \sqrt[3]{x}$.
  - *Both functions include all real numbers in their domain and range since a cubed number can be positive or negative, as well as the cube root of a number.*

Exploratory Challenge 3

Create a data table for $y = x^3$ and $y = \sqrt[3]{x}$, and graph both functions on the same set of axes. Round decimal answers to the nearest hundredth.

x	$y = x^3$	$y = \sqrt[3]{x}$
-8	-512	-2
-2	-8	-1.26
-1	-1	-1
0	0	0
1	1	1
2	8	1.26
8	512	2



- Using the table and graphs, what observations can you make about the relationships between $y = x^3$ and $y = \sqrt[3]{x}$?
  - *They share the points* $(0, 0)$, $(1, 1)$, *and* $(-1, -1)$. *The domain and range of both functions are all real numbers. The functions are symmetrical about the origin. Each of these two functions is a reflection of the other across the line* $y = x$.

If students do not arrive at the responses above, use the following prompts:

- What are the domain and range of each function?
- How are the graphs related to each other?
- Describe the symmetry that exists within the tables and graphs.

Closing (2 minutes)

- The square root function is a reflection of the quadratic function across the line $y = x$, when x is nonnegative.
- The domain of $y = x^2$, $y = x^3$, and $y = \sqrt[3]{x}$ is all real numbers. The domain of $y = \sqrt{x}$ is $x \geq 0$.
- The range of $y = x^2$ and $y = \sqrt{x}$ is $y \geq 0$. The range of $y = x^3$ and $y = \sqrt[3]{x}$ is all real numbers.
- $y = x^3$ and $y = \sqrt[3]{x}$ are each symmetrical about the origin and are reflections of each other across the line $y = x$; the two operations reverse each other. Note that inverse functions have not yet been introduced, but this is an opportunity to offer a preview, depending on the ability and interest level of your students.

Lesson Summary

- **The square root parent function is a reflection of the quadratic parent function across the line $y = x$, when x is nonnegative.**
- **The domain of quadratic, cubic, and cube root parent functions is all real numbers. The domain of the square root parent function is $x \geq 0$.**
- **The range of quadratic and square root parent functions is $[0, \infty)$. The range of the cubic and cube root parent functions is all real numbers.**
- **The cube root and cubic parent functions are symmetrical about the origin and are reflections of each other across the line $y = x$; the two operations reverse each other.**

Exit Ticket (3 minutes)

Name ___________________________________ Date ____________________

Lesson 18: Graphing Cubic, Square Root, and Cube Root Functions

Exit Ticket

1. Describe the relationship between the graphs of $y = x^2$ and $y = \sqrt{x}$. How are they alike? How are they different?

2. Describe the relationship between the graphs of $y = x^3$ and $y = \sqrt[3]{x}$. How are they alike? How are they different?

Exit Ticket Sample Solutions

1. **Describe the relationship between $y = x^2$ and $y = \sqrt{x}$. How are they alike? How are they different?**

   The square root function is a reflection of the quadratic function about $y = x$ when x is nonnegative. The domain of $y = x^2$ is all real numbers. The domain of $y = \sqrt{x}$ is $x \geq 0$. The range of $y = x^2$ and $y = \sqrt{x}$ is $y \geq 0$.

2. **Describe the relationship between the graphs of $y = x^3$ and $y = \sqrt[3]{x}$. How are they alike? How are they different?**

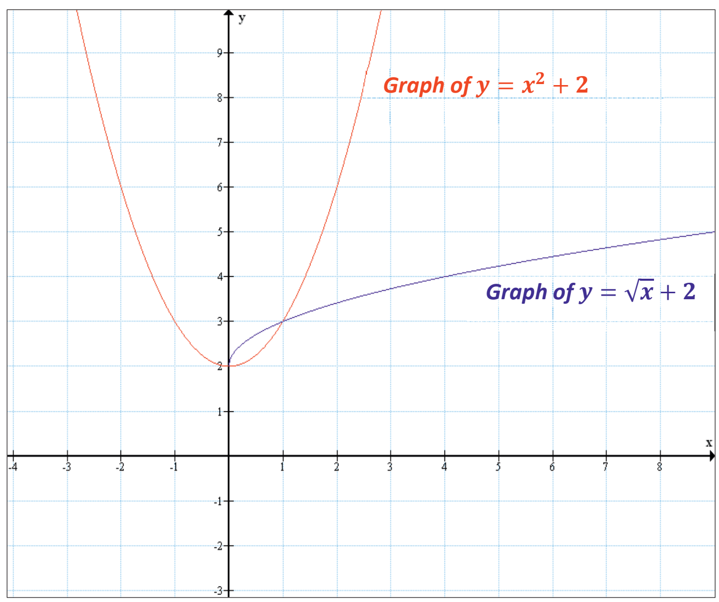
   The domain and range of $y = x^3$ and $y = \sqrt[3]{x}$ are all real numbers. The shape of the graphs of $y = x^3$ and $y = \sqrt[3]{x}$ are the same, but they are oriented differently. (Some students may be able to articulate that each graph appears to be a reflection of the other across a diagonal line going through the origin.)

Problem Set Sample Solutions

1. **Create the graphs of the functions $f(x) = x^2 + 2$ and $g(x) = \sqrt{x} + 2$ using the given values. Use a calculator to help with decimal approximations.**

   See the values in the table.

x	$f(x)$	$g(x)$
-4	18	
-2	6	
-1	3	
0	2	2
1	3	3
2	6	$3.4142\ldots$
4	18	4



2. **What can be said about the first three values for $g(x)$ in the table?**

   The domain of $g(x) = \sqrt{x} + 2$ is limited to nonnegative numbers since the square root of a negative number is not real, so there is no value for $g(-4)$, $g(-2)$, or $g(-1)$.

3. **Describe the relationship between the graphs given by the equations $y = x^2 + 2$ and $y = \sqrt{x} + 2$. How are they alike? How are they different?**

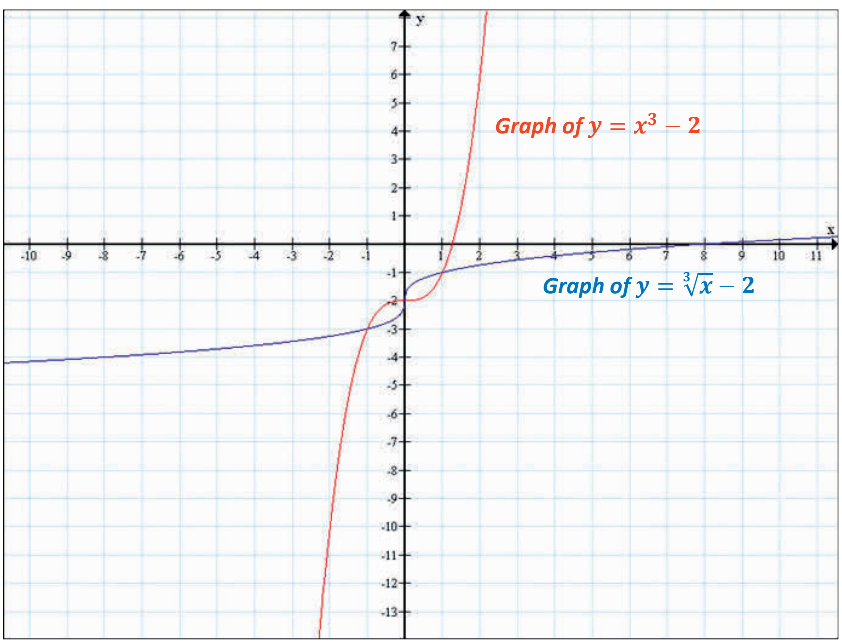
   The graph of the square root function is a reflection of the graph of the quadratic function when x is nonnegative. The reflection is about the line given by the graph of the equation $y = x + 2$. The domain of $y = x^2 + 2$ is all real numbers. The domain of $y = \sqrt{x} + 2$ is $x \geq 0$. The range of $y = x^2 + 2$ and $y = \sqrt{x} + 2$ is $y \geq 2$.

4. Refer to your class notes for the graphs of $y = x^2$ and $y = \sqrt{x}$. How are the graphs of $y = x^2$ and $y = \sqrt{x}$ transformed to generate the graphs of $y = x^2 + 2$ and $y = \sqrt{x} + 2$?

The graph of $y = x^2 + 2$ is the graph of $y = x^2$ translated vertically 2 units up. The graph of $y = \sqrt{x} + 2$ is also the vertical translation of the graph of $y = \sqrt{x}$ translated 2 units up.

5. Create the graphs of $p(x) = x^3 - 2$ and $q(x) = \sqrt[3]{x} - 2$ using the given values for x. Use a calculator to help with decimal approximations.

x	$p(x)$	$q(x)$
-8	-514	-4
-2	-10	$-3.2599 \ldots$
-1	-3	-3
0	-2	-2
1	-1	-1
2	6	$-0.74007 \ldots$
8	510	0



6. For the table in Problem 5, explain why there were no function values that resulted in an error.

Unlike square roots, the domain of a cube root function includes all real numbers since the product of three (or any other odd number) factors of a negative number, yields a negative number. Since the domains for both functions include all real numbers, there are no excluded rows in the table.

7. Describe the relationship between the domains and ranges of the functions $p(x) = x^3 - 2$ and $q(x) = \sqrt[3]{x} - 2$. Describe the relationship between their graphs.

The domain and range of $p(x) = x^3 - 2$ and $q(x) = \sqrt[3]{x} - 2$ are all real numbers. The graphs of $y = x^3 - 2$ and $y = \sqrt[3]{x} - 2$ are each symmetrical about the line given by the equation $y = x - 2$.

8. **Refer to your class notes for the graphs of $y = x^3$ and $y = \sqrt[3]{x}$. How are the graphs of $y = x^3$ and $y = \sqrt[3]{x}$ transformed to generate the graphs of $y = x^3 - 2$ and $y = \sqrt[3]{x} - 2$?**

    The graph of $y = x^3 - 2$ is the graph of $y = x^3$ translated vertically 2 units down. The graph of $y = \sqrt[3]{x} - 2$ is also the vertical translation of the graph of $y = \sqrt[3]{x}$ translated 2 units down.

9. **Using your responses to Problems 4 and 8, how do the functions given in Problems 1 and 5 differ from their parent functions? What effect does that difference seem to have on the graphs of those functions?**

    In Problem 1, f is the squaring function x^2 plus 2, and g is the square root function $\sqrt{x}$ plus 2. Adding 2 to a function translates the graph of the function 2 units up vertically. In Problem 5, p is the cubing function x^3 minus 2, and q is the cube root function $\sqrt[3]{x}$ minus 2. Subtracting 2 from a function translates the graph of the function 2 units down vertically.

10. **Create your own functions using $r(x) = x^2 - \square$ and $s(x) = \sqrt{x} - \square$ by filling in the box with a positive or negative number. Predict how the graphs of your functions will compare to the graphs of their parent functions based on the number that you put in the blank boxes. Generate a table of solutions for your functions, and graph the solutions.**

    Answers will vary. If k is the number inserted into $\square$, then the graph of the function will be translated vertically k units down for positive k values and $-k$ units up for negative k values.

Lesson 19: Translating Graphs of Functions

Student Outcomes

- Students recognize and use parent functions for linear, absolute value, quadratic, square root, and cube root functions to perform vertical and horizontal translations. They identify how the graph of $y = f(x)$ relates to the graphs of $y = f(x) + k$ and $y = f(x + k)$ for any specific values of k, positive or negative, and find the constant value, k, given the parent functions and the translated graphs. Students write the function representing the translated graphs.

Lesson Notes

In the Opening Exercise, students sketch the graphs of the equation $y = f(x)$ for a parent function, f, and graph the equations, $y = f(x) + k$ and $y = f(x + k)$, representing transformations of f. The functions in this example are linear, absolute value, and quadratic. In Example 1, students identify the value of k by looking at the graphs of the *parent function* and the translated functions. Students also write the function representing the translated graph. Concepts in this lesson relate directly to those in Module 3, Lessons 17 and 18. In this lesson, the functions will expand to include quadratic, square root, cube root, and absolute value.

Classwork

Opening Exercise (10 minutes)

After providing students with graphing calculators, have them graph each set of three functions in the same window. If graphing calculators are not accessible, provide the graphs of the functions, or project the graphs on the board. Ask students to explain what similarities and differences they see among the graphs. Remind students that part (c) is the same type of problem they solved in Module 3.

> *Scaffolding:*
>
> It may be helpful for struggling students to also create a table of values for each set of graphs. Ask them to compare the y-coordinates of the parent function and the translated functions. For example, in the linear functions for part (a), they should see that for the function $g(x) = x + 5$, the y-coordinates are 5 more than the y-coordinates of $f(x) = x$. In the same manner, the y-coordinates of $h(x) = x - 6$ are 6 less than those of $f(x) = x$.

Opening Exercise

Graph each set of three functions in the same coordinate plane (on your graphing calculator or a piece of graph paper). Then, explain what similarities and differences you see among the graphs.

a. $f(x) = x$, $g(x) = x + 5$, $h(x) = x - 6$

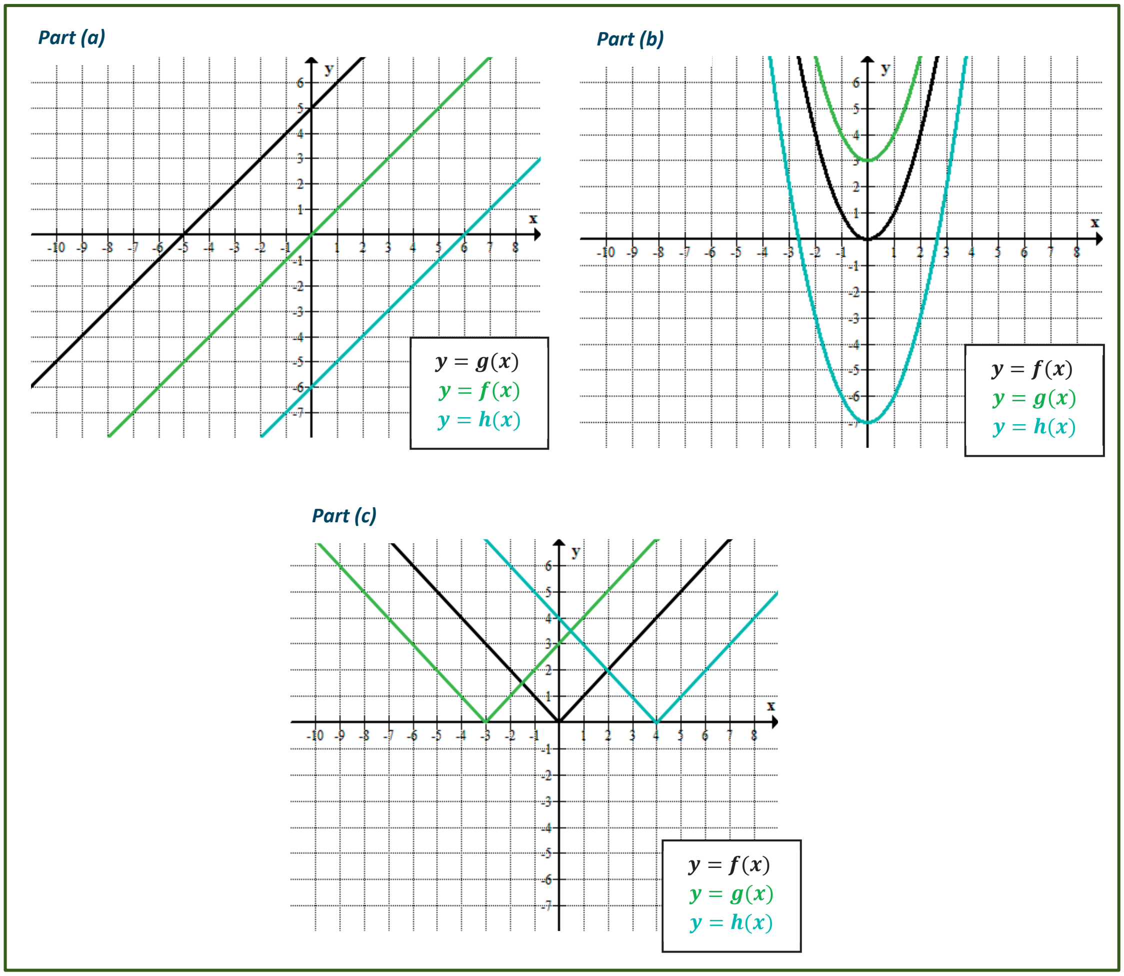
b. $f(x) = x^2$, $g(x) = x^2 + 3$, $h(x) = x^2 - 7$

c. $f(x) = |x|$, $g(x) = |x + 3|$, $h(x) = |x - 4|$

Part (a)—The graphs are parallel lines, but they have different x- and y-intercepts.

Part (b)—The graphs look the same (because they are congruent), but they have different vertices, which in this case means different minimum values. They are related by vertical translations.

Part (c)—The overall shapes of the graphs look the same (because they are congruent), but they have different vertices. They are related by horizontal translations.



- What do you notice about the coordinates of the points of the translated graphs in relation to the graph of their respective parent functions? How are the y-coordinates or x-coordinates of each of the three graphs related?
    - *For part (a), at any given x, the y-coordinate of the graph of g is 5 greater than the corresponding point on the graph of f, and the y-coordinate of the graph of h is 6 less than the corresponding point on the graph of f.*
    - *For part (b), at any given x, the y-coordinate of the graph of g is 3 greater than the corresponding point on the graph of f, and the y-coordinate of the graph of h is 7 less than the corresponding point on the graph of f. The constant values represent the different y-intercepts of different graphs.*
    - *For part (c), at a given value for y, the x-coordinates of the graph of g are 3 less than the corresponding point on the graph of f, and the x-coordinates of the graph of h are 4 greater than the corresponding points on the graph of f.*
    - *When comparing the corresponding points on the graphs of g and h, we find that for part (a), the y-coordinate of the point on the graph of g is 11 more than that of the graph of h, for part (b), the y-coordinate of the point on the graph of g is 10 more than that of the graph of h, and for part (c), the x-coordinates of points on the graph of g are 7 less than the corresponding points on the graph of h.*

- How are the graphs of g and h translated compared to that of f *and to each other*?
  - *For part (a), the graph of g is 5 units above the graph of f, while the graph of h is 6 units below the graph of f, and the graph of g is 11 units above the graph of h.*
  - *For part (b), the graph of g is 3 units above the graph of f, while the graph of h is 7 units below the graph of f, making the graph of g 10 units above the graph of h.*
  - *For part (c), the graph of g is 3 units left of the graph of f, the graph of h is 4 units right of the graph of f, and the graph of g is 7 units to the left of the graph of h.*
- Say you have a function defined by $(x) = \sqrt{x}$, what can you conclude about the graph of $p(x) = \sqrt{x} + 4$?
  - *The graph of p is congruent to that of f, but it is translated 4 units up.*
- If you have the function, $f(x) = \sqrt[3]{x}$, what can you conclude about the graph of $q(x) = \sqrt[3]{x+5}$?
  - *The general shape of the graph of q is congruent to that of f but is translated 5 units left.*

Students analyze the similarities and differences of the graphs to look for patterns. They then generalize the patterns to determine how vertical and horizontal shifts of the graphs are related to the structure of the equations. They also relate the structure of the equations for each function to that of the parent function and make the connection to the structure of the translated graphs of each.

Example (10 minutes)

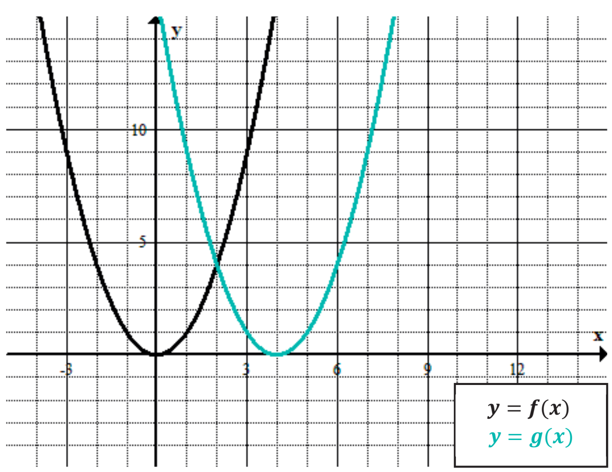
On the board, post the following sets of graphs. For each set, ask the list of questions below.

Example

For each graph, answer the following:

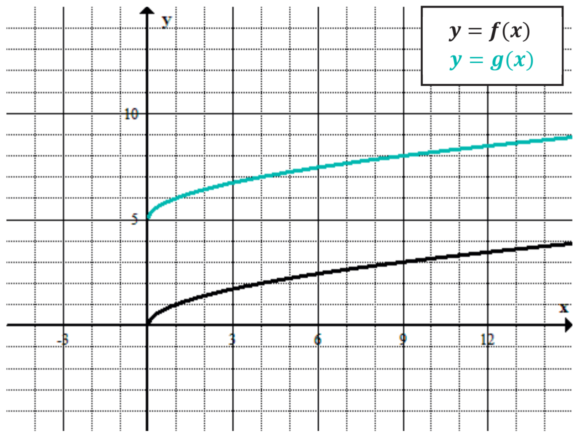
- **What is the parent function?**
- **How does the translated graph relate to the graph of the parent function?**
- **Write the formula for the function depicted by the translated graph.**

a.



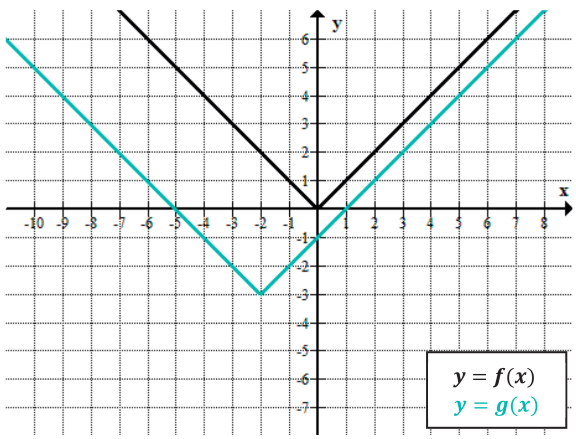
The parent function is $f(x) = x^2$. The graph is shifted 4 units to the right. The function defined by the translated graph is $g(x) = (x-4)^2$.

b.



The parent function is $f(x) = \sqrt{x}$. The constant value added to $f(x)$ is 5 because the graph is shifted 5 units up. The function defined by the translated graph is $g(x) = \sqrt{x} + 5$.

c.



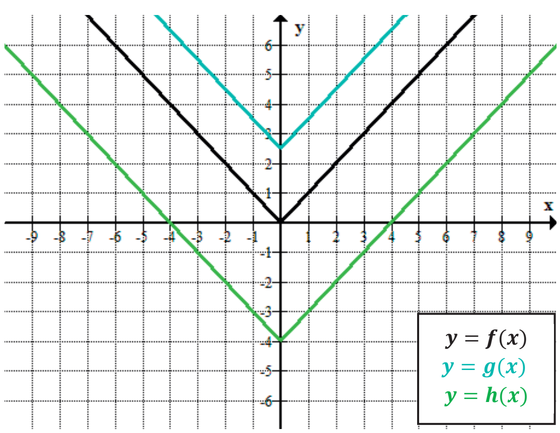
The parent function is $f(x) = |x|$. The constant values added to $f(x)$ are -3 and $+2$ because the graph is shifted 3 units down and 2 units to the left. The function defined by the translated graph is $g(x) = |x + 2| - 3$.

Exercises (20 minutes)

Exercises

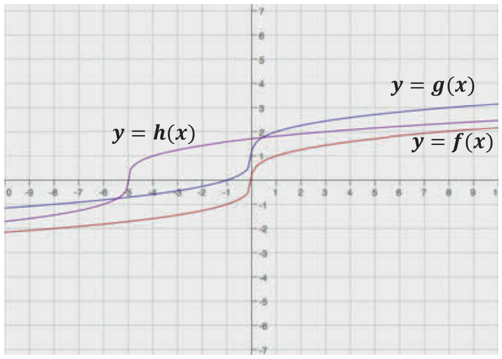
1. **For each of the following graphs, use the formula for the parent function f to write the formula of the translated function.**

   a.



Parent Function: $f(x) = |x|$
Translated Functions: $g(x) = |x| + 2.5$, $h(x) = |x| - 4$

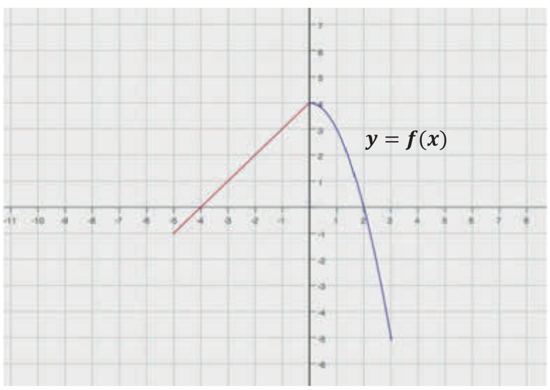
b.

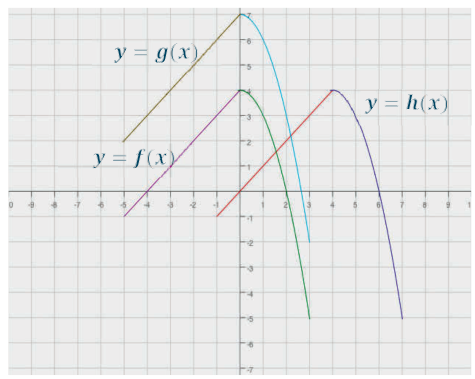


Parent Function: $f(x) = \sqrt[3]{x}$
Translated Functions: $g(x) = \sqrt[3]{x} + 1$, $h(x) = \sqrt[3]{x+5}$

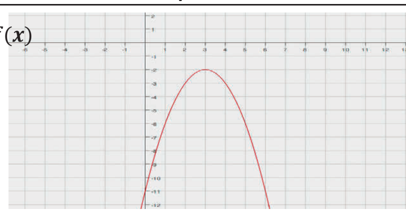
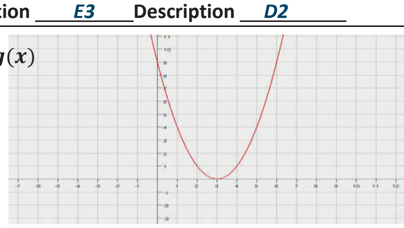
2. Below is a graph of a piecewise function f whose domain is $-5 \le x \le 3$. Sketch the graphs of the given functions on the same coordinate plane. Label your graphs correctly.

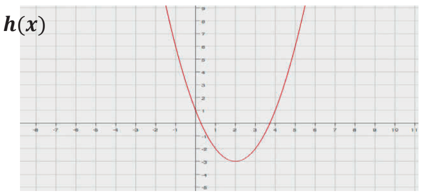
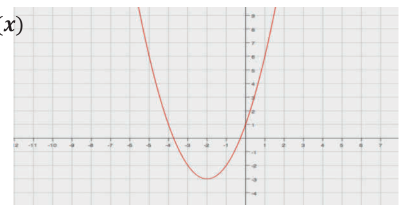
$g(x) = f(x) + 3$ $h(x) = f(x-4)$





3. Match the correct equation and description of the function with the given graphs.

Graphs	Equation	Description
$y = f(x)$  Equation *E3* Description *D2*	**E1.** $y = (x-3)^2$ **E2.** $y = (x+2)^2 - 3$ **E3.** $y = -(x-3)^2 - 2$	**D1.** The graph of the parent function is translated down 3 units and left 2 units. **D2.** The graph of the function does not have an x-intercept.
$y = g(x)$  Equation *E1* Description *D4*	**E4.** $y = (x-2)^2 - 3$	**D3.** The coordinate of the y-intercept is $(0, 1)$, and both x-intercepts are positive. **D4.** The graph of the function has only one x-intercept.

$y = h(x)$  Equation ___*E4*___ Description ___*D3*______		
$y = p(x)$  Equation ___*E2*___ Description ___*D1*______		

Closing (3 minutes)

- Given any function, how does adding a positive or negative value, k, to $f(x)$ or x affect the graph of the parent function?
  - *The value of the constant k shifts the graph of the original function k units up (if $k > 0$) and k units down (if $k < 0$) if k is added to $f(x)$ such that the new function is $g(x) = f(x) + k$. The value of k shifts the graph of the original function k units to the left (if $k > 0$) and k units to the right (if $k < 0$) if k is added to x such that the new function is $g(x) = f(x + k)$.*

Exit Ticket (5 minutes)

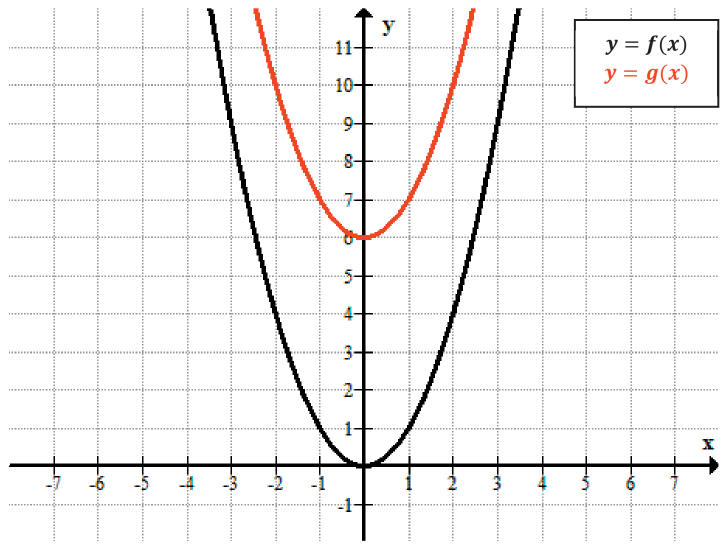
Students analyze the graph and critique the reasoning of others. Then, they provide a valid explanation about their argument.

Name ______________________________ Date ______________

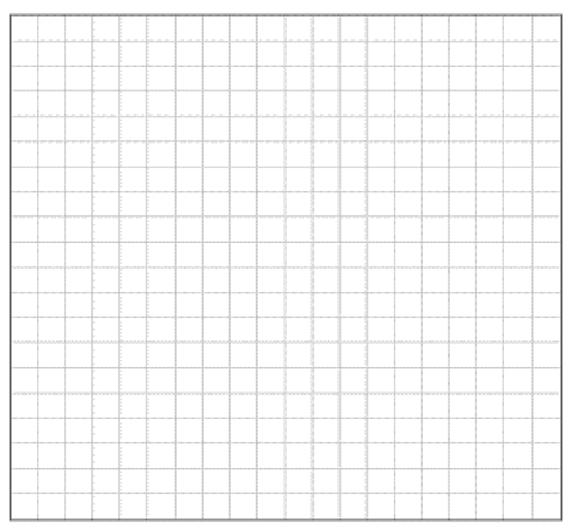
Lesson 19: Translating Graphs of Functions

Exit Ticket

1. Ana sketched the graphs of $f(x) = x^2$ and $g(x) = x^2 - 6$ as shown below. Did she graph both of the functions correctly? Explain how you know.



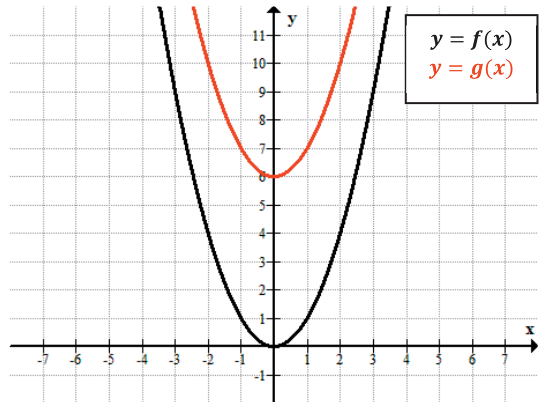
2. Use transformations of the graph of $f(x) = \sqrt{x}$ to sketch the graph of $f(x) = \sqrt{x-1} + 3$.



Exit Ticket Sample Solutions

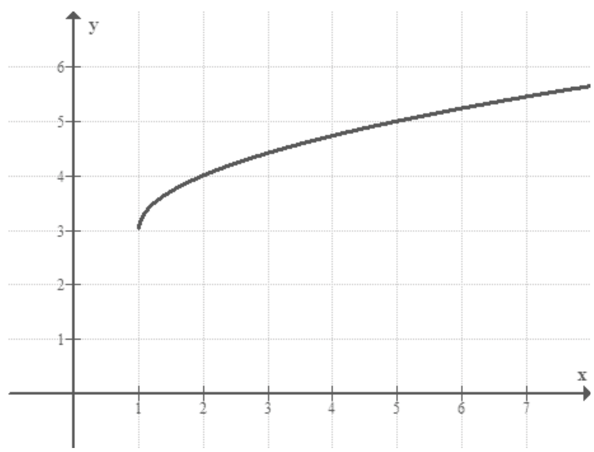
1. Ana sketched the graphs of $f(x) = x^2$ and $g(x) = x^2 - 6$ as shown below. Did she graph both of the functions correctly? Explain how you know.

The function f was graphed correctly, but not g. The graph of g should have been translated 6 units below the graph of f.



2. Use transformations of the graph of $f(x) = \sqrt{x}$ to sketch the graph of $f(x) = \sqrt{x-1} + 3$.

The graph should depict the graph of the square root function translated 1 unit right and 3 units up.



Problem Set Sample Solutions

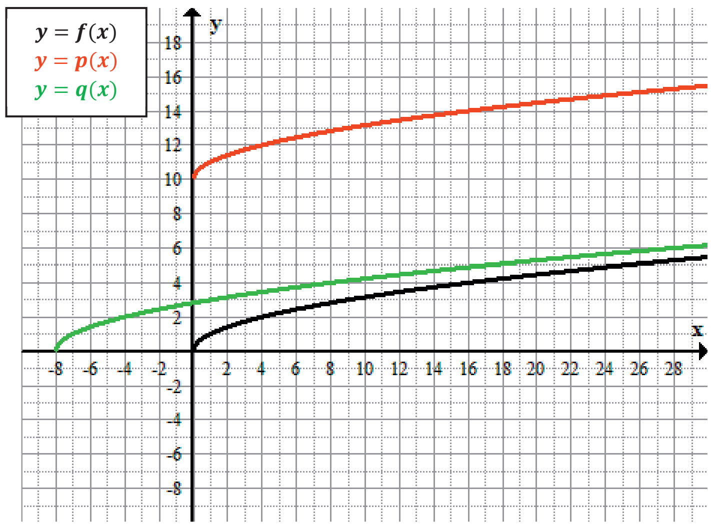
Students should complete these problems without using a graphing calculator. The following solutions indicate an understanding of the lesson objectives.

1. **Graph the functions in the same coordinate plane. Do not use a graphing calculator.**

   $f(x) = \sqrt{x}$

   $p(x) = 10 + \sqrt{x}$

   $q(x) = \sqrt{x + 8}$



2. **Write a function that translates the graph of the parent function $f(x) = x^2$ down 7.5 units and right 2.5 units.**

   $f(x) = (x - 2.5)^2 - 7.5$

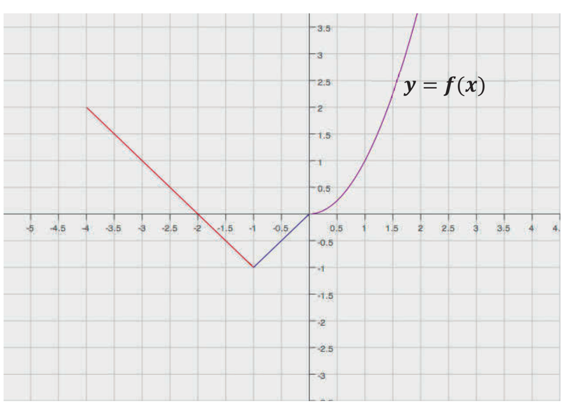
3. **How would the graph of $f(x) = |x|$ be affected if the function were transformed to $f(x) = |x + 6| + 10$?**

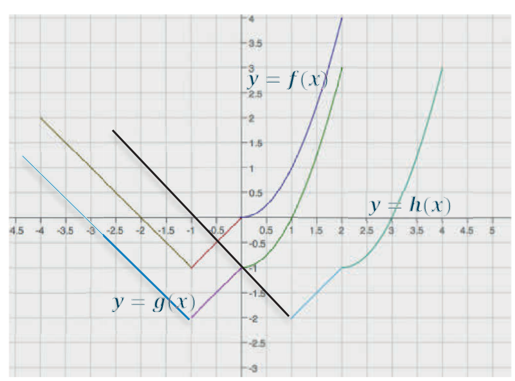
   The graph would be shifted 10 units up and 6 units to the left.

4. **Below is a graph of a piecewise function f whose domain is the interval $-4 \le x \le 2$. Sketch the graph of the given functions below. Label your graphs correctly.**

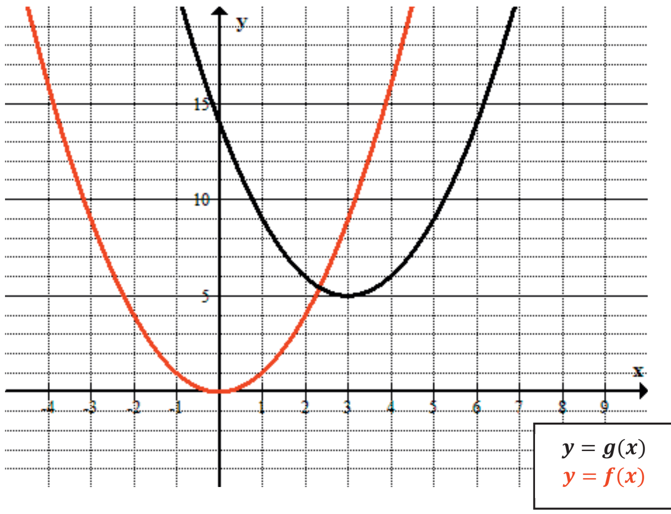
   $g(x) = f(x) - 1$ $\quad$ $h(x) = g(x - 2)$ **[Be careful; this one might be a challenge.]**

   Point out that the graph of h is related to g rather than f. Make sure students recognize that they must find the graph of g first, and then translate it to find h.





5. **Study the graphs below. Identify the parent function and the transformations of that function depicted by the second graph. Then, write the formula for the transformed function.**



The parent function is $f(x) = x^2$, in red. The graph of the transformed function, in black, is the graph of $y = f(x)$ shifted 3 units to the right and 5 units up. The function defined by the translated graph is $g(x) = (x - 3)^2 + 5$.

Lesson 20: Stretching and Shrinking Graphs of Functions

Student Outcomes

- Students recognize and use parent functions for absolute value, quadratic, square root, and cube root to perform transformations that stretch and shrink the graphs of the functions. They identify the effect on the graph of $y = f(x)$ when $f(x)$ is replaced with $kf(x)$ and $f(kx)$, for any specified value of k, positive or negative, and identify the constant value, k, given the graphs of the parent functions and the transformed functions. Students write the formulas for the transformed functions given their graphs.

Lesson Notes

In Lesson 19, students learned how to write the formulas for the graphs of parent functions (including quadratic, square root, and cube root) that were translated up, down, right, or left by k units. In this lesson, students extend what they learned in Module 3 about how multiplying the parent function by a constant or multiplying the x-values of the parent function results in the shrinking or stretching (scaling) of the graph of the parent function and, in some cases, results in the reflection of the function about the y- or x-axis. In this lesson, we will review some of Module 3's work with quadratic functions but will focus on cubic, square root, and cube root functions.

Classwork

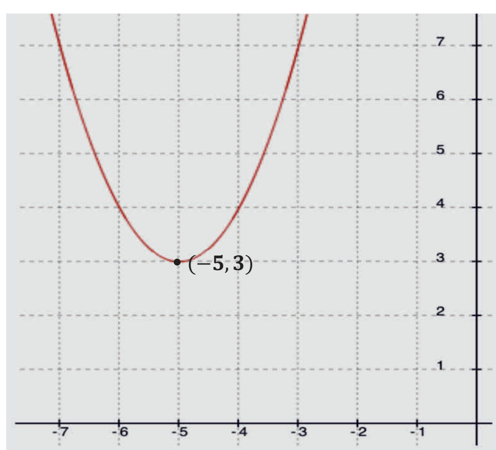
Opening Exercise (4 minutes)

Opening Exercise

The graph of a quadratic function defined by $f(x) = x^2$ has been translated 5 units to the left and 3 units up. What is the formula for the function, g, depicted by the translated graph?

$g(x) = (x + 5)^2 + 3$

Sketch the graph of the equation $y = g(x)$.

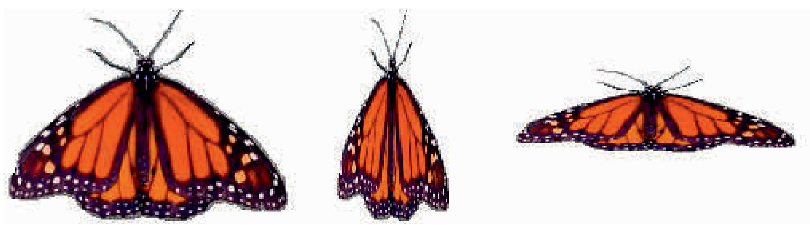


Example (1 minute)

Have students compare the photographs below of a monarch butterfly; then, ask a few to share their observations.

- What do you notice about the three pictures of the same monarch butterfly?
  - *The picture of the butterfly has been stretched (enlarged) and shrunk (compressed).*

Example



> *Scaffolding:*
>
> If students do not readily see the possibilities of stretching or shrinking a function, you might try asking them to look at the graphs of $y = x^2$, $y = \frac{1}{2}x^2$, and $y = 2x^2$.
>
> You may need to guide them to the conclusion that multiplying by a number greater than 1 makes the curve narrower, while multiplying by a number between 0 and 1 makes the graph wider. While on the subject, try $y = -x^2$ to show that multiplying by a negative number turns the function upside down.

- Is it possible to shrink or stretch the graph of a function? If so, how might that happen?
  - *Yes, since we discovered that adding or subtracting a value to the parts of a parent function shifts its graph horizontally or vertically, it is possible that multiplying or dividing will shrink or stretch a function. Note that students may respond with comments about the points of the graph being pushed together or spread apart.*

In Exercise 1, students analyze the graphs and tables of parent functions and their transformations. They make use of the structure of the equations representing the functions and look for patterns in the tables and graphs that allow them to make generalizations about how to recognize when a function is being enlarged or compressed and how to quickly sketch a graph of a function under those circumstances.

Exploratory Challenge (20 minutes)

Have students work in pairs or small groups. Have the groups pause after each part to have a class discussion and to compare their findings. Make sure students have calculators and enough graph paper for at least four good-sized graphs. Remind them that this is largely a review of work done in Module 3.

Exploratory Challenge

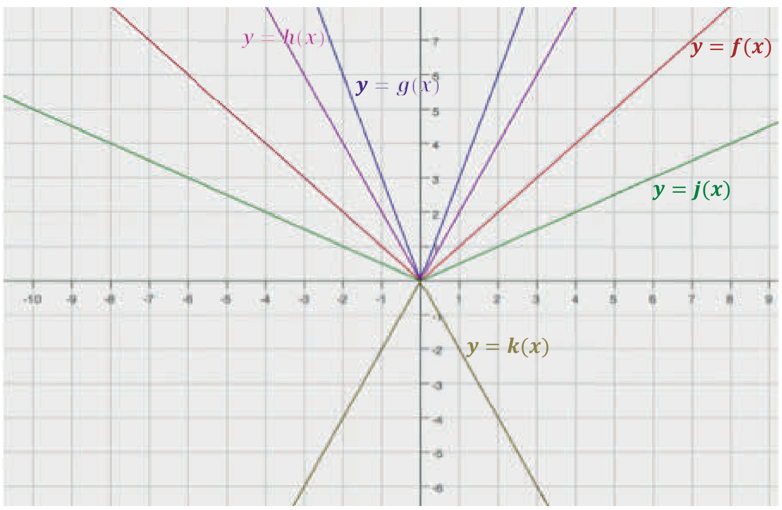
Complete the following to review Module 3 concepts:

a. **Consider the function $f(x) = |x|$. Complete the table of values for $f(x)$. Then, graph the equation $y = f(x)$ on the coordinate plane provided for part (b).**

x	$f(x)$
-4	4
-2	2
0	0
2	2
4	4

b. Complete the following table of values for each transformation of the function f. Then, graph the equations $y = g(x)$, $y = h(x)$, $y = j(x)$, and $y = k(x)$ on the same coordinate plane as the graph of $y = f(x)$. Label each graph.

x	$f(x)$	$g(x) = 3f(x)$	$h(x) = 2f(x)$	$j(x) = 0.5f(x)$	$k(x) = -2f(x)$
-4	4	12	8	2	-8
-2	2	6	4	1	-4
0	0	0	0	0	0
2	2	6	4	1	-4
4	4	12	8	2	-8



c. Describe how the graph of $y = kf(x)$ relates to the graph of $y = f(x)$ for each case.

i. $k > 1$

The graph is stretched vertically by a factor equal to k.

ii. $0 < k < 1$

The graph is shrunk vertically by a factor equal to k.

iii. $k = -1$

The graph is reflected across the x-axis.

iv. $-1 < k < 0$

The graph is reflected across the x-axis and shrunk vertically by a factor equal to $|k|$.

v. $k < -1$

The graph is reflected across the x-axis and stretched vertically by a factor equal to $|k|$.

Scaffolding:

For visual learners, ask students to identify two points on the parent function, say $(1, 1)$ and $(2, 2)$. Then, have them draw vertical lines passing through these points. Now name the points on the graph of $y = h(x)$. These points are $(1, 2)$ and $(2, 4)$. Have them find the ratio of the y-values of the transformed graph and the y-values of the parent function as follows:

For $(1, 1)$ and $(1, 2)$:

$$\frac{2}{1} = 2;$$

For $(2, 2)$ and $(2, 4)$:

$$\frac{4}{2} = 2.$$

This common ratio, 2, is the stretch or shrink-scaling factor.

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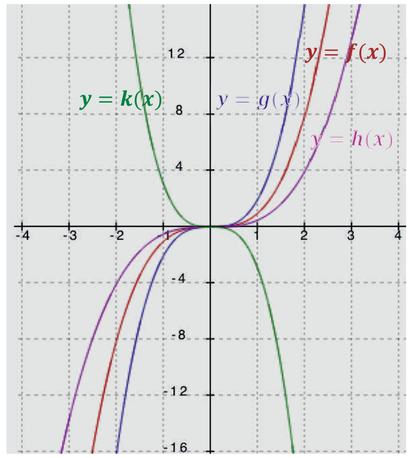
d. **Describe the transformation of the graph of f that results in the graphs of g, h, and k given the following formulas for each function. Then, graph each function and label each graph.**

$f(x) = x^3$

$g(x) = 2x^3$

$h(x) = 0.5x^3$

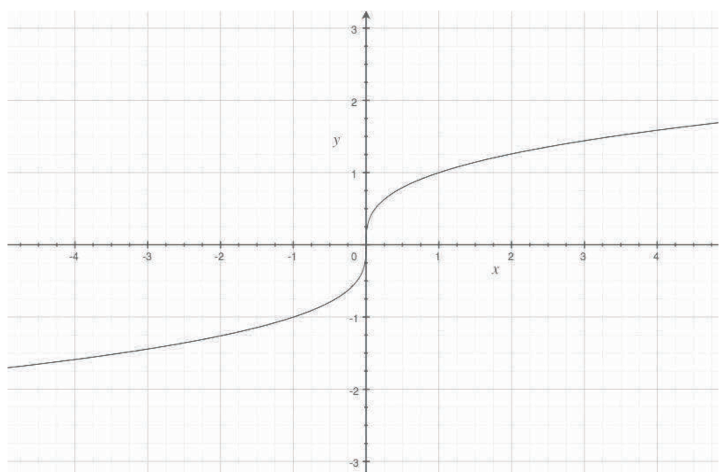
$k(x) = -3x^3$



The graph of g shows a vertically stretched graph of f with a scale factor of 2. The graph of h is a vertically shrunk, or compressed, graph of f with a scale factor of 0.5. The graph of k shows a vertically stretched graph of f with a scale factor of 3 and is reflected across the x-axis.

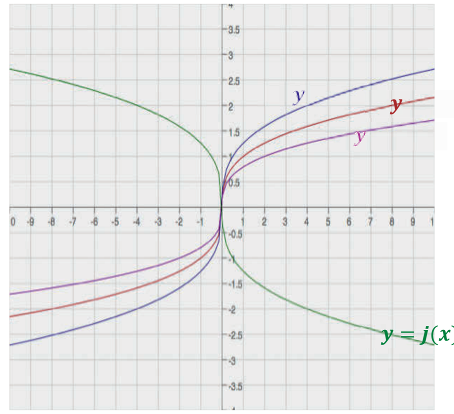
e. **Consider the function $f(x) = \sqrt[3]{x}$. Complete the table of values; then graph the equation $y = f(x)$.**

x	$f(x)$
-8	-2
-1	-1
0	0
1	1
8	2



f. Complete the following table of values, rounding each value to the nearest hundredth. Graph the equations $y = g(x)$, $y = h(x)$, and $y = j(x)$ on the same coordinate plane as your graph of $y = f(x)$ above. Label each graph.

x	$f(x)$	$g(x) = f(2x)$	$h(x) = f(0.5x)$	$j(x) = f(-2x)$
-8	-2	-2.52	-1.59	2.52
-1	-1	-1.26	-0.79	1.26
0	0	0	0	0
1	1	1.26	0.79	-1.26
8	2	2.52	1.59	-2.52



g. Describe the transformations of the graph of f that result in the graphs of g, h, and j.

When the x-values of f are multiplied by 2, the graph is shrunk horizontally by a factor of 0.5. When the x-values of f are multiplied by 0.5, the graph is stretched horizontally by a factor of 2. When the x-values of f are multiplied by -2, the graph is shrunk horizontally by a factor of 0.5 and is reflected about the y-axis.

h. Describe how the graph of $y = f\left(\frac{1}{k}x\right)$ relates to the graph of $y = f(x)$ for each case.

i. $k > 1$

The graph is stretched horizontally by a factor equal to k.

ii. $0 < k < 1$

The graph is shrunk horizontally by a factor equal to k.

iii. $k = -1$

The graph is reflected across the y-axis.

iv. $-1 < k < 0$

The graph is shrunk horizontally by a factor equal to $|k|$ and is reflected across the y-axis.

v. $k < -1$

The graph is stretched horizontally by a factor equal to $|k|$ and is reflected across the y-axis.

- Is it possible to transform the square root function by a horizontal stretch or shrink using a negative scale factor? Why or why not?
    - *Yes, it will work for a different limited domain. For example, $f(x) = \sqrt{x}$ has all nonnegative numbers as its domain. However, multiplying the x-values by -1 gives us $g(x) = \sqrt{(-x)}$, which is a congruent graph but with a domain of all numbers less than or equal to 0, and so is a reflection of f across the y-axis.*

Exercise 1 (8 minutes)

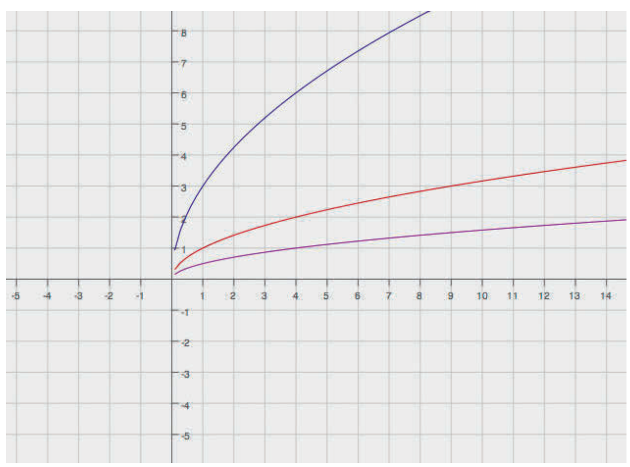
Work through Exercise 1 as a class, perhaps posting the graphs on the board as you go through the questions.

Exercise 1

For each set of graphs below, answer the following questions:

- **What are the parent functions?**
- **How does the transformed graph relate to the graph of the parent function?**
- **Write the formula for the function depicted by the transformed graph.**

a.

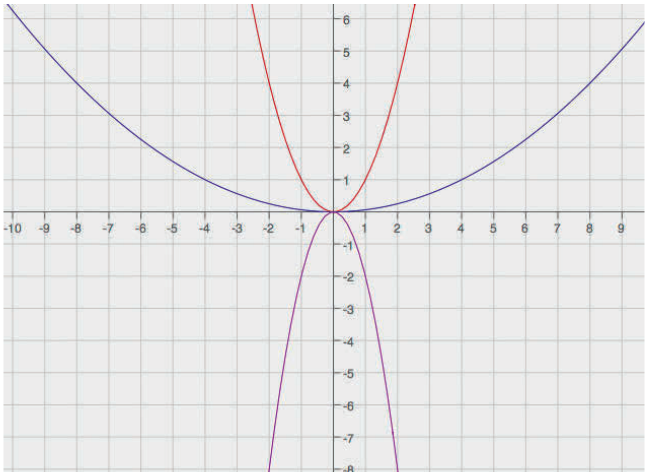


The parent function (in red) is $f(x) = \sqrt{x}$. The graph in blue is a vertical scaling of the graph of f with a scale factor of 3. The function depicted by the blue graph is $g(x) = 3\sqrt{x}$. The other graph (in pink) is a vertical scaling of the graph of f with a scale factor of 0.5. The function depicted by the graph is $h(x) = 0.5\sqrt{x}$.

Scaffolding:

Lead struggling students to identify the stretch or shrink factor of the blue graph. They might not be able to identify the points readily from the graph of the parent function (red), so have them identify points on the blue graph first. Then, have them solve for the y-values on the red graph using the same x-value on the blue graph. Once the points are identified, they can find the ratio.

b.



The parent function (in red) is $f(x) = x^2$. The graph in blue is a horizontal scaling of the graph of f with a scale factor of 4. The function depicted by the blue graph is $g(x) = \left(\frac{1}{4}x\right)^2$. The other graph (in pink) is a vertical scaling of the graph of f with a scale factor of 2 and is reflected over the x-axis. The function depicted by the graph is $h(x) = -2x^2$.

For the graph in blue, the function could also be written as $g(x) = 0.0625x^2$. In this case, the students could also say that the graph of $f(x) = x^2$ has been shrunk vertically by a factor of 0.0625. The two interpretations of f and g are both correct. This reflects the nature of the specific (quadratic) function; however, it is not a general property of all functions.

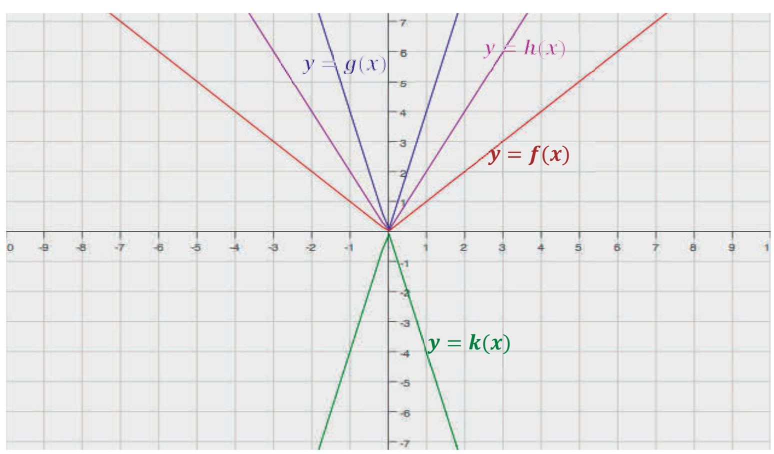
Exercise 2 (8 minutes)

Exercise 2

Graph each set of functions in the same coordinate plane. Do not use a graphing calculator.

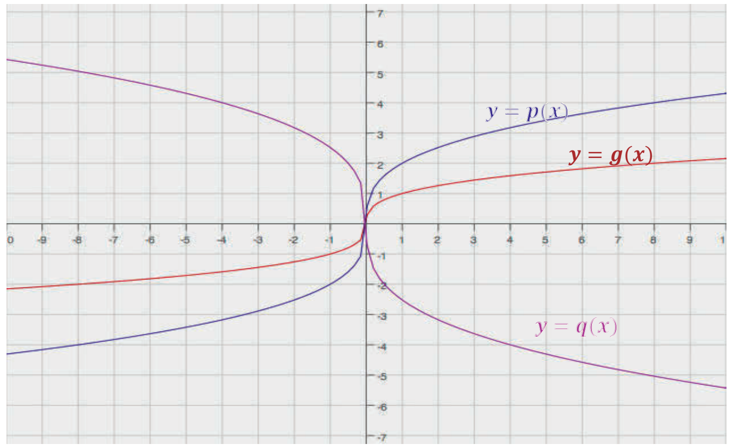
a. $f(x) = |x|$
$g(x) = 4|x|$
$h(x) = |2x|$
$k(x) = -2|2x|$

$y = f(x)$ in red
$y = g(x)$ in purple
$y = h(x)$ in pink
$y = k(x)$ in green



b. $g(x) = \sqrt[3]{x}$
$p(x) = 2\sqrt[3]{x}$
$q(x) = -2\sqrt[3]{2x}$

$y = g(x)$ in red
$y = p(x)$ in purple
$y = q(x)$ in pink



Closing (2 minutes)

Discuss how the vertical scaling by a scale factor of k of the graph of a function $y = f(x)$ corresponds to changing the equation of the graph from $y = f(x)$ to $y = kf(x)$. Investigate the four cases of k:

1. $k > 1$
2. $0 < k < 1$
3. $-1 < k < 0$
4. $k < -1$

Then, discuss how the horizontal scaling by a scale factor of k of the graph of a function $y = f(x)$ corresponds to changing the equation of the graph from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$. Investigate the four cases of k:

1. $k > 1$
2. $0 < k < 1$
3. $-1 < k < 0$
4. $k < -1$

Exit Ticket (2 minutes)

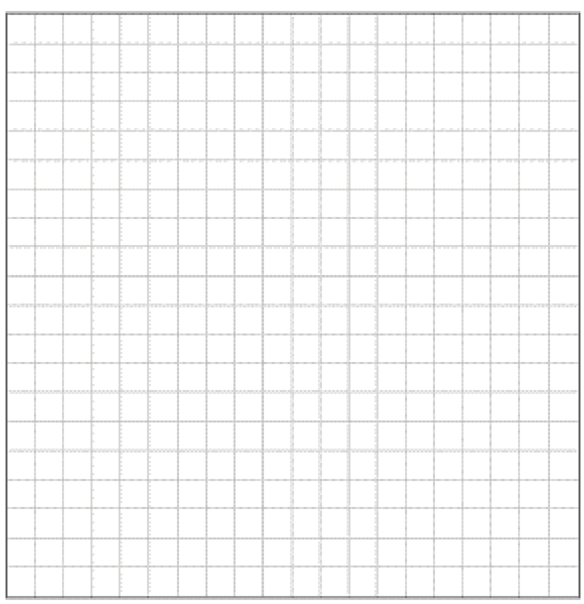
Name ______________________________ Date ______________

Lesson 20: Stretching and Shrinking Graphs of Functions

Exit Ticket

1. How would the graph of $f(x) = \sqrt{x}$ be affected if it were changed to $g(x) = -2\sqrt{x}$?

2. Sketch and label the graphs of both f and g on the grid below.

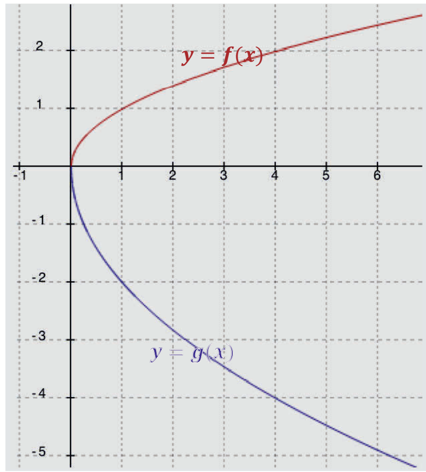


Exit Ticket Sample Solutions

1. How would the graph of $f(x) = \sqrt{x}$ be affected if it were changed to $g(x) = -2\sqrt{x}$?

   The graph of f would be stretched vertically by a factor of 2 and reflected across the x-axis.

2. Sketch and label the graphs of both f and g on the grid below.



Problem Set Sample Solutions

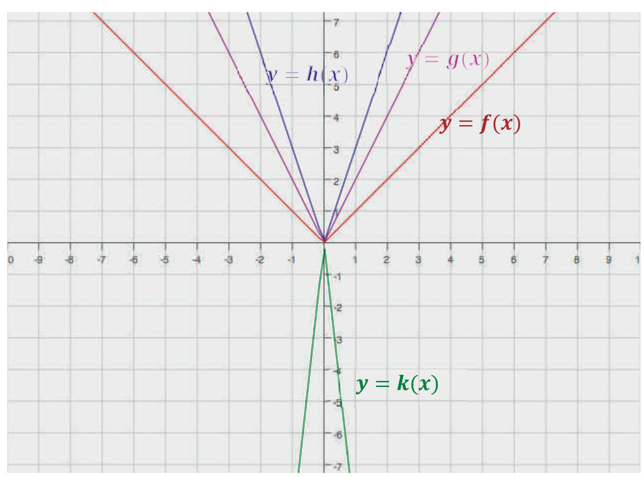
1. Graph the functions in the same coordinate plane. Do not use a graphing calculator.

   $f(x) = |x|$

   $g(x) = 2|x|$

   $h(x) = |3x|$

   $k(x) = -3|3x|$



2. Explain how the graphs of functions $j(x) = 3|x|$ and $h(x) = |3x|$ are related.

   Each of these transformations of the absolute value functions creates the same graph.

3. **Explain how the graphs of functions $q(x) = -3|x|$ and $r(x) = |-3x|$ are related.**

The two graphs have the same scaling factor of 3, but they are reflections of each other across the x-axis. Multiplying an absolute value by a negative number will reflect it across the x-axis. However, multiplying by a negative number INSIDE the absolute value has the same effect as multiplying by a positive number on the outside.

4. **Write a function, g, in terms of another function, f, such that the graph of g is a vertical shrink of the graph f by a factor of 0.75.**

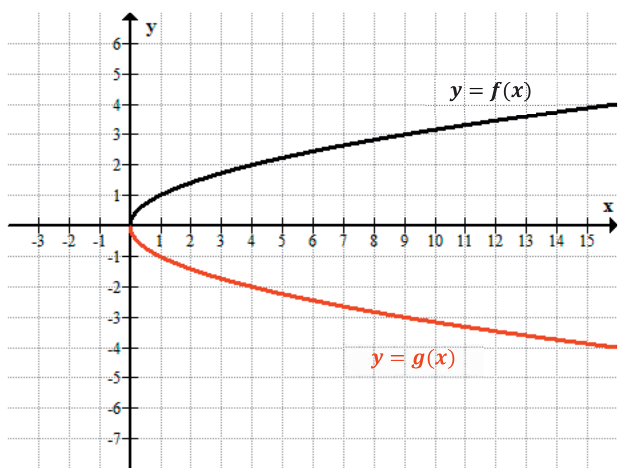
$g(x) = 0.75\,f(x)$

In Problem 5, students critique the reasoning of each answer, determine which of the two is correct, and provide a justification for their response.

5. **A teacher wants the students to write a function based on the parent function $f(x) = \sqrt[3]{x}$. The graph of f is stretched vertically by a factor of 4 and shrunk horizontally by a factor of $\frac{1}{3}$. Mike wrote $g(x) = 4\sqrt[3]{3x}$ as the new function, while Lucy wrote $h(x) = 3\sqrt[3]{4x}$. Which one is correct? Justify your answer.**

Mike is correct. A vertical stretch by a factor of 4 means multiplying $f(x)$ by 4, and a horizontal shrink by a factor of $\frac{1}{3}$ means that the x-values of $f(x)$ must be multiplied by 3.

6. **Study the graphs of two different functions below. Which is a parent function? What is the constant value(s) multiplied to the parent function to arrive at the transformed graph? Now write the function defined by the transformed graph.**



The parent function is $f(x) = \sqrt{x}$. The graph of $y = g(x)$ is the graph of $y = f(x)$ reflected across the x-axis. The function depicted by the transformed graph is $g(x) = -\sqrt{x}$.

Lesson 21: Transformations of the Quadratic Parent Function, $f(x) = x^2$

Student Outcomes

- Students make a connection between the symbolic and graphic forms of quadratic equations in the completed-square (vertex) form. They efficiently sketch a graph of a quadratic function in the form, $f(x) = a(x-h)^2 + k$, by transforming the quadratic parent function, $f(x) = x^2$, without the use of technology. They then write a function defined by a quadratic graph by transforming the quadratic parent function.

Lesson Notes

In the two preceding lessons, students learned how to translate the graph of the parent function by adding or subtracting a constant k to it or to its x-values and how to stretch or shrink the graph of the parent function by multiplying a constant k by it or by its x-values. In this lesson, the students are expected to do a combination of both, that is, translating and stretching or shrinking of the graph of the quadratic parent function, $f(x) = x^2$.

Throughout this lesson, students use the structure of the equations that are used to represent functions to determine the transformations of the quadratic parent function. They complete the square for quadratic functions given in other forms in order to identify when and by how much a function shifts and stretches or shrinks.

Classwork

Have students work in pairs or small groups to complete the square for the function below. Ask for justification for each step, but definitely pause at Step 3 to remind students about how balancing the equality should work for this problem.

Scaffolding:

- For students who struggle with this process, it may be helpful to guide them through the steps. At first, give a simpler function, such as $f(x) = x^2 + 6x$, and help them complete the square. Then, they can try Example 1.
- In Step 3 of this example, pause to ask students why there is a need to subtract 18 outside the parentheses after adding 9 on the inside.

Example 1 (8 minutes): Quadratic Expression Representing a Function

Example 1: Quadratic Expression Representing a Function

a. **A quadratic function is defined by $g(x) = 2x^2 + 12x + 1$. Write this in the completed-square (vertex) form and show all the steps.**

	$g(x) = 2x^2 + 12x + 1$	
Step 1	$= (2x^2 + 12x) + 1$	*Gather variable terms.*
Step 2	$= 2(x^2 + 6x) + 1$	*Factor out the GCF.*
Step 3	$= 2(x^2 + 6x + 9) + 1 - 18$	*Complete the square and balance the equality.*
Step 4	$= 2(x+3)^2 - 17$	*Factor the perfect square.*
	$g(x) = 2(x+3)^2 - 17$	

b. **Where is the vertex of the graph of this function located?**

The vertex is at $(-3, -17)$.

c. **Look at the completed-square form of the function. Can you name the parent function? How do you know?**

The parent function is $f(x) = x^2$. *The function is quadratic.*

d. **What transformations have been applied to the parent function to arrive at function g? Be specific.**

The parent function $f(x) = x^2$ *is translated* 3 *units to the left, stretched vertically by a factor of* 2, *and translated* 17 *units down.*

e. **How does the completed-square form relate to the quadratic parent function $f(x) = x^2$?**

The completed-square form can be understood through a series of transformations of the quadratic parent function, f.

Example 2 (5 minutes)

Have students work with a partner or small group to determine the function.

> *Scaffolding:*
>
> Visual learners may benefit from using their graphing calculator to verify that their function in Example 2 is indeed the correct transformation of $f(x) = x^2$.

Example 2

The graph of a quadratic function $f(x) = x^2$ has been translated 3 units to the right, vertically stretched by a factor of 4, and moved 2 units up. Write the formula for the function that defines the transformed graph.

$g(x) = 4(x - 3)^2 + 2$

- How did you arrive at your answer?
  - *The parent function is* $f(x) = x^2$. *Below are the steps in the process:*

    Translating 3 *units to the right:* $(x - 3)^2$

    Stretching vertically by a factor of 4: $4(x - 3)^2$

    Translating 2 *units up:* $4(x - 3)^2 + 2$

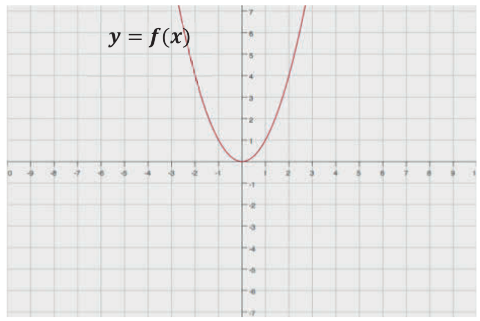
    New function: $g(x) = 4(x - 3)^2 + 2$

Exercise 1 (8 minutes)

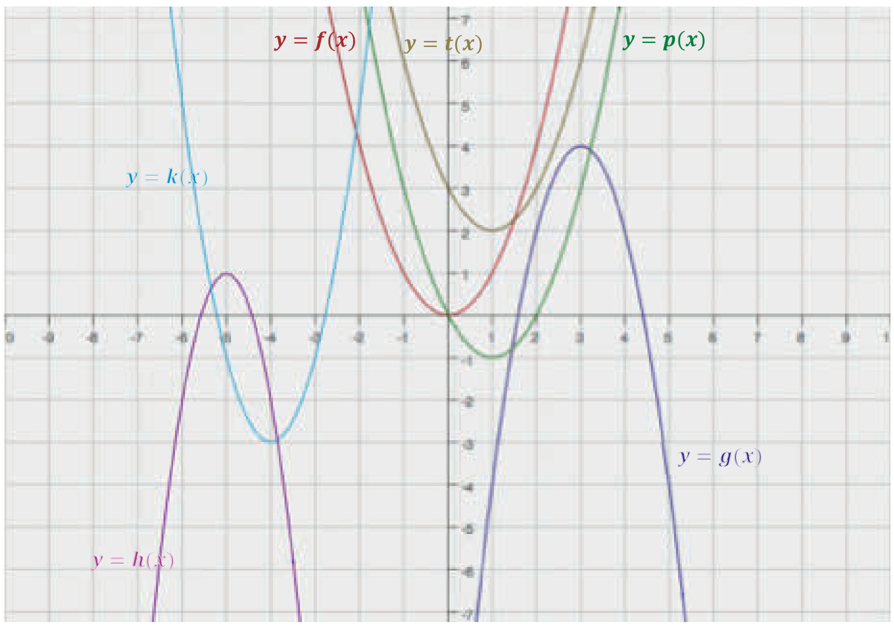
Have students work with a partner or small group to sketch the graphs of the following quadratic functions using the transformations of the parent function $f(x) = x^2$. Remind them that some of the functions need to be written in the completed-square form. Do not allow graphing calculators for this exercise.

Exercises

1. **Without using a graphing calculator, sketch the graph of the following quadratic functions on the same coordinate plane, using transformations of the graph of the parent function $f(x) = x^2$.**
   a. $g(x) = -2(x-3)^2 + 4$
   b. $h(x) = -3(x+5)^2 + 1$
   c. $k(x) = 2(x+4)^2 - 3$
   d. $p(x) = x^2 - 2x$
   e. $t(x) = x^2 - 2x + 3$



Note: By completing the square, we have $p(x) = (x-1)^2 - 1$ and $t(x) = (x-1)^2 + 2$.

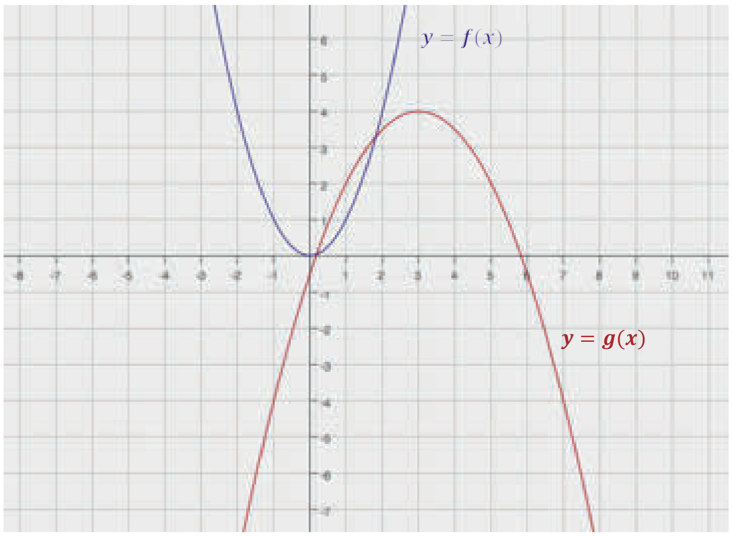


Exercises 2–4 (15 minutes)

2. Write a formula for the function that defines the described transformations of the graph of the quadratic parent function $f(x) = x^2$.
   a. 3 units shift to the right
   b. Vertical shrink by a factor of 0.5
   c. Reflection across the x-axis
   d. 4 units shift up

   Then, graph both the parent and the transformed functions on the same coordinate plane.

   $g(x) = -0.5(x-3)^2 + 4$



3. Describe the transformation of the quadratic parent function $f(x) = x^2$ that results in the quadratic function $g(x) = 2x^2 + 4x + 1$.

   First, rewrite $g(x)$ into the completed-square form.

$$\begin{aligned} g(x) &= 2x^2 + 4x + 1 \\ &= (2x^2 + 4x) + 1 \\ &= 2(x^2 + 2x) + 1 \\ &= 2(x^2 + 2x + 1) + 1 - 2 \\ &= 2(x^2 + 2x + 1) - 1 \\ g(x) &= 2(x+1)^2 - 1 \end{aligned}$$

   This means that the graph of f is translated 1 unit to the left, vertically stretched by a factor of 2, and translated 1 unit down.

4. **Sketch the graphs of the following functions based on the graph of the function $f(x) = x^2$. If necessary, rewrite some of the functions in the vertex (completed-square) form. Label your graphs.**

   a. $g(x) = -(x-4)^2 + 3$

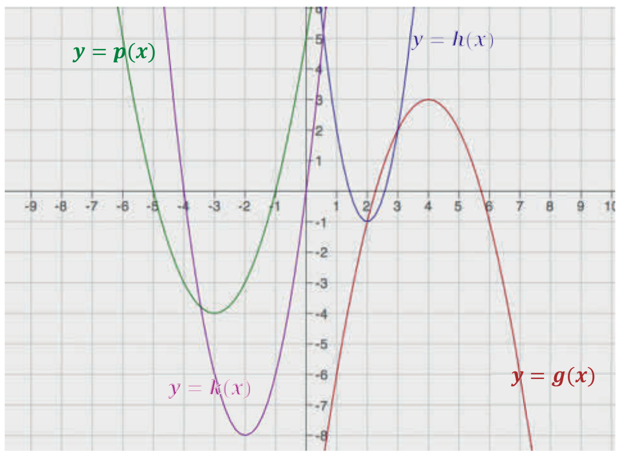
   b. $h(x) = 3(x-2)^2 - 1$

   c. $k(x) = 2x^2 + 8x$

   d. $p(x) = x^2 + 6x + 5$

   c. $k(x) = 2(x+2)^2 - 8$

   d. $p(x) = (x+3)^2 - 4$



Closing (4 minutes)

- How would you sketch the graph of any non-parent quadratic function written in the standard form without using a calculator or creating a table of values?
  - *For any non-parent quadratic function in standard form, we need to rewrite it in the completed-square form, and then identify the translations and the vertical shrink or stretch factor. We can also determine whether or not the graph faces up or down by the sign of the shrink or stretch factor.*

Lesson Summary

Transformations of the quadratic parent function, $f(x) = x^2$, can be rewritten in form $g(x) = a(x-h)^2 + k$, where (h, k) is the vertex of the translated and scaled graph of f, with the scale factor of a, the leading coefficient. We can then quickly and efficiently (without the use of technology) sketch the graph of any quadratic function in the form $f(x) = a(x-h)^2 + k$ using transformations of the graph of the quadratic parent function, $f(x) = x^2$.

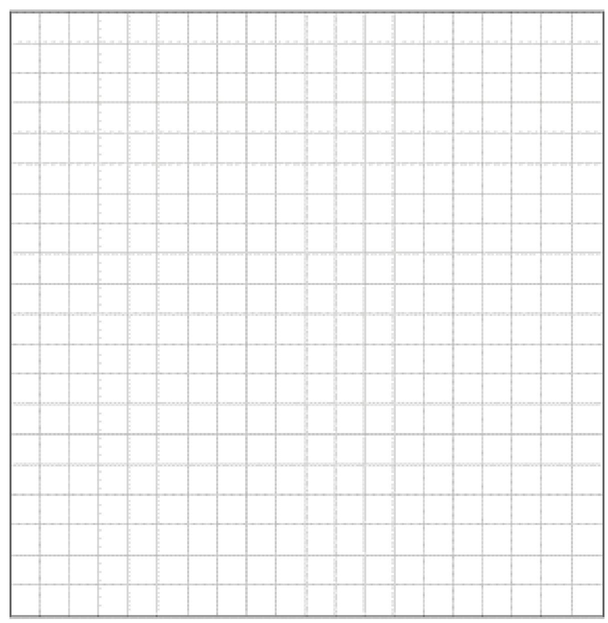
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 21: Transformations of the Quadratic Parent Function, $f(x) = x^2$

Exit Ticket

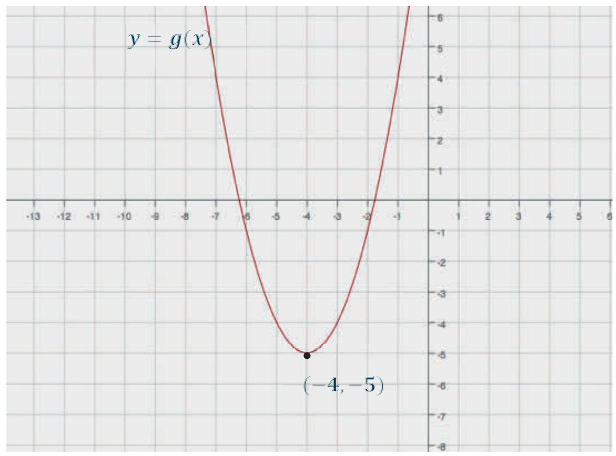
Describe in words the transformations of the graph of the parent function $f(x) = x^2$ that would result in the graph of $g(x) = (x + 4)^2 - 5$. Graph the equation $y = g(x)$.



Exit Ticket Sample Solutions

Describe in words the transformations of the graph of the parent function $f(x) = x^2$ that would result in the graph of $g(x) = (x+4)^2 - 5$. Graph the equation $y = g(x)$.

The graph of g is a translation of the graph of f, 4 units to the left and 5 units down.



Problem Set Sample Solutions

This Problem Set should be given as homework to reinforce what has been learned in the classroom. Encourage students to try working without calculators. The following solutions indicate an understanding of the objectives of this lesson.

1. **Write the function $g(x) = -2x^2 - 20x - 53$ in completed-square form. Describe the transformations of the graph of the parent function $f(x) = x^2$ that result in the graph of g.**

$$\begin{aligned} g(x) &= -2x^2 - 20x - 53 \\ &= (-2x^2 - 20x) - 53 \\ &= -2(x^2 + 10x) - 53 \\ &= -2(x^2 + 10x + 25) - 53 + 50 \\ &= -2(x^2 + 10x + 25) - 3 \\ g(x) &= -2(x+5)^2 - 3 \end{aligned}$$

The graph of f is translated 5 units to the left, vertically stretched by a factor of 2, and translated 3 units down. The graph of f is facing up, while the graph of g is facing down because of the negative value of a.

2. **Write the formula for the function whose graph is the graph of $f(x) = x^2$ translated 6.25 units to the right, vertically stretched by a factor of 8, and translated 2.5 units up.**

$g(x) = 8(x - 6.25)^2 + 2.5$

3. Without using a graphing calculator, sketch the graphs of the functions below based on transformations of the graph of the parent function $f(x) = x^2$. Use your own graph paper, and label your graphs.

a. $g(x) = (x+2)^2 - 4$

b. $h(x) = -(x-4)^2 + 2$

c. $k(x) = 2x^2 - 12x + 19$

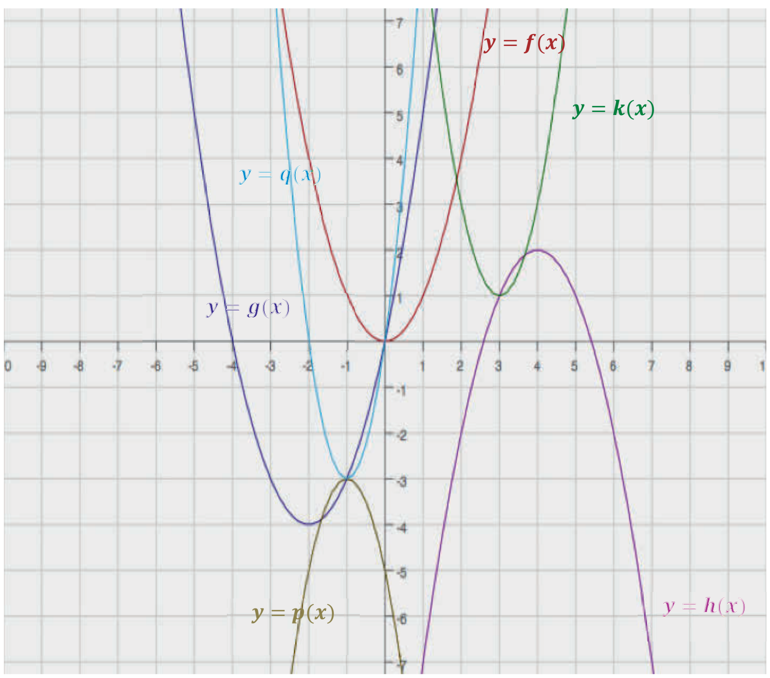
d. $p(x) = -2x^2 - 4x - 5$

e. $q(x) = 3x^2 + 6x$

c. $k(x) = 2(x-3)^2 + 1$

d. $p(x) = -2(x+1)^2 - 3$

e. $q(x) = 3(x+1)^2 - 3$



Lesson 22: Comparing Quadratic, Square Root, and Cube Root Functions Represented in Different Ways

Student Outcomes

- Students compare two different quadratic, square root, or cube root functions represented as graphs, tables, or equations. They interpret, contextualize, and abstract various scenarios to complete the comparative analysis.

Classwork

Opening Exercise (10 minutes)

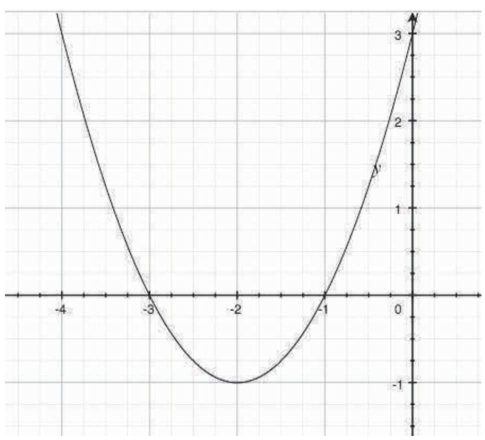
Project the graph on the board or screen. Have students work in pairs or small groups to select domain values and fill in the table based on the graph.

Scaffolding:

Provide students with the discussion questions ahead of time so that they have time to brainstorm responses prior to the class discussion.

Opening Exercise

Populate the table on the right with values from the graph.



x	y
-4	3
-3	0
-2	-1
-1	0
0	3

Briefly discuss ways to recognize key features in both representations of this function.

- What is the vertex for the function? Find it and circle it in both the table and the graph.
  - $(-2, -1)$
- What is the y-intercept for the function? Find it and circle it in both the table and the graph.
  - $(0, 3)$
- What are the x-intercepts for the function? Find them and circle them in both the table and the graph.
  - $(-3, 0)$ *and* $(-1, 0)$

- What components of the equation for a function give us clues for identifying the key features of a graph?
  - *Given a quadratic function in the form $f(x) = ax^2 + bx + c$, the y-intercept is represented by the constant, c; the vertex (h, k) can be seen in the completed-square form, $f(x) = a(x - h)^2 + k$; and the zeros of the function are found most readily in the factored form, $f(x) = a(x - m)(x - n)$.*
- How can the key features of the graph of a quadratic function give us clues about how to write the function the graph represents?
  - *Given a graph of a quadratic function, (h, k) represents the vertex (i.e., maximum or minimum point). These values, h and k, can be substituted into the vertex form. Then, substituting any other ordered pair for (x, y), which represents a point on the quadratic curve (e.g., the y-intercept), will allow you to solve for the leading coefficient, a, of the vertex form of a quadratic function.*
  - *When both x-intercepts are visible, we can write the equation of the graph in factored form using the coordinates of any other point to determine the leading coefficient. And when the y-intercept and only one x-intercept are visible, we can most easily write the function by using standard form. In this case, the y-intercept tells us the value of the constant term, c, and we can use two other points to substitute for x and y into the form $f(x) = ax^2 + bx + c$ to determine the specific values for a and b.*

Exploratory Challenges 1–3 (25 minutes)

Have students work on the exercises below in pairs or small groups. Note that the equation for $S(t)$ will be messy to complete the square. Some students may need help with the fractions or decimals, and all will need a calculator.

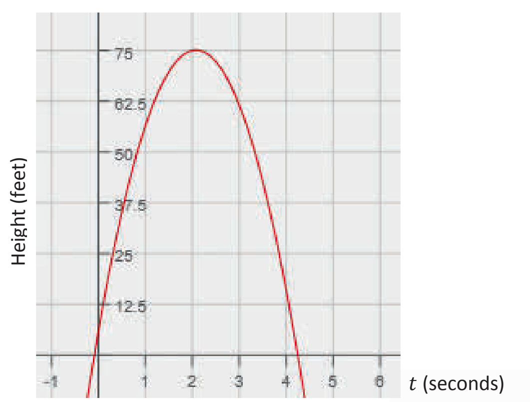
Exploratory Challenges 1–3

Solve each problem, and show or explain how you found your answers.

1. **Xavier and Sherleese each threw a baseball straight up into the air. The relationship between the height (distance from the ground in feet) of Sherleese's ball with respect to the time since it was thrown, in seconds, is given by the function:**

$$S(t) = -16t^2 + 79t + 6.$$

The graph of the height as a function of time of Xavier's ball is represented below.



Xavier claims that his ball went higher than Sherleese's. Sherleese disagrees. Answer the questions below, and support your answers mathematically by comparing the features found in the equation to those in the graph.

Scaffolding:
Struggling students may use graphing calculators to compare the equation with the graph or to check their answers.

a. **Who is right?**

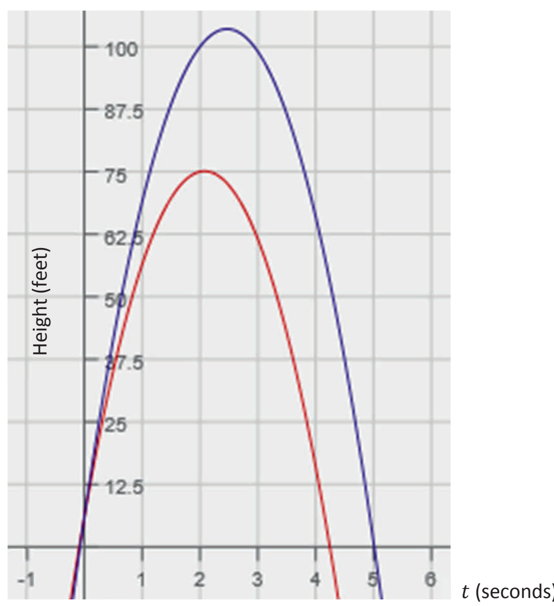
Sherleese is right. Xavier's baseball went 75 ft. at its highest point (based on the maximum point of the graph), which was lower than Sherleese's. The maximum height of Sherleese's ball is approximately 103.5 ft. This could be determined by completing the square, $S(t) = -16\left(t - \frac{79}{32}\right)^2 + 103.5$, using the vertex formula to find the vertex, or using a table to approximate.

b. **For how long was each baseball airborne?**

Sherleese's baseball was airborne for approximately 5 sec.; the zeros of S are $(5.01, 0)$ and $(-0.07, 0)$. Since the ball lands on the ground at about 5 sec., that is approximately how long it was in flight. Xavier's baseball was airborne for approximately 4.3 sec.

c. **Construct a graph of Sherleese's throw as a function of time (t) on the same set of axes as the graph of Xavier's, and use the graph to support your answers to parts (a) and (b).**

In the graph below, you can see that Sherleese's baseball clearly has a higher vertex. You can also read the t-axis to find the length of time each baseball was airborne (the distance between $t = 0$ and the positive t-intercept).



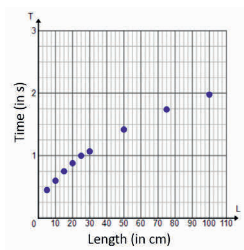
2. In science class, students constructed pendulums of various lengths and then recorded the time required for the pendulum to complete one full oscillation (out and back). The results are displayed in the table shown below.

Length (in cm)	Time (in seconds)
5	0.45
10	0.60
15	0.75
20	0.88
25	1.00
30	1.07
50	1.42
75	1.74
100	1.98

a. **Jack looks at the first three rows of the table and says that a linear function should be used to model the data. Based on the data, do you agree with him? Justify your reasoning.**

No, I do not agree with Jack. Looking at the first three rows, the data does appear to be linear because the average rate of change is the same for the intervals $[5, 10]$ and $[10, 15]$. However, after that, the average rate of change does not remain constant; it decreases for each successive interval. This indicates that a linear model is not appropriate for the data.

b. **Create a scatterplot of length versus oscillation time.**



c. **Based on the scatterplot, what sort of function might be used to model the data?**

It appears that the data could be modeled using a square root function or possibly a cube root function.

d. **Mr. Williams, the science teacher, tells the students that the oscillation time for a pendulum can be found using the formula $T = 2\pi\sqrt{\frac{L}{9.8}}$ where L is the length of the pendulum, in meters, and T is the oscillation time, in seconds. Does this formula support the results from the table?**

Yes. When values for L from the table are substituted into the formula, the values for T are approximately equal to the values in the table. For example, for a length of 25 cm, the oscillation time is approximately 1.0035 sec. (as shown in the calculation below), which is very close to the value recorded in the table (1.00).

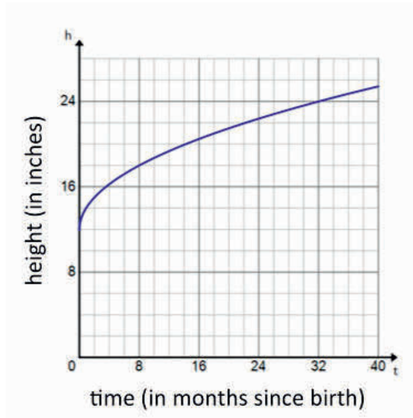
$$L = 0.25 \rightarrow T = 2\pi\sqrt{\frac{0.25}{9.8}} \approx 1.0035$$

e. **Looking at the table of values, what effect does quadrupling the length of the pendulum have on the oscillation time? Use the formula from part (d) to demonstrate why this is the case.**

Quadrupling the length of the pendulum, doubles the oscillation time.

$$T = 2\pi\sqrt{\frac{4L}{9.8}} = 2\pi\sqrt{4 \cdot \frac{L}{9.8}} = 2\pi \cdot \sqrt{4} \cdot \sqrt{\frac{L}{9.8}} = 2\left(2\pi\sqrt{\frac{L}{9.8}}\right)$$

3. **The growth of a Great Dane puppy can be represented by the graph below, where y represents the shoulder height (in inches) and x represents the puppy's age (in months).**



The growth of a lion cub can be modeled by the function represented in the table below.

x (monthssince birth)	y (height in inches)
0	8
8	18
16	20.599
24	22.422
32	23.874

a. **Which animal has the greater shoulder height at birth?**

The Great Dane has the taller shoulder height at birth, which is around 12 in. The lion cub has a shoulder height of only 8 in. at birth.

b. **Which animal will have the greater shoulder height at 3 years of age (the age each animal is considered full-grown)?**

Estimating from the graph, the lion's shoulder height at 36 months appears to be close to 25 in. Looking at the table of values, the average rate of change for each interval is decreasing. On the interval $[24, 32]$, the average rate of change is 1.452. So on the interval $[32, 36]$, the average rate of change should be less than 0.7. Therefore, I think the Great Dane's height at 36 months will be closer to 4 in. than to 25 in., and the lion will have the greater shoulder height at 3 years of age.

Scaffolding:

- As an extension, ask students the month when both animals will have the same shoulder height or what the shoulder height difference is at 64 months.
- For students who love a challenge, ask whether this graph will work for the entire life of the animal (and why or why not?). Have them think about whether a piecewise function may be needed to indicate that the animals may actually begin to shrink in height as they reach old age.

For part (b) of this exercise, students must estimate from a table and from a graph. In comparing those estimates, they will need to make some decisions about the precision of each answer.

c. **If you were told that the domain for these functions is the set of all real numbers, would you agree? Why or why not?**

There are physical limitations on time and height, so the domain must be greater than or equal to zero but must not exceed the finite limit of their life spans, and the range must both be greater than or equal to the finite heights recorded at birth, and less than or equal to their maximum possible heights. Both the domain and range may have other limitations as well. For example, neither animal will grow continuously for his lifetime; rather, both are likely to shrink if they live to an old age.

Closing (5 minutes)

- The critical values of a function, which are the zeros (roots), the vertex, and the leading coefficient, can be used to create and interpret the function in a context (e.g., the vertex represents the maximum or minimum value of a quadratic function).
- Graphing calculators and bivariate data tables are useful tools when comparing functions of the same type.

> **Lesson Summary**
>
> **The key features of a quadratic function, which are the zeros (roots), the vertex, and the leading coefficient, can be used to interpret the function in a context (e.g., the vertex represents the maximum or minimum value of the function). Graphing calculators and bivariate data tables are useful tools when comparing functions.**

Exit Ticket (5 minutes)

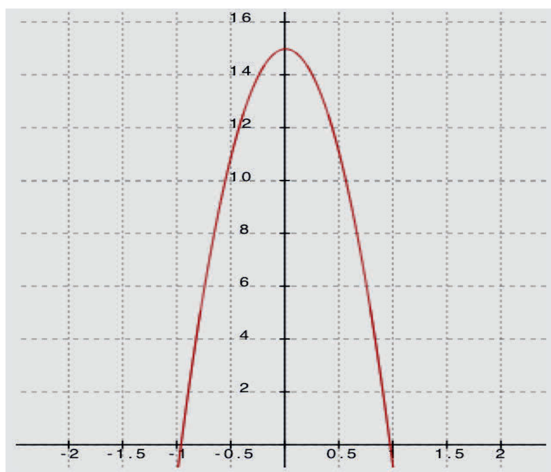
Name ____ Date ____________________

Lesson 22: Comparing Quadratic, Square Root, and Cube Root Functions Represented in Different Ways

Exit Ticket

Two people, each in a different apartment building, have buzzers that don't work. They both must throw their apartment keys out of the window to their guests, who will then use the keys to enter.

Tenant 1 throws the keys such that the height-time relationship can be modeled by the graph below. On the graph, time is measured in seconds, and height is measured in feet.



Tenant 2 throws the keys such that the relationship between the height of the keys (in feet) and the time that has passed (in seconds) can be modeled by $h(t) = -16t^2 + 18t + 9$.

a. Whose window is higher? Explain how you know.

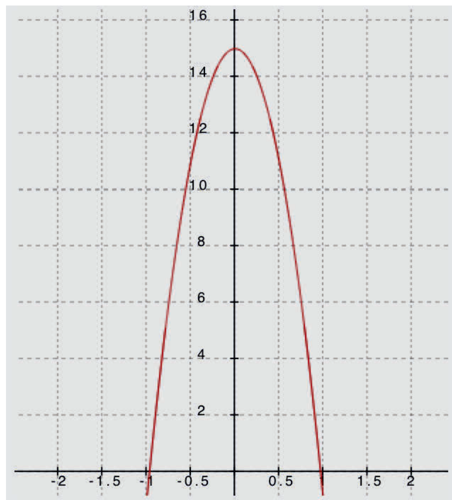
b. Compare the motion of Tenant 1's keys to that of Tenant 2's keys.

c. In this context, what would be a sensible domain for these functions?

Exit Ticket Sample Solutions

Two people, each in a different apartment building, have buzzers that don't work. They both must throw their apartment keys out of the window to their guests, who will then use the keys to enter.

Tenant 1 throws the keys such that the height-time relationship can be modeled by the graph below. On the graph, time is measured in seconds, and height is measured in feet.



Tenant 2 throws the keys such that the relationship between the height of the keys (in feet) and the time that has passed (in seconds) can be modeled by $h(t) = -16t^2 + 18t + 9$.

a. **Whose window is higher? Explain how you know.**

The window for Tenant 1 is higher (15 ft.) than that of Tenant 2 (9 ft.), which can be seen by comparing the values of the y-intercepts. You can see this in the graph below, showing both functions on the same coordinate plane.

b. **Compare the motion of Tenant 1's keys to that of Tenant 2's keys.**

Tenant 2's keys reach a maximum height at the vertex and then fall back toward the ground. (See the graph on the previous page.) The vertex for the graph of h can be found by completing the square:

$$\begin{aligned} h(t) &= -16\left(t^2 - \left(\frac{18}{16}\right)t + \quad \right) + 9 \\ &= -16\left(t^2 - \left(\frac{9}{8}\right)t + \left(\frac{9}{16}\right)^2\right) + 9 + 16\left(\frac{9}{16}\right)^2 \\ &= -16\left(t - \frac{9}{16}\right)^2 + \frac{225}{16}. \end{aligned}$$

So, the keys will reach a height of $\frac{225}{16}$ ft., or about 14 ft., before beginning the descent.

By comparison, Tenant 1's keys' motion is free falling. No linear term for Tenant 1 means an initial velocity of 0 ft/sec initial velocity; this is a quadratic graph whose axis of symmetry is the y-axis.

c. **In this context, what would be a sensible domain for these functions?**

Both domains would be positive. For Tenant 2, the zeros are $(-0.375, 0)$ and $(1.5, 0)$, so the domain is $[0, 1.5]$. For Tenant 1, the domain is $[0, 1]$ since the keys would be on the ground at about 1 second.

Problem Set Sample Solutions

1. **One type of rectangle has lengths that are always two inches more than their widths. The function f describes the relationship between the width of this rectangle in x inches and its area, $f(x)$, in square inches and is represented by the table below.**

x	$f(x)$
0	0
1	3
2	8
3	15
4	24
5	35

A second type of rectangle has lengths that are always one half of their widths. The function $g(x) = \frac{1}{2}x^2$ describes the relationship between the width given in x inches and the area, $g(x)$, given in square inches of such a rectangle.

a. **Use g to determine the area of a rectangle of the second type if the width is 20 inches.**

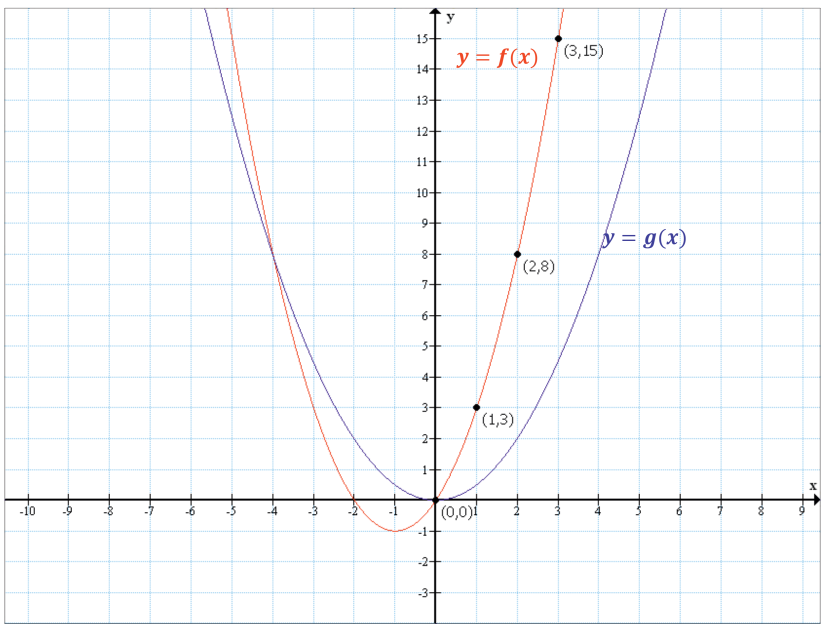
$$g(x) = \frac{1}{2}x^2$$
$$g(20) = \frac{1}{2}(20)^2$$
$$g(20) = 200$$

The area of the rectangle is 200 in^2.

b. **Why is $(0,0)$ contained in the graphs of both functions? Explain the meaning of $(0,0)$ in terms of the situations that the functions describe.**

For the first type of rectangle (with length two inches more than its width), a width of 0 in. means that the rectangle's length is 2 in. However, if the width is 0 in., its area is 0 in^2, and there is no rectangle. For the second type of rectangle (represented by g), a width of 0 in. means that the rectangle's area is 0 in^2. Although the point $(0,0)$ is contained in the graphs of both functions, it does not represent any rectangle width area pair because rectangles by definition have positive widths and lengths.

c. **Determine which function has a greater average rate of change on the interval $0 \le x \le 3$.**



On the interval $0 \le x \le 3$, f has a greater average rate of change than g, as seen in the graphs of the functions above. The average rate of change for f is $\frac{f(3)-f(0)}{3-0} = 5$, while the average rate of change for g is $\frac{g(3)-g(0)}{3-0} = \frac{3}{2}$.

d. **Interpret your answer to part (c) in terms of the situation being described.**

The area of rectangles, on average, on the interval $0 \le x \le 3$ is growing a little more than 3 times faster for rectangles described by the function f than those described by the function g of a certain width.

e. **Which type of rectangle has a greater area when the width is 5 inches? By how much?**

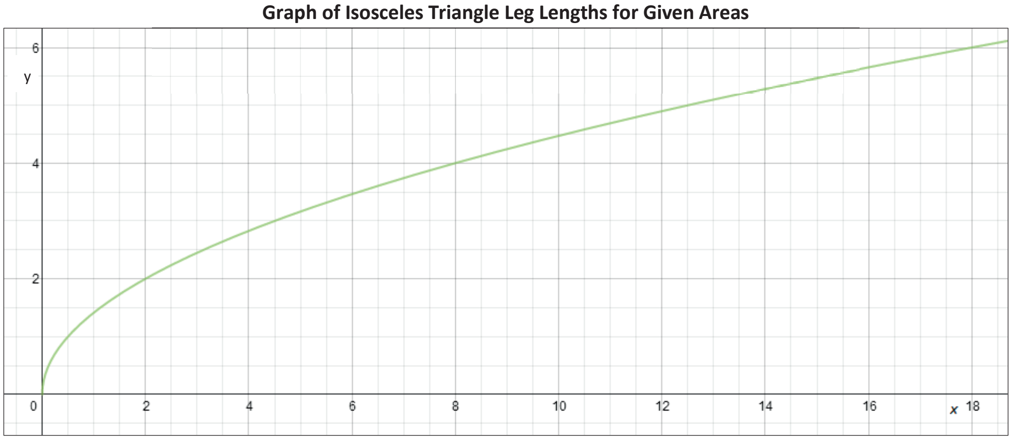
According to the table, the first rectangle will have an area of 35 in^2 if its width is 5 in. Using the formula for g, $g(5) = 12.5$, the second rectangle will have an area of 12.5 in^2 if its width is 5 in. The first type of rectangle has an area that is 22.5 in^2 greater than the second type of rectangle when their widths are 5 in.

f. **Will the first type of rectangle always have a greater area than the second type of rectangle when widths are the same? Explain how you know.**

If we consider rectangles with width c, where $c > 0$, then $c + 2 > \frac{1}{2}c$, so $(c + 2) \cdot c > \left(\frac{1}{2}c\right) \cdot c$. This means that the areas of the first type of rectangle will always be greater than the areas of the second type of rectangle when they have the same width.

2. **The function given by the equation $y = \sqrt{x}$ gives the edge length, y units, of a square with area x square units. Similarly, the graph below describes the length of a leg, y units, of an isosceles right triangle whose area is x square units.**

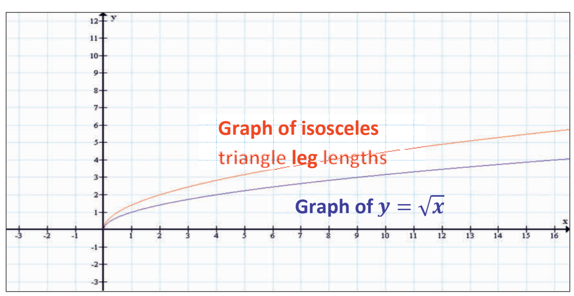
Graph of Isosceles Triangle Leg Lengths for Given Areas



a. **What is the length of a leg of an isosceles right triangle with an area of 12 square units?**

The length of a leg of an isosceles triangle with an area of 12 square units is approximately 4.9 units.

b. **Graph the function that represents a square with area x square units using the same graph that was given. Which function has a greater average rate of change on the interval $0 \leq x \leq 3$?**



The graphs of both functions intersect at the origin. According to the provided graph, when the area of an isosceles triangle is 3 square units, the length of a leg is approximately 2.4 units. Thus, the average rate of change is approximately 0.8. Using the equation $y = \sqrt{x}$, if $x = 3$, $y \approx 1.7$. This means that when the area of a square is 3 square units, the length of a leg is approximately 1.7 units. Thus, the average rate of change is approximately 0.6. The average rate of change of the legs of an isosceles triangle is greater than the average rate of change of the sides of a square on the interval $0 \leq x \leq 3$.

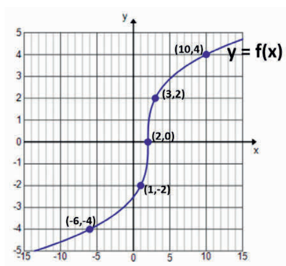
c. **Interpret your answer to part (b) in terms of the situation being described.**

Over the interval $0 \le x \le 3$, the average rate of change for the lengths of the legs of an isosceles triangle per a given area is approximately $\frac{4}{3}$ greater than the average rate of change for the sides of the square. In other words, for every unit of change in area, the legs grow (on average) by 0.2 units more over the interval $0 \le x \le 3$.

d. **Which will have a greater value: the edge length of a square with area 16 square units or the length of a leg of an isosceles right triangle with an area of 16 square units? Approximately by how much?**

According to the graph, the length of the leg of an isosceles right triangle with an area of 16 square units is approximately 5.7 units. Using the equation $y = \sqrt{x}$, the length of the sides of a square with an area of 16 square units is 4 units. The leg of the isosceles right triangle is approximately 1.7 units greater than the side lengths of the square when their areas are 16 square units.

3. **A portion of a graph of a cube root function, f, and select values of a square root function, g, are given below. The domain of g is ≥ 0 .**



x	$g(x)$
0	3
1	3.5
4	4
9	4.5
16	5

Fill in each blank with one of the following: $>$, $<$, or $=$.

a. $f(2)$ ________ $g(2)$

$f(2) = 0$ and $g(2)$ is between 3.5 and 4.

$f(2)$ __$<$__ $g(2)$

b. **y-intercept of f ______ y-intercept of g**

The y-intercept of f is between -2 and -3. The y-intercept of g is 3.

y-intercept of f __$<$__ y-intercept of g

c. **Average rate of change of f on interval $[0, 16]$ ______ Average rate of change of g on interval $[0, 16]$**

$f(x) = 2\sqrt[3]{x-2}$

Average rate of change of f on interval $[0, 16] = \frac{f(16)-f(0)}{16-0} = \frac{2\sqrt[3]{14}-2\sqrt[3]{-2}}{16} \approx 0.459$

Average rate of change of g on interval $[0, 16] = \frac{g(16)-g(0)}{16-0} = \frac{5-3}{16} = 0.125$

Average rate of change of f on interval $[0, 16]$ __$>$__ Average rate of change of g on interval $[0, 16]$

Lesson 23: Modeling with Quadratic Functions

Student Outcomes

- Students write the quadratic function described verbally in a given context. They graph, interpret, analyze, check results, draw conclusions, and apply key features of a quadratic function to real-life applications in business and physics.

Lesson Notes

Throughout this lesson, students make sense of problems by analyzing the given information; make sense of the quantities in the context, including the units involved; look for entry points to a solution; consider analogous problems; create functions to model situations; use graphs to explain or validate their reasoning; monitor their own progress and the reasonableness of their answers; and report their results accurately and with an appropriate level of precision.

This real-life descriptive modeling lesson is about using quadratic functions to understand the problems of the business world and of the physical world (i.e., objects in motion). For this lesson, students will need calculators (not necessarily graphing calculators) and graph paper.

Notes to the teacher about objects in motion:

Any object that is free falling or projected into the air without a power source is under the influence of gravity. All free-falling objects (on Earth) accelerate toward the center of the earth (downward) at a rate of 9.8 m/s^2 or 32 ft/s^2.

The model representing the motion of falling or thrown objects, using standard units, is a quadratic function, $h(t) = -16t^2 + v_0t + h_0$, where h represents the distance from the ground (height of the object in feet) and t is the number of seconds the object has been in motion, or if units are metric, $h(t) = -4.9t^2 + v_0t + h_0$. In each case, v_0 represents the initial velocity of the object and h_0 the initial position (i.e., initial height). Note that this section will be included in the student materials for this lesson.

Scaffolding:

Students with a high interest in physics may benefit from some independent study of motion problems. Send them to websites such as The Physics Classroom http://www.physicsclassroom.com/ for more information.

Classwork

Opening (5 minutes): The Mathematics of Objects in Motion

Opening: The Mathematics of Objects in Motion

Read the following explanation of the Mathematics of Objects in Motion:

Any object that is free falling or projected into the air without a power source is under the influence of gravity. All free-falling objects on Earth accelerate toward the center of Earth (downward) at a constant rate ($-32\ \text{ft/s}^2$, or $-9.8\ \text{m/s}^2$) because of the constant force of Earth's gravity (represented by g). That acceleration rate is included in the physics formula used for all objects in a free-falling motion. It represents the relationship of the height of the object (distance from Earth) with respect to the time that has passed since the launch or fall began. That formula is

$$h(t) = \frac{1}{2}gt^2 + v_0t + h_0.$$

For this reason, the leading coefficient for a quadratic function that models the position of a falling, launched, or projected object must either be -16 or -4.9. Physicists use mathematics to make predictions about the outcome of a falling or projected object.

The mathematical formulas (equations) used in physics commonly use certain variables to indicate quantities that are most often used for motion problems. For example, the following are commonly used variables for an event that includes an object that has been dropped or thrown:

- **h is often used to represent the function of height (how high the object is above Earth in feet or meters);**
- **t is used to represent the time (number of seconds) that have passed in the event;**
- **v is used to represent *velocity* (the rate at which an object changes position in ft/s or m/s);**
- **s is used to represent the object's change in position, or *displacement* (how far the object has moved in feet or meters).**

We often use subscripts with the variables, partly so that we can use the same variables multiple times in a problem without getting confused, but also to indicate the passage of time. For example:

- **v_0 indicates the initial velocity (i.e., the velocity at 0 seconds);**
- **h_0 tells us the height of the object at 0 seconds, or the initial position.**

So putting all of that together, we have a model representing the motion of falling or thrown objects, using U.S. standard units, as a quadratic function:

$$h(t) = -16t^2 + v_0t + h_0,$$

where h represents the height of the object in feet (distance from Earth), and t is the number of seconds the object has been in motion. Note that the negative sign in front of the 16 (half of $g = 32$) indicates the downward pull of gravity. We are using a convention for quantities with direction here; upward is positive and downward is negative. If units are metric, the following equation is used:

$$h(t) = -4.9t^2 + v_0t + h_0,$$

where everything else is the same, but now the height of the object is measured in meters and the velocity in meters per second.

These physics functions can be used to model many problems presented in the context of free-falling or projected objects (objects in motion without any inhibiting or propelling power source, such as a parachute or an engine).

Mathematical Modeling Exercise 1 (15 minutes)

After students have read the explanation in the student materials of the physics of free-falling objects in motion, discuss the variables and parameters used in the function to describe projectile motion on Earth.

Provide graph paper for the following problem. Have students work in pairs or small groups; read the problem from the student materials and discuss an entry point for answering the related questions. Then, walk students through the problem-solving process using the guiding questions provided below.

Mathematical Modeling Exercise 1

Use the information in the Opening to answer the following questions.

Chris stands on the edge of a building at a height of 60 ft. and throws a ball upward with an initial velocity of 68 ft/s. The ball eventually falls all the way to the ground. What is the maximum height reached by the ball? After how many seconds will the ball reach its maximum height? How long will it take the ball to reach the ground?

a. **What units will we be using to solve this problem?**

Feet for height, seconds for time, and feet per second for velocity

b. **What information from the contextual description do we need to use in the function equation?**

Gravity $= -32$ ft/s^2

Initial Velocity $(v_0) = 68$ ft/s

Initial Height $(h_0) = 60$ ft.

So, the function is $h(t) = -16t^2 + 68t + 60$.

c. **What is the maximum point reached by the ball? After how many seconds will it reach that height? Show your reasoning.**

The maximum function value is at the vertex. To find this value, we first notice that this function is factorable and is not particularly friendly for completing the square. So, we will rewrite the function in factored form, $f(t) = -4(4t^2 - 17t - 15) \rightarrow f(t) = -4(4t + 3)(t - 5)$.

Second, we find the zeros, or the t-intercepts, of the function by equating the function to zero (zero height).

So, $-4(4t + 3)(t - 5) = 0$.

Then, $(4t + 3) = 0$ *or* $(t - 5) = 0$.

t-intercepts are $t = -\frac{3}{4}$ *or* $t = 5$.

Then, using the concept of symmetry, we find the midpoint of the segment connecting the two t-intercepts (find the average of the t-coordinates): $t = \frac{-\frac{3}{4}+5}{2} = 2\frac{1}{8}$, *or* 2.125.

Now, we find $h(2.125)$ *in the original function,* $h(t) = -16t^2 + 68t + 60$, $h(2.125) = 132.25$.

Therefore, the ball reached its maximum height of 132.25 ft. *after* 2.125 sec.

(Note that students may try to complete the square for this function. The calculations are very messy, but the results will be the same:
$h(t) = -16\left(t - \frac{17}{8}\right)^2 + \frac{529}{4} = -16(t - 2.125)^2 + 132.25$. *Students may also opt to use the vertex formula.)*

Scaffolding:

Visual learners might benefit from a graphing tool to see this function's graph. An online tool can be found here: www.desmos.com/calculator.

d. **How long will it take the ball to land on the ground after being thrown? Show your work.**

We factored in the previous question and found the zeros of this function to be $t = -\frac{3}{4}$ and $t = 5$. The ball begins its flight at 0 sec. and ends at 5 sec. Therefore, it will be in flight for 5 sec.

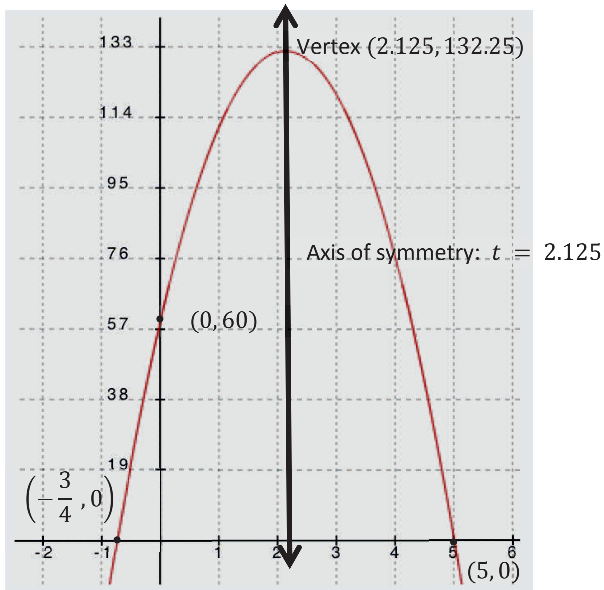
e. **Graph the function of the height, h, of the ball in feet to the time, t, in seconds. Include and label key features of the graph such as the vertex, axis of symmetry, and t- and y-intercepts.**

The graph should include identification of the intercepts, vertex, and axis of symmetry.

Vertex: $(2.125, 132.25)$

t-intercepts: $(-0.75, 0)$ *and* $(5, 0)$

y-intercept: $(0, 60)$ *(See the graph of $y = h(t)$.)*



Mathematical Modeling Exercise 2 (10 minutes)

Have students review the terminology for business applications and read the context of the problem. Discuss the quantities in the problem and the entry point for solving the problem. Then, solve the problem below. Walk them through the steps to solve the problem using the guiding questions and commentary provided.

Notes to the teacher about business applications:

The following business formulas are used in business applications in this module. Refer students to Lesson 12 if they need a more detailed review.

$$\text{Total Production Cost} = (\text{cost per item})(\text{total number of items sold})$$

$$\text{Total Revenue} = (\text{price per item})(\text{total number of items sold})$$

$$\text{Profit} = \text{Total Revenue} - \text{Total Production Cost}$$

Mathematical Modeling Exercise 2

Read the following information about Business Applications:

Many business contexts can be modeled with quadratic functions. This is because the expressions representing price (price per item), the cost (cost per item), and the quantity (number of items sold) are typically linear. The product of any two of those linear expressions will produce a quadratic expression that can be used as a model for the business context. The variables used in business applications are not as traditionally accepted as variables that are used in physics applications, but there are some obvious reasons to use c for cost, p for price, and q for quantity (all lowercase letters). For total production cost we often use C for the variable, R for total revenue, and P for total profit (all uppercase letters). You have seen these formulas in previous lessons, but we will review them here since we use them in the next two lessons.

Business Application Vocabulary

UNIT PRICE (PRICE PER UNIT): The price per item a business sets to sell its product, sometimes represented as a linear expression.

QUANTITY: The number of items sold, sometimes represented as a linear expression.

REVENUE: The total income based on sales (but without considering the cost of doing business).

UNIT COST (COST PER UNIT) OR PRODUCTION COST: The cost of producing one item, sometimes represented as a linear expression.

PROFIT: The amount of money a business makes on the sale of its product. Profit is determined by taking the total revenue (the quantity sold multiplied by the price per unit) and subtracting the total cost to produce the items (the quantity sold multiplied by the production cost per unit): $\textbf{Profit} = \textbf{Total Revenue} - \textbf{Total Production Cost.}$

The following business formulas will be used in this lesson:

$\textbf{Total Production Costs} = (\textbf{cost per unit})(\textbf{quantity of items sold})$

$\textbf{Total Revenue} = (\textbf{price per unit})(\textbf{quantity of items sold})$

$\textbf{Profit} = \textbf{Total Revenue} - \textbf{Total Production Costs}$

Now answer the questions related to the following business problem:

A theater decided to sell special event tickets at $\$10$ per ticket to benefit a local charity. The theater can seat up to $1,000$ people, and the manager of the theater expects to be able to sell all $1,000$ seats for the event. To maximize the revenue for this event, a research company volunteered to do a survey to find out whether the price of the ticket could be increased without losing revenue. The results showed that for each $\$1$ increase in ticket price, 20 fewer tickets will be sold.

a. **Let x represent the number of $\$1.00$ price-per-ticket increases. Write an expression to represent the expected price for each ticket.**

   Let x represent the number of $\$1$ increases. If each ticket is $\$10$, plus a possible price increase in $\$1$ increments, the price per ticket will be $10 + 1x$.

b. **Use the survey results to write an expression representing the possible number of tickets sold.**

   Since 20 fewer seats will be sold for each $\$1$ increase in the ticket price, $20x$ represents the number of seats fewer than $1,000$ that will be sold.

   $1000 - 20x$ represents the expected number of tickets sold at this higher price.

Point out that if there are no price increases, we would expect to sell all 1,000 seats ($x = 0$), but there will be 20 fewer for each \$1.00 in price-per-ticket increase.

c. **Using x as the number of \$1-ticket price increases and the expression representing price per ticket, write the function, R, to represent the total revenue in terms of the number of \$1-ticket price increases.**

$\text{Total Revenue} = (\text{price per ticket})(\text{number of tickets})$

$R(x) = (10 + x)(1000 - 20x) = 10\,000 + 1000x - 200x - 20x^2 = -20x^2 + 800x + 10\,000$

Point out that quadratic expressions are usually written in standard form with exponents in descending order. However, there is no requirement to do so, and students would be equally correct to leave the equation in factored form.

d. **How many \$1-ticket price increases will produce the maximum revenue? (In other words, what value for x produces the maximum R value?)**

We need to find the vertex of the revenue equation. The equation is originally in factored form, so we can just go back to that form, or we can complete the square (which seems to be pretty efficient).

1. *By completing the square:* $R(x) = -20\left(x^2 - 40x + \quad\right) + 10\,000$

$$-20(x^2 - 40x + 400) + 10\,000 + 8000$$

$$-20(x - 20)^2 + 18\,000, \textit{ so the vertex is } (20, 18\,000).$$

2. *Using the factors, set the equation equal to zero to find the zeros of the function:*

$$R(x) = (1000 - 20x)(10 + x) = 0$$

$x = -10$ and 50 are the zeros, and the vertex will be on the axis of symmetry (at the midpoint between -10 and 50), which is $x = 20$.

Finally, we reach the conclusion that after 20 price increases, the theater will maximize its revenue.

e. **What is the price of the ticket that will provide the maximum revenue?**

Price per ticket expression is $10 + x$. For $x = 20$, we have $10 + 20 = 30$. The price per ticket that will provide the maximum revenue is \$30.

f. **What is the maximum revenue?**

According to the R-value at the vertex, the maximum revenue will be \$18,000.

g. **How many tickets will the theater sell to reach the maximum revenue?**

With the maximum revenue of \$18,000 at \$30/ticket, the theater is selling 600 tickets.

h. **How much more will the theater make for the charity by using the results of the survey to price the tickets than they would had they sold the tickets for their original \$10 price?**

At \$10 per ticket, the theater would have brought in \$10,000 after selling all 1,000 seats. The theater will make an additional \$8,000 by using the survey results to price their tickets.

The next two exercises may be completed in class (if time permits) or added to the Problem Set. Working in pairs or small groups, have students strategize entry points to the solutions and the necessary problem-solving process. They should finish as much as possible in about 5 minutes for each exercise. Set a 5-minute timer after the start of Exercise 1 and again for Exercise 2. Remind them to refer to the two examples just completed if they get stuck. They should not think that they can finish the tasks in the 5-minute time but should get a good start. What they do not finish in class, should be completed as part of the Problem Set.

Exercise 1 (5 minutes)

Use a timer and start on Exercise 2 after 5 minutes.

Exercise 1

Two rock climbers try an experiment while scaling a steep rock face. They each carry rocks of similar size and shape up a rock face. One climbs to a point 400 ft. above the ground, and the other climbs to a place below her at 300 ft. above the ground. The higher climber drops her rock, and 1 second later the lower climber drops his. *Note that the climbers are not vertically positioned. No climber is injured in this experiment.*

a. Define the variables in this situation, and write the two functions that can be used to model the relationship between the heights, h_1 and h_2, of the rocks, in feet, after t seconds.

h_1 represents the height of the rock dropped by the higher climber,
h_2 represents the height of the rock dropped by the lower climber,
t represents the number of seconds passed since the higher climber dropped her rock,
$t-1$ represents the number of seconds since the lower climber dropped his rock.
$h_1(t) = -16t^2 + 400$ and $h_2(t) = -16(t-1)^2 + 300$

b. Assuming the rocks fall to the ground without hitting anything on the way, which of the two rocks will reach the ground last? Show your work, and explain how you know your answer is correct.

We are looking for the zeros in this case. Setting each function equal to zero we get:

$$h_1(t) = -16t^2 + 400 = 0$$
$$-16(t^2 - 25) = 0$$
$$-16(t+5)(t-5) = 0$$
$$t = -5 \text{ or } 5$$

For this context, it will take 5 seconds for the higher climber's rock to hit the ground.

$$h_2(t) = -16(t-1)^2 + 300 = 0$$

The standard form for this equation is $h_2(t) = -16t^2 + 32t + 284$, which is not factorable. We will solve from the completed-square form.

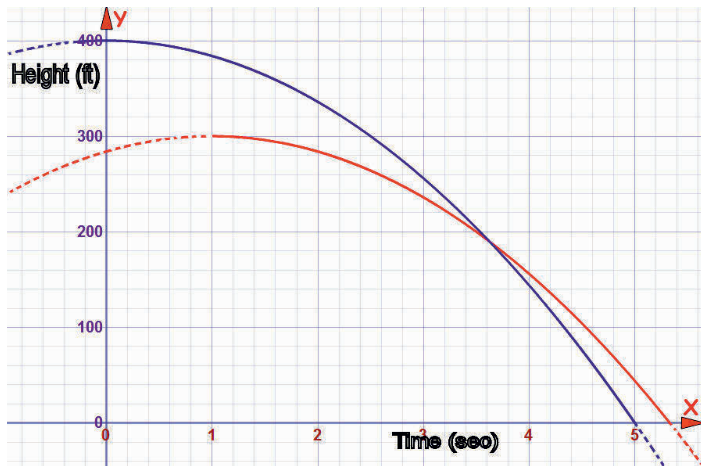
$$-16(t-1)^2 + 300 = 0$$
$$-16(t-1)^2 = -300$$
$$(t-1)^2 = \frac{300}{16} \text{ [square root both sides]}$$
$$t - 1 = \pm\frac{\sqrt{300}}{4}$$
$$t = 1 \pm \frac{\sqrt{300}}{4} \approx -3.3 \text{ or } 5.3$$

For this context, it will take approximately 5.3 seconds for the lower climber's rock to hit the ground.

The rock dropped from the higher position will hit the ground approximately 0.3 seconds before the rock dropped from the lower position. (Notice that the first function equation is easy to factor, but the other is not. Students may try to factor but may use the completed-square form to solve or may opt to use the quadratic formula on the second one.)

c. **Graph the two functions on the same coordinate plane, and identify the key features that show that your answer to part (b) is correct. Explain how the graphs show that the two rocks hit the ground at different times.**

The two times are indicated on the x-axis as the x-intercepts. The red graph shows the height-to-time relationship for the rock dropped from 300 ft., and the blue graph shows the same for the rock dropped from 400 ft. For the red graph, $h(t) = 0$ when $t = 5.3$, and for the blue graph, $h(t) = 0$ when $t = 5$.



d. **Does the graph show how far apart the rocks were when they landed? Explain.**

No, the graph only shows the height of the rocks with respect to time. Horizontal position and movement are not indicated in the function or the graph.

Exercise 2 (5 minutes)

Use a timer and start on the Exit Ticket after 5 minutes.

Exercise 2

Amazing Photography Studio takes school pictures and charges $\$20$ for each class picture. The company sells an average of 12 class pictures in each classroom. They would like to have a special sale that will help them sell more pictures and actually increase their revenue. They hired a business analyst to determine how to do that. The analyst determined that for every reduction of $\$2$ in the cost of the class picture, there would be an additional 5 pictures sold per classroom.

a. **Write a function to represent the revenue for each classroom for the special sale.**

Let x represent the number of $\$2$ reductions in price.

Then the price expression would be $20 - 2x$.

The quantity expression would be $12 + 5x$.

So, the revenue is $R(x) = (20 - 2x)(12 + 5x) = 240 + 100x - 24x - 10x^2 = -10x^2 + 76x + 240$.

b. What should the special sale price be?

Find the vertex for $R(x) = -10x^2 + 76x + 240$.

$$-10(x^2 - 7.6x + \_\_\_) + 240$$
$$-10(x^2 - 7.6x + 3.8^2) + 240 + 3.8^2(10) = -10(x - 3.8)^2 + 240 + 144.40$$

So, $R(x) = -10(x - 3.8)^2 + 384.4$, and the studio should reduce the price by between three or four $\$2$-increments.

If we check the revenue amount for 3 reductions, the price is $\$20 - \$2(3) = \$14$. The quantity will be $12 + 5(3) = 27$ pictures per classroom, so the revenue would be $\$378$.

Now check 4 reductions: The price is $\$20 - \$2(4) = \$12$. The quantity will be $12 + 5(4) = 32$ pictures per classroom, and the revenue for 4 reductions would be $\$384$.

The special sale price should be $\$12$ since the revenue was greater than when the price was $\$14$.

c. How much more will the studio make than they would have without the sale?

The revenue for each class will be $\$384$ during the sale. Without the sale, they would make $\$20$ per picture for 12 pictures, or $\$240$. They will increase their revenue by $\$144$ per classroom.

To ensure students understand, have them look at the revenue for five \$2-increments of price reduction. The price expression is $20 - 2(5) = 10$. The quantity will be $12 + 5(5) = 37$. That makes the revenue for five \$2-increments in price reduction \$370. The revenue is going back down. Are you surprised?

Closing (1 minute)

Lesson Summary

We can write quadratic functions described verbally in a given context. We can also graph, interpret, analyze, or apply key features of quadratic functions to draw conclusions that help us answer questions taken from the problem's context.

- **We find quadratic functions commonly applied in physics and business.**
- **We can substitute known x- and y-values into a quadratic function to create a linear system that, when solved, can identify the parameters of the quadratic equation representing the function.**

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 23: Modeling with Quadratic Functions

Exit Ticket

What is the relevance of the vertex in physics and business applications?

Exit Ticket Sample Solutions

What is the relevance of the vertex in physics and business applications?

By finding the vertex, we know the highest or lowest value for the function and also the x-value that gives that minimum or maximum. In physics, that could mean the highest point for an object in motion. In business, that could mean the minimum cost or the maximum profit or revenue.

Problem Set Sample Solutions

1. **Dave throws a ball upward with an initial velocity of 32 ft/s. The ball initially leaves his hand 5 ft. above the ground and eventually falls back to the ground. In parts (a)–(d), you will answer the following questions: What is the maximum height reached by the ball? After how many seconds will the ball reach its maximum height? How long will it take the ball to reach the ground?**

   a. **What units will we be using to solve this problem?**

      Height is measured in feet, time is measured in seconds, and velocity is measured in feet per second.

   b. **What information from the contextual description do we need to use to write the formula for the function h of the height of the ball versus time? Write the formula for height of the ball in feet, $h(t)$, where t stands for seconds.**

      Gravity: -32 ft/s^2

      Initial height (h_0): 5 ft. ***above the ground***

      Initial velocity (v_0): 32 ft/s

      Function: $h(t) = -16t^2 + 32t + 5$

   c. **What is the maximum point reached by the ball? After how many seconds will it reach that height? Show your reasoning.**

      Complete the square:

      $h(t) = -16t^2 + 32t + 5$

      $h(t) = -16(t^2 - 2t + \square) + 5 + \square$

      $h(t) = -16(t^2 - 2t + 1) + 5 + 16$ ***Completing the square***

      $h(t) = -16(t - 1)^2 + 21$ ***The vertex (maximum height) is 21 ft. and is reached at 1 sec.***

d. **How long will it take for the ball to land on the ground after being thrown? Show your work.**

The ball will land at a time t when $h(t) = 0$, *that is, when* $0 = -16t^2 + 32t + 5$:

$$0 = -16\left(t^2 - 2t - \frac{5}{16}\right)$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)\left(-\frac{5}{16}\right)}}{2(1)}$$

$$t = \frac{2 \pm \sqrt{4 + \frac{20}{16}}}{2}$$

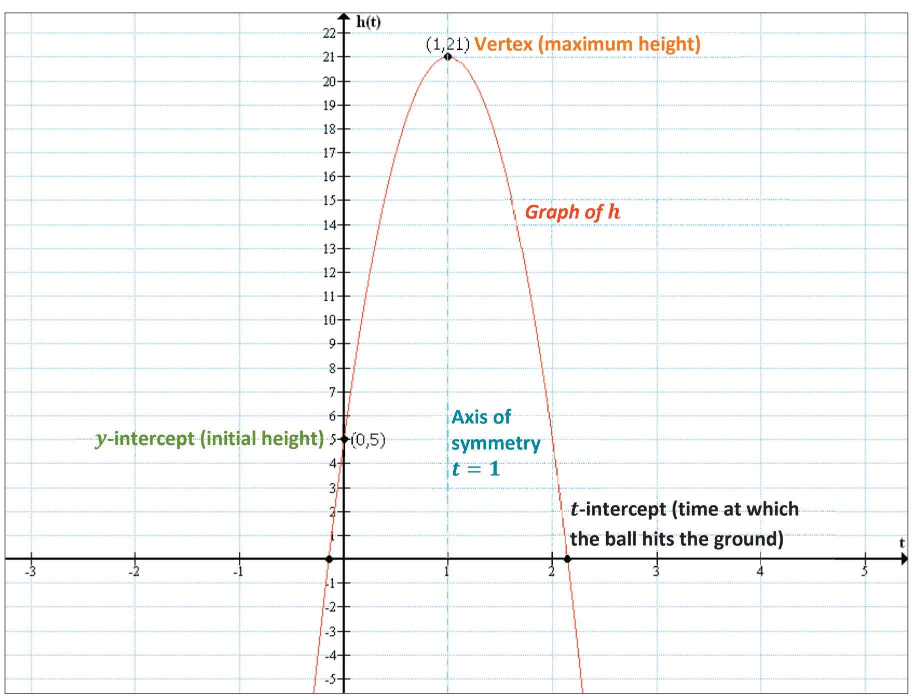
$$t = \frac{2 \pm \sqrt{\frac{21}{4}}}{2}$$

$$t = 1 \pm \frac{\sqrt{21}}{4}$$

$t \approx 2.146$ *and* $t \approx -0.146$

The negative value does not make sense in the context of the problem, so the ball reaches the ground in approximately 2.1 sec.

e. **Graph the function of the height of the ball in feet to the time in seconds. Include and label key features of the graph such as the vertex, axis of symmetry, and** t**- and** y**-intercepts.**



2. **Katrina developed an app that she sells for $\$5$ per download. She has free space on a website that will let her sell 500 downloads. According to some research she did, for each $\$1$ increase in download price, 10 fewer apps are sold. Determine the price that will maximize her profit.**

Profit equals total revenue minus total production costs. Since the website that Katrina is using allows up to 500 downloads for free, there is no production cost involved, so the total revenue is the total profit.

Let x represent the number of $\$1$ increases to the cost of a download.

Cost per download: $5 + 1x$

Apps sold: $500 - 10x$

$\text{Revenue} = (\text{unit price})(\text{quantity sold})$

$R(x) = (5 + 1x)(500 - 10x)$

$0 = (5 + 1x)(500 - 10x)$

$0 = 5 + x$ *or* $0 = 500 - 10x$

$x = -5$ *or* $x = 50$

The average of the zeros represents the axis of symmetry.

$\frac{-5+50}{2} = 22.5$ *Katrina should raise the cost by $\$22.50$ to earn the greatest revenue.*

$R(x) = (5 + x)(500 - 10x)$

$R(x) = -10x^2 + 450x + 2500$ *Written in standard form*

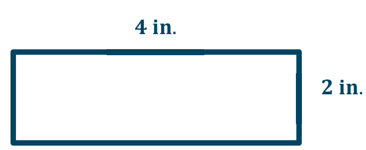
$R(22.5) = -10(22.5)^2 + 450(22.5) + 2500$

$R(22.5) = 7562.50$

Katrina will maximize her profit if she increases the price per download by $\$22.50$ to $\$27.50$ per download. Her total revenue (and profit) would be $\$7,562.50$.

3. **Edward is drawing rectangles such that the sum of the length and width is always six inches.**

a. **Draw one of Edward's rectangles, and label the length and width.**



b. **Fill in the following table with four different possible lengths and widths.**

Width (inches)	*Length (inches)*
1	5
2	4
3	3
2.5	3.5

c. **Let x be the width. Write an expression to represent the length of one of Edward's rectangles.**

If x represents the width, then the length of the rectangle would be $6 - x$.

d. **Write an equation that gives the area, y, in terms of the width, x.**

$$\text{Area} = \text{length} \times \text{width}$$
$$y = x(6 - x)$$

e. **For what width and length will the rectangle have maximum area?**

$y = x(6 - x)$

$y = -x^2 + 6x$

$y = -1(x^2 - 6x + \square) + \square$

$y = -1(x^2 - 6x + 9) + 9$ *By completing the square*

$y = -1(x - 3)^2 + 9$ *The vertex is* $(3, 9)$.

The rectangle with the maximum area has a width of 3 in. *(also length* 3 in.*) and an area of* 9 in^2.

f. **Are you surprised by the answer to part (e)? What special name is given for the rectangle in your answer to part (e)?**

Responses will vary. The special rectangle in part (e) is a square.

4. **Chase is standing at the base of a 60-foot cliff. He throws a rock in the air hoping to get the rock to the top of the cliff. If the rock leaves his hand 6 ft. above the base at a velocity of 80 ft/s, does the rock get high enough to reach the top of the cliff? How do you know? If so, how long does it take the rock to land on top of the cliff (assuming it lands on the cliff)? Graph the function, and label the key features of the graph.**

I will consider the top of the cliff as 0 ft. *so that I can find the time when the rock lands by finding the zeros of the function. Since Chase is standing at the bottom of the cliff, his initial height is negative; therefore, the initial height of the rock is negative.*

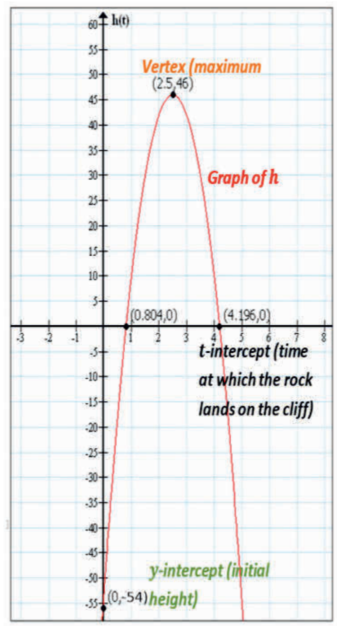
Gravity: $-32\ \text{ft/s}^2$

Initial height: -54 ft.

Initial velocity: $80\ \text{ft/s}$

$$h(t) = -16t^2 + 80t - 54$$
$$h(t) = -16(t^2 - 5t + \square) - 54 + \square$$
$$h(t) = -16\left(t^2 - 5t + 6\frac{1}{4}\right) - 54 + 100 \qquad \textit{By completing the square}$$
$$h(t) = -16\left(t - \frac{5}{2}\right)^2 + 46$$

In completed-square form, the vertex of the function is $\left(\frac{5}{2}, 46\right)$.



The rock reaches the top of the cliff because it reaches a maximum height of 46 ft. *above the cliff at* $2\frac{1}{2}$ sec. *after the rock was thrown.*

To find how much time it takes to reach the top of the cliff, I found the zeros of the function. I can see by the graph that the function has two possible zeros; however, given the context of the problem, only the latter of the two makes sense because the rock must go beyond the top of the cliff in order to land on the top of the cliff.

$$h(t) = -16t^2 + 80t - 54$$
$$h(t) = -2(8t^2 - 40t + 27)$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(8)(27)}}{2(8)}$$
$$t = \frac{40 \pm \sqrt{1600 - 864}}{16}$$
$$t = \frac{5}{2} \pm \frac{\sqrt{46}}{4}$$
$$t \approx 0.804 \text{ or } t \approx 4.196$$

The rock lands on the top of the cliff at approximately 4.2 sec. *after it was thrown.*

Lesson 24: Modeling with Quadratic Functions

Student Outcomes

- Students create a quadratic function from a data set based on a contextual situation, sketch its graph, and interpret both the function and the graph in context. They answer questions and make predictions related to the data, the quadratic function, and graph.

Lesson Notes

Throughout this lesson, students make sense of problems by analyzing the given information; make sense of the quantities in the context, including the units involved; look for entry points to a solution; consider analogous problems; create functions to model situations; use graphs to explain or validate their reasoning; monitor their own progress and the reasonableness of their answers; and report their results accurately and with an appropriate level of precision.

In this lesson, students understand that it takes three points to determine a unique quadratic function. They use data sets to write quadratic functions with and without context. For convenience, the points used in the following exercises have known y-intercepts and can be modeled precisely by quadratic functions with rational coefficients; however, teachers should remind students that in real life, data sets are unlikely to be able to be modeled with any function with 100% accuracy.

Classwork

Opening Exercise (10 minutes)

Project the graph on the board or screen, and ask students to draw as many quadratic graphs as possible through the following two points on the graph, which is also found in their student materials. Encourage them to check with their neighbors for ideas. These points are $(0, 4)$ and $(1, 9)$.

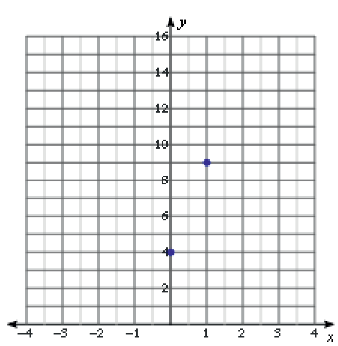
Scaffolding:

- Students may incorrectly draw U-shaped graphs that are not quadratic graphs. Remind them that quadratic graphs must be symmetrical: x-values on either side of the vertex must have matching y-values, and the curves continually grow wider for increasing values of $|x|$.
- Encourage students to draw quadratic graphs that are concave down as well as up; there are many different quadratics sharing these two points.

Opening Exercise

Draw as many quadratic graphs as possible through the following two points on the graph. Check with your neighbors for ideas. These points are $(0, 4)$ and $(1, 9)$.

Two Points



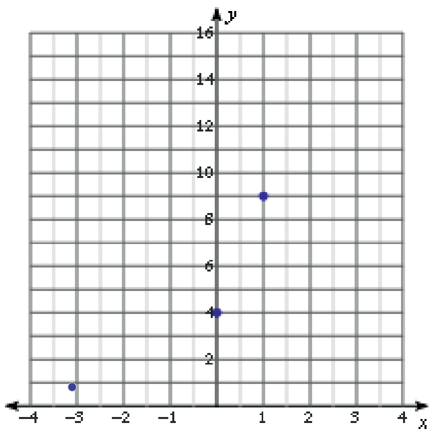
After a few minutes, gather the class together, and have students share some of their graphs. You might have three or four students come to the board and sketch one of their graphs, each in a different color. There are an infinite number of solutions. Make sure that some of the sketches have one of the points as a vertex and that some open up and some down.

Now, introduce a third point and ask students to repeat the exercise. Now the points are $(0, 4)$, $(1, 9)$, and $(-3, 1)$.

Scaffolding:

Unlike in the previous example, advanced students may notice that when three points are known, the value of the *second difference* is fixed; therefore, the quadratic function is uniquely defined.

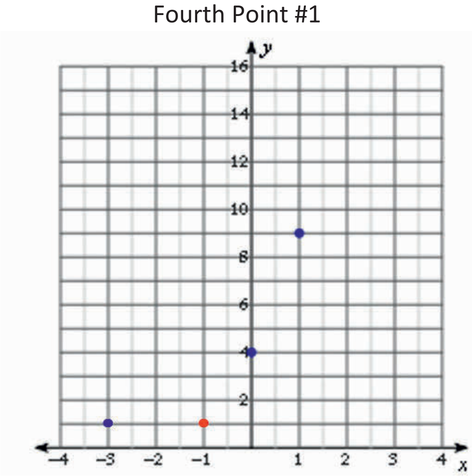
Three Points



Ultimately, students should conclude that only one quadratic graph can pass through all three points simultaneously. Therefore, it requires no less than three points to determine a quadratic function.

Students may be curious about what happens if a fourth point is introduced. Add a fourth point in two different places, and have them study the possibilities. Try adding a point in another color that is on the quadratic graph, $(-1, 1)$, and then add one that is not, $(2, 5)$.

Fourth Point #1



Scaffolding:

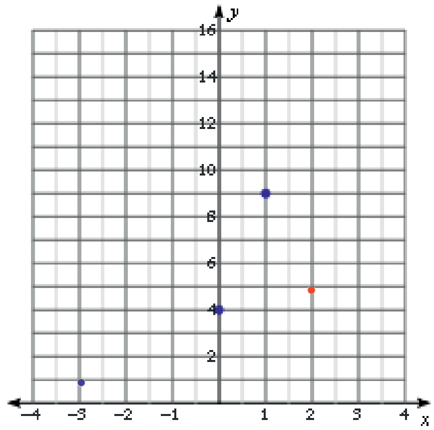
- Students may remember from earlier lessons that in quadratic equations, *second differences* are equal. This supports the idea that any number of quadratic equations can be drawn through two points because the value of the second difference is not well defined. Here is an example showing a quadratic function with its 1st and 2nd differences.

$f(x) = -5(x-1)^2 + 9$				
$x_2 - x_1$	x	y	1st f Diff $y_2 - y_1$	2nd Diff
	0	4		
1	1	9	5	
1	2	4	-5	-10
1	3	-11	-15	-10
1	4	-36	-25	-10

Point out that the differences in the x-values do not have to be 1 but must be regular. Ask why.

- Why must the differences in the x-values for the selected data points be at regular intervals?
  - *We are comparing rates of change. We need a constant change in x so that we are comparing equal intervals.*
- If the first differences represent the average rate of change for an interval (slope), how would you describe the second differences?
  - *They can be described as the average rate of change of the slope, or the slope of the slope.*

Fourth Point #2



Explain that a fourth point, in this case $(2, 5)$, may either belong to the quadratic graph (see: Fourth Point #1 graph) or not (see: Fourth Point #2 graph), but the function has already been determined by the first three (blue) points.

Example 1 (10 minutes)

Example 1

Use the points $(0, 4)$, $(1, 9)$, and $(-3, 1)$ to write the equation for the quadratic function whose graph contains the three points.

Demonstrate for students how, if we know the y-intercept and two other points for a quadratic function, we can form a system of linear equations to determine the standard form of the quadratic function defined by those points. Use the example with the blue points above: $(0, 4)$, $(1, 9)$, and $(-3, 1)$.

- Notice that we have the y-intercept, which allows us to find the value of c quickly and first. After that, we can substitute the other two coordinates into the equation, giving us two linear equations to solve simultaneously.

Using $(0, 4)$

$$f(x) = ax^2 + bx + c$$
$$4 = a(0)^2 + b(0) + c$$
$$4 = c$$

Using $(1, 9)$

$$f(x) = ax^2 + bx + 4$$
$$9 = a(1)^2 + b(1) + 4$$
$$9 = a + b + 4$$
$$a + b = 5$$

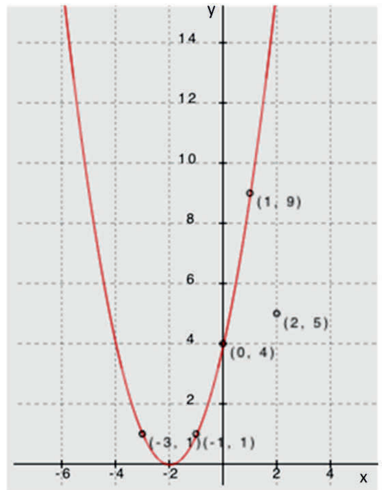
Using $(-3, 1)$

$$f(x) = ax^2 + bx + 4$$
$$1 = a(-3)^2 + b(-3) + 4$$
$$1 = 9a - 3b + 4$$
$$9a - 3b = -3$$

- Since $c = 4$, the resulting system has two variables: $\begin{cases} a + b = 5 \\ 9a - 3b = -3 \end{cases}$. Use substitution or elimination to determine that $a = 1$ and $b = 4$.
- Substitute $a = 1$, $b = 4$, and $c = 4$ into standard form: $f(x) = x^2 + 4x + 4$ is the quadratic function that contains the given points.

Demonstrate that the graph of the function we just found does, in fact, pass through all three points by showing the graph on the board or screen.

- Notice that in the graph below, we have included the two different fourth points from the Opening Exercise, $(-1, 1)$ and $(2, 5)$. Clearly $(-1, 1)$ is on the graph of the function, but $(2, 5)$ is not.



Exercise 1 (10 minutes)

Have students complete the following exercise independently.

Exercise 1

Write in standard form the quadratic function defined by the points $(0, 5)$, $(5, 0)$, and $(3, -4)$.

Using $(0, 5)$

$$f(x) = ax^2 + bx + c$$
$$5 = a(0)^2 + b(0) + c$$
$$5 = c$$

Using $(5, 0)$

$$f(x) = ax^2 + bx + 5$$
$$0 = a(5)^2 + b(5) + 5$$
$$0 = 25a + 5b + 5$$
$$25a + 5b = -5$$
$$5a + b = -1$$

Using $(3, -4)$

$$f(x) = ax^2 + bx + 5$$
$$-4 = a(3)^2 + b(3) + 5$$
$$-4 = 9a + 3b + 5$$
$$9a + 3b = -9$$
$$3a + b = -3$$

Since $c = 5$, the resulting system has two variables: $\begin{cases} 5a + b = -1 \\ 3a + b = -3 \end{cases}$.

Use substitution or elimination, and find that $a = 1$ and $b = -6$.

Substitute $a = 1$, $b = -6$, and $c = 5$ into standard form: $f(x) = x^2 - 6x + 5$ is the quadratic function that contains the given points.

Exercise 2 (10 minutes)

Have students work with a partner or in small groups to write the quadratic equation for the function defined by the following data set. Have them read the description of the experiment and study the collected data. Then, use the guiding questions to walk the students through the process of writing the quadratic equation to represent the data.

Exercise 2

Louis dropped a watermelon from the roof of a tall building. As it was falling, Amanda and Martin were on the ground with a stopwatch. As Amanda called the seconds, Martin recorded the floor the watermelon was passing. They then measured the number of feet per floor and put the collected data into this table. Write a quadratic function to model the following table of data relating the height of the watermelon (distance in feet from the ground) to the number of seconds that had passed.

Height (distance from the ground) for a Watermelon That Was Dropped from a Tall Building					
Time (t)	0	1	2	3	4
Height $f(t)$	300	284	236	156	44

a. **How do we know this data will be represented by a quadratic function?**

The relationship between height and time for all free-falling objects is represented by a quadratic equation. Also, we can see mathematically that the function values have a first difference of -16, -48, -80, and -112. The second differences are constant at -32.

b. **Do we need to use all five data points to write the equation?**

No, only three are needed.

c. **Are there any points that are particularly useful? Does it matter which we use? Write the quadratic function that models the data.**

$(0, 300)$ is useful because it is the y-intercept. We will need to use $(0, 300)$, but the other two can be selected based on efficiency (the least messy or smallest numbers).

Encourage different groups of students to use different sets of three points and then compare their results.

Use $(0, 300)$	*Use* $(1, 284)$	*Use* $(2, 236)$
$f(t) = at^2 + bt + c$	$f(t) = at^2 + bt + c$	$f(t) = at^2 + bt + c$
$300 = a(0)^2 + b(0) + c$	$284 = a(1)^2 + b(1) + 300$	$236 = a(2)^2 + b(2) + 300$
$300 = c$	$-16 = a + b$	$-64 = 4a + 2b$

Since $c = 300$, the resulting system has two variables: $\begin{cases} -16 = a + b \\ -64 = 4a + 2b \end{cases}$.

Use substitution or elimination and find that $a = -16$ and $b = 0$.

Substitute $a = -16$, $b = 0$, and $c = 300$ into standard form: $f(t) = -16t^2 + 300$.

Note: The same values for a, b, and c will occur no matter which points are used to write the function. However, the point $(0, 300)$ is particularly useful because it solves the system for c right away. Not using $(0, 300)$ first means that the students will need to solve a 3×3 system of equations. Students learn in Grade 8 to solve a 2×2 system of equations, but solving a 3×3 system is considered an advanced topic in Algebra II. Students could also point out that smaller values for t yield smaller coefficients for the system, making it easier to solve.

d. **How does this equation for the function match up with what you learned about physics in Lesson 23? Is there a more efficient way to find this equation?**

It matches perfectly. This equation shows that the initial position (height) of the object is 300 ft. and that the initial velocity is 0. It correctly uses -16 as the leading coefficient. We could have written the equation directly from the information provided since we already know the initial height and velocity.

e. **Can you use your quadratic function to predict at what time, t, the watermelon will hit the ground (i.e., $f(t) = 0$)?**

Yes.

$$f(t) = -16t^2 + 300$$
$$0 = -16t^2 + 300$$
$$-300 = -16t^2$$
$$18.75 = t^2$$
$$\pm 4.33 \approx t$$

So, the watermelon hit the ground after about 4.33 sec.

Closing (1 minute)

To determine a unique quadratic function from a table or graph, we must know at least three distinct points.

Lesson Summary

We can create a quadratic function from a data set based on a contextual situation, sketch its graph, and interpret both the function and the graph in context. We can then answer questions and make predictions related to the data, the quadratic function, and graph.

To determine a unique quadratic function from a table or graph, we must know at least three distinct points.

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 24: Modeling with Quadratic Functions

Exit Ticket

Write a quadratic function from the following table of data.

Fertilizer Impact on Corn Yields					
Fertilizer, x (kg/m^2)	0	100	200	300	400
Corn Yield, y (1000 bushels)	4.7	8.7	10.7	10.7	8.7

Exit Ticket Sample Solutions

Write a quadratic function from the following table of data.

Fertilizer Impact on Corn Yields					
Fertilizer, x (kg/m^2)	0	100	200	300	400
Corn Yield, y (1000 bushels)	4.7	8.7	10.7	10.7	8.7

Using the three points:

Use $(0, 4.7)$

$$f(x) = ax^2 + bx + c$$
$$4.7 = a(0)^2 + b(0) + c$$
$$4.7 = c$$

Use $(100, 8.7)$

$$f(x) = ax^2 + bx + c$$
$$8.7 = a(100)^2 + b(100) + 4.7$$
$$4 = 10{,}000a + 100b$$

Use $(200, 10.7)$

$$f(x) = ax^2 + bx + c$$
$$10.7 = a(200)^2 + b(200) + 4.7$$
$$6 = 40{,}000a + 200b$$

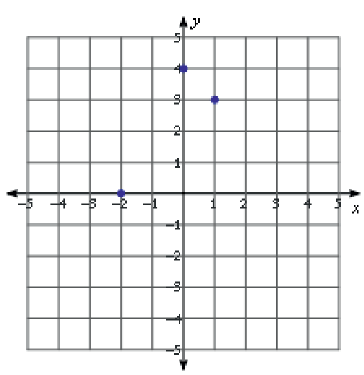
Since $c = 4.7$***, the resulting system has two variables:*** $\begin{cases} 4 = 10{,}000a + 100b \\ 6 = 40{,}000a + 200b \end{cases}$.

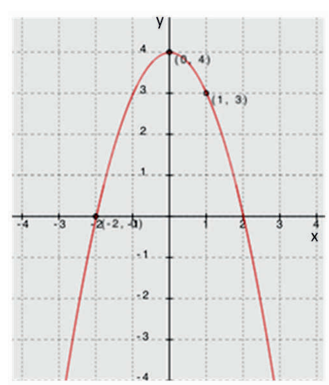
Use substitution or elimination and find that $a = \frac{-1}{10{,}000} = -0.0001$ ***and*** $b = \frac{1}{20} = 0.05$.

Substitute $a = -0.0001$, $b = 0.05$***, and*** $c = 4.7$ ***into standard form:*** $f(x) = -0.0001x^2 + 0.05x + 4.7$.

Problem Set Sample Solutions

1. **Write a quadratic function to fit the following points, and state the x-values for both roots. Then, sketch the graph to show that the equation includes the three points.**





Using the three points:

Use $(0, 4)$

$$f(x) = ax^2 + bx + c$$
$$4 = a(0)^2 + b(0) + c$$
$$4 = c$$

Use $(-2, 0)$

$$f(x) = ax^2 + bx + c$$
$$0 = a(-2)^2 + b(-2) + 4$$
$$-4 = 4a - 2b$$

Use $(1, 3)$

$$f(x) = ax^2 + bx + c$$
$$3 = a(1)^2 + b(1) + 4$$
$$-1 = a + b$$

Since $c = 4$***, the resulting system has two variables:*** $\begin{cases} -4 = 4a - 2b \\ -1 = a + b \end{cases}$.

Use substitution or elimination and find that $a = -1$ ***and*** $b = 0$.

Substitute $a = -1$, $b = 0$***, and*** $c = 4$ ***into standard form:*** $f(x) = -x^2 + 4$.

2. Write a quadratic function to fit the following points: $(0, 0.175)$, $(20, 3.575)$, $(30, 4.675)$.

Use $(0, 0.175)$

$$f(x) = ax^2 + bx + c$$
$$0.175 = a(0)^2 + b(0) + c$$
$$0.175 = c$$

Use $(20, 3.575)$

$$f(x) = ax^2 + bx + c$$
$$3.575 = a(20)^2 + b(20) + 0.175$$
$$3.4 = 400a + 20b$$

Use $(30, 4.675)$

$$f(x) = ax^2 + bx + c$$
$$4.675 = a(30)^2 + b(30) + 0.175$$
$$4.5 = 900a + 30b$$

Since $c = 0.175$, the resulting system has two variables: $\begin{cases} 3.4 = 400a + 20b \\ 4.5 = 900a + 30b \end{cases}$.

Use substitution or elimination and find that $a = -0.002$ and $b = 0.21$.

Substitute $a = -0.002$, $b = 0.21$, and $c = 0.175$ into standard form: $f(x) = -0.002x^2 + 0.21x + 0.175$.

Lagrange's Interpolation Method: An Extension for Accelerated Students

Lagrange's Interpolation Method allows mathematicians to write a polynomial from a given set of points. Because three points determine a unique quadratic function, students can use interpolation to write a quadratic function without having to solve a system of equations to find the coefficients.

Given the points (a, b), (c, d), (e, f), the quadratic function defined by these points can be written as follows:

$$f(x) = b \cdot \frac{(x-c)(x-e)}{(a-c)(a-e)} + d \cdot \frac{(x-a)(x-e)}{(c-a)(c-e)} + f \cdot \frac{(x-a)(x-c)}{(e-a)(e-c)}.$$

This works because, for each x substituted into the function, two of the terms disappear by the zero–multiplication rule, and the third term divides to $f(x) \cdot 1$. For example, write the quadratic function uniquely defined by the points: $(-1, 2)$, $(2, 23)$, $(-4, -1)$.

$$f(x) = 2 \cdot \frac{(x-2)(x+4)}{(-3)(3)} + 23 \cdot \frac{(x+1)(x+4)}{(3)(6)} - 1 \cdot \frac{(x+1)(x-2)}{(-3)(-6)}.$$

Then, $f(2) = 2 \cdot \frac{(2-2)(2+4)}{(-3)(3)} + 23 \cdot \frac{(2+1)(2+4)}{(3)(6)} - 1 \cdot \frac{(2+1)(2-2)}{(-3)(-6)}$,

and $f(2) = 0 + 23 \cdot \frac{(3)(6)}{(3)(6)} - 0$,

so $f(2) = 23 \cdot 1 = 23$.

This process can be repeated for each of the three points, and so this function is clearly a degree two polynomial containing the three given points. This form may be considered perfectly acceptable; however, by multiplying out and collecting like terms, we can rewrite this function in standard form.

$$f(x) = \frac{-2}{9}(x^2 + 2x - 8) + \frac{23}{18}(x^2 + 5x + 4) - \frac{1}{18}(x^2 - x - 2)$$

$$18f(x) = -4x^2 - 8x + 32 + 23x^2 + 115x + 92 - x^2 + x + 2$$

$$18f(x) = 18x^2 + 108x + 126$$

$$f(x) = x^2 + 6x + 7$$

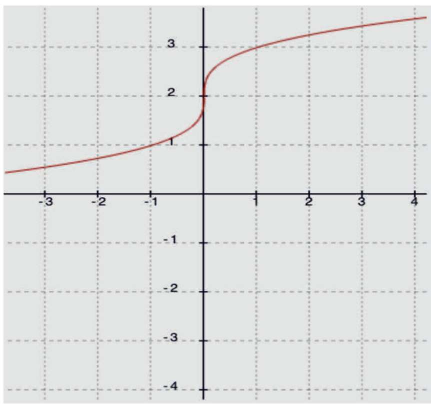
For students who love a challenge, design a short set of exercises with which accelerated students may practice interpolation. These exercises should not necessarily reduce to integer or even rational coefficients in standard form, and students may want to consider the potential pros and cons of leaving the function in its original interpolated form.

Name ____ Date ____________________

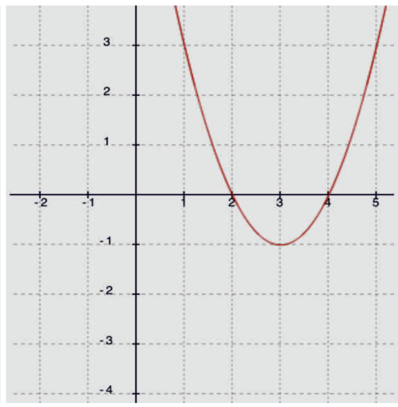
1. Label each graph with the function it represents; choose from those listed below.

$f(x) = 3\sqrt{x}$ $\qquad g(x) = \frac{1}{2}\sqrt[3]{x}$ $\qquad h(x) = -5x^2$

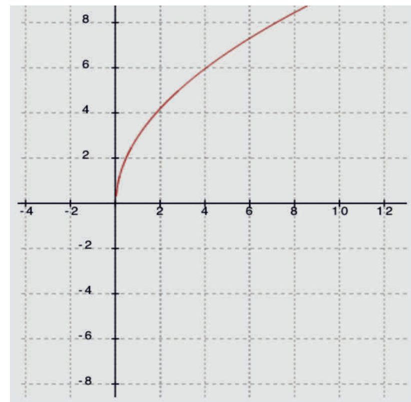
$k(x) = \sqrt{x+2} - 1$ $\qquad m(x) = \sqrt[3]{x} + 2$ $\qquad n(x) = (x-3)^2 - 1$



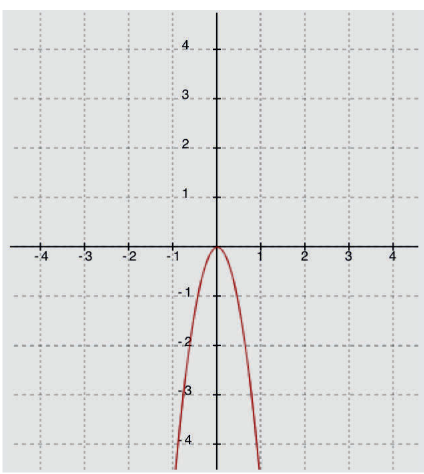
Function ________________



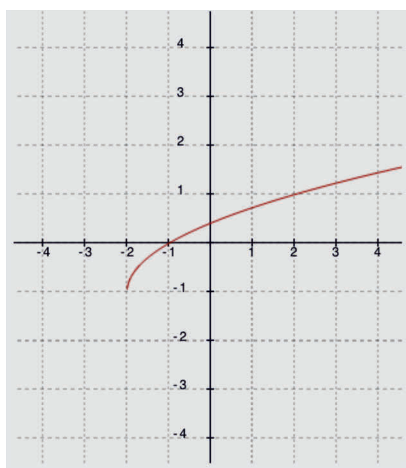
Function ________________



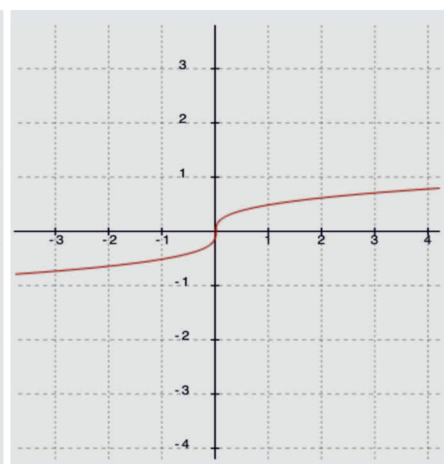
Function ________________



Function ________________



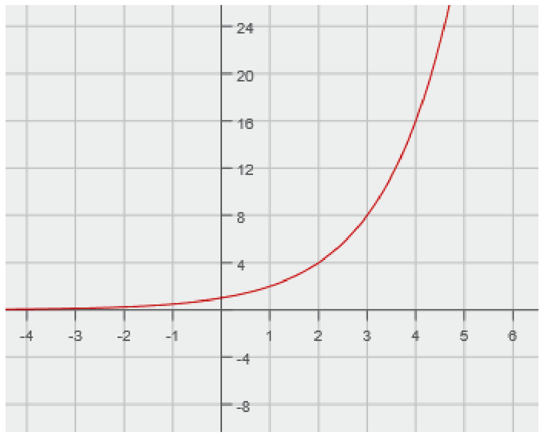
Function ________________



Function ________________

2. Compare the following three functions.

   i. A function f is represented by the graph below.



   ii. A function g is represented by the following equation.

$$g(x) = (x - 6)^2 - 36$$

   iii. A linear function h is represented by the following table.

x	-1	1	3	5	7
$h(x)$	10	14	18	22	26

For each of the following, evaluate the three expressions given, and identify which expression has the largest value and which has the smallest value. Show your work.

a. $f(0)$, $g(0)$, $h(0)$

b. $\dfrac{f(4)-f(2)}{4-2}, \dfrac{g(4)-g(2)}{4-2}, \dfrac{h(4)-h(2)}{4-2}$

c. $f(1000), g(1000), h(1000)$

3. An arrow is shot into the air. A function representing the relationship between the number of seconds it is in the air, t, and the height of the arrow in meters, h, is given by

$$h(t) = -4.9t^2 + 29.4t + 2.5.$$

a. Complete the square for this function. Show all work.

b. What is the maximum height of the arrow? Explain how you know.

c. How long does it take the arrow to reach its maximum height? Explain how you know.

d. What is the average rate of change for the interval from $t = 1$ to $t = 2$ seconds? Compare your answer to the average rate of change for the interval from $t = 2$ to $t = 3$ seconds, and explain the difference in the context of the problem.

e. How long does it take the arrow to hit the ground? Show your work, or explain your answer.

f. What does the constant term in the original equation tell you about the arrow's flight?

g. What do the coefficients on the second- and first-degree terms in the original equation tell you about the arrow's flight?

4. Rewrite each expression below in expanded (standard) form:

a. $\left(x+\sqrt{3}\right)^2$

b. $\left(x-2\sqrt{5}\right)\left(x-3\sqrt{5}\right)$

c. Explain why, in these two examples, the coefficients of the linear terms are irrational and the constants are rational.

Factor each expression below by treating it as the difference of squares:

d. $q^2 - 8$

e. $t - 16$

5. Solve the following equations for r. Show your method and work. If no solution is possible, explain how you know.

a. $r^2 + 12r + 18 = 7$

b. $r^2 + 2r - 3 = 4$

c. $r^2 + 18r + 73 = -9$

6. Consider the equation $x^2 - 2x - 6 = y + 2x + 15$ and the function $f(x) = 4x^2 - 16x - 84$ in the following questions:

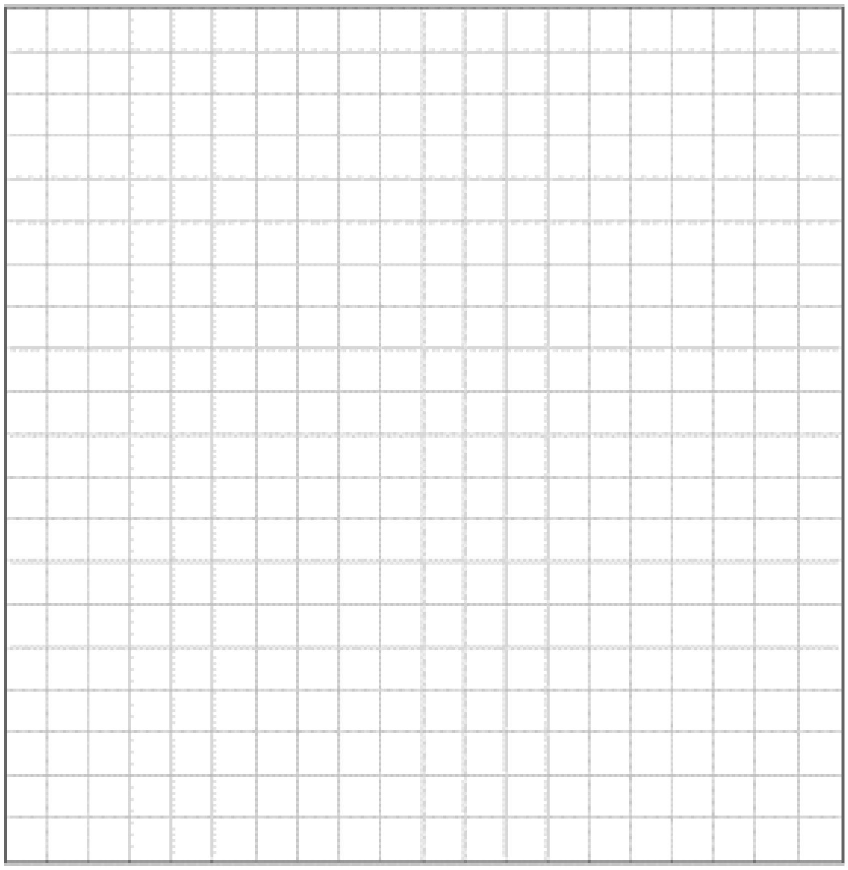
a. Show that the graph of the equation $x^2 - 2x - 6 = y + 2x + 15$ has x-intercepts at $x = -3$ and 7.

b. Show that the zeros of the function $f(x) = 4x^2 - 16x - 84$ are the same as the x-values of the x-intercepts for the graph of the equation in part (a) (i.e., $x = -3$ and 7).

c. Explain how this function is different from the equation in part (a).

d. Identify the vertex of the graphs of each by rewriting the equation and function in the completed-square form, $a(x - h)^2 + k$. Show your work. What is the same about the two vertices? How are they different? Explain why there is a difference.

e. Write a new quadratic function with the same zeros but with a maximum rather than a minimum. Sketch a graph of your function, indicating the scale on the axes and the key features of the graph.



A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1		Student matches two or fewer of the six graphs and functions accurately.	Student matches three of the six graphs and functions accurately.	Student matches four or five of the six graphs and functions accurately.	Student matches all six graphs and functions accurately.
2	**a–c**	Student marks each part separately, using the same scoring criteria for each. Student does not order the function values for each part correctly, and there is no evidence to show an understanding of the three representations as being exponential, quadratic, and linear or of how to determine the functions' values at the indicated x-values.	Student marks each part separately, using the same scoring criteria for each. Student provides work that shows an understanding of the graphic representation as being exponential, the tabular as linear, and the symbolic as quadratic. Only one of the three function values is correctly determined for the indicated values of x, but the function values are not ordered correctly.	Student marks each part separately, using the same scoring criteria for each. Student provides work that shows an understanding of the graphic representation as being exponential, the tabular as linear, and the symbolic as quadratic. The values for x are substituted correctly, but there are one or more errors in calculating the values of the expressions. The function values are not ordered correctly.	Student marks each part separately, using the same scoring criteria for each. Student provides work that shows an understanding of the graphic representation as being exponential, the tabular as linear, and the symbolic as quadratic. The values for x are substituted correctly, and the expressions are ordered correctly.

3	**a**	Student shows little or no understanding of the process required to complete the square.	Student attempts to rewrite the function in completed-square form, using a correct process. However, there are errors in calculations and steps missing in the process.	Student writes the function in completed-square form, but critical steps in the work are missing, or there are errors in the calculations.	Student correctly writes the function in completed-square form, and the correct steps for the process are included.
	b–g	Student shows little evidence of interpreting the function. OR Student makes little or no attempt to answer the question.	Student uses the function form found in part (a) but shows only some understanding of interpreting the key features of the function. Errors are made in calculations, and there is limited explanation.	Student provides an explanation that indicates an understanding of the key features of the function. Student uses the function form found in part (a) but incorrectly interprets the function or makes minor errors in calculation. Student explains the process but leaves gaps in the explanation.	Student provides an explanation that indicates an understanding of the key features of the function. Student uses the function form found in part (a) and correctly interprets the function features. Student provides evidence of the process and gives an accurate explanation of the reasoning used.
4	**a–b**	Student shows little evidence of understanding multiplication of binomials that include radicals. OR Student makes little or no attempt to rewrite the expression in standard form.	Student shows some evidence of understanding multiplication of binomials that include radicals. There are errors in the calculations and work.	Student shows evidence of understanding multiplication of binomials that include radicals. There are no errors in the calculations, but radical calculations are left unfinished, such as $(\sqrt{3})^2$ or $-3\sqrt{5} - 2\sqrt{5}$.	Student performs expansions accurately. The terms are in simplest radical form, and the work supports the solutions.
	c	Student makes no attempt to provide an explanation.	Student shows little evidence of understanding the properties of irrational numbers. The explanation uses numerical examples for both questions but provides no further explanation, or an explanation is provided for only one part of the question.	Student shows some evidence of understanding the properties of irrational numbers in the explanation. The explanation is partially correct and is attempted for both parts of the question but is missing one or more aspects.	Student clearly shows an understanding of the properties of irrational numbers based on the explanation.

	d–e	Student shows no evidence of understanding factoring the difference of squares. OR Student makes little or no attempt to factor the expression.	Student shows some evidence of understanding factoring the difference of squares. There are errors in the calculations that lead to incorrect solutions.	Student shows strong evidence of understanding factoring the difference of squares. There are no errors in the calculations, but the radical is left off (e.g., $\sqrt{t}$), or irrational calculations are left unfinished (e.g., $\sqrt{16}$).	Student provides accurate factors, the terms are in simplest radical form, and the work supports the solutions. (Note: In part (d), full credit is given for either form of the radical: $\sqrt{8}$ or $2\sqrt{2}$. However, in part (e), $\sqrt{16}$ must be changed to 4.)
5	**a–b**	Student shows no evidence of understanding the process of solving a quadratic equation. OR Student makes little or no attempt to solve the equation.	Student shows some evidence of understanding the process of solving a quadratic equation in the work shown. There are errors in calculations, and an incorrect method is used to find the solutions, or the process is aborted before completion.	Student completes the equation solving process using an appropriate method for each part. There are errors in calculations (e.g., factored incorrectly) or the equation is set up incorrectly (e.g., student fails to set the expression equal to 0), or there is only one solution found.	Student correctly solves the equations using an efficient method with accurate and supportive work shown.
	c	Student shows no evidence of understanding the process of solving a quadratic equation. OR Student makes little or no attempt to solve the equation.	Student shows some evidence of understanding the process of solving a quadratic equation. There are errors in calculations, and an incorrect method is used to find the solutions, or the process is aborted before completion.	Student understands the nature of this quadratic equation as having no real solutions. However, the explanation does not include using the discriminant or a graphic representation to justify the reasoning.	Student correctly sets up the equation for solution. The discriminant value shows there are no real solutions. Accurate explanation is included. (Note: The explanation may include references to the graphic representation.)
6	**a–b**	Student shows no evidence of understanding the concept of verifying the zeros of a function. OR Student makes little or no attempt to answer the question.	Student attempts to determine whether $x = -3$ and 7 are x-intercepts for the function. Errors are made in method selection or in calculations that lead to an inconsistency.	Student uses a valid method to show that the function has x-intercepts at $x = -3$ and 7 and includes an explanation to *show* the solutions are correct. Errors are made in the calculations that do not affect the final result.	Student uses a valid method to show that the function has x-intercepts at $x = -3$ and 7 and includes an explanation to *show* the solutions are correct.

	c	Student shows no evidence of understanding the relationship between the function and the equation and how that relationship is manifested in the graphs.	Student shows an understanding of how the graphs relate but does not mention how the expression used for the formula for f is 4 times the expression that defines y based on the given equation.	Student shows an understanding of how the graphs relate and of how the expression used for the formula for f is 4 times the expression that defines y based on the given equation, but there are minor errors or misuse of vocabulary.	Student shows an understanding of how the graphs relate and of how the expression used for the formula for f is 4 times the expression that defines y based on the given equation. The ideas are communicated with accurate use of vocabulary.
	d	Student shows no evidence of understanding the process of completing the square. OR Student makes little or no attempt to solve the problem.	Student shows some evidence of understanding the process of completing the square, but the attempt contains errors that lead to incorrect solutions. A valid explanation of the difference between the two vertices is not included.	Student shows evidence of understanding the process of completing the square. The attempt contains errors that lead to incorrect coordinates for the vertices. However, the explanation of the differences is based on the vertices found and is logical.	Student accurately performs the process of completing the square; the coordinates of the vertices are correct, and the explanation of their differences is accurate and logical.
	e	Student shows little evidence of understanding the concept of creating an equation with a maximum. OR Student makes little or no attempt to create the equation or sketch its graph.	Student shows evidence of understanding that the leading coefficient for the function must be negative. However, the function does not have the same zeros as those in parts (a) and (b), and the graph does not match the function created. The scale is not indicated on the graph, and the key features are not identified.	Student shows evidence of understanding that the leading coefficient for the function must be negative. However, the function does not have the same zeros as those in parts (a) and (b), or the graph does not match the function created. The scale is not indicated on the graph, or the key features are not identified.	Student shows evidence of understanding that the leading coefficient for the function must be negative. The function created has the same zeros as those in parts (a) and (b), and the graph matches the function created. The scale is indicated on the graph, and the key features are identified.

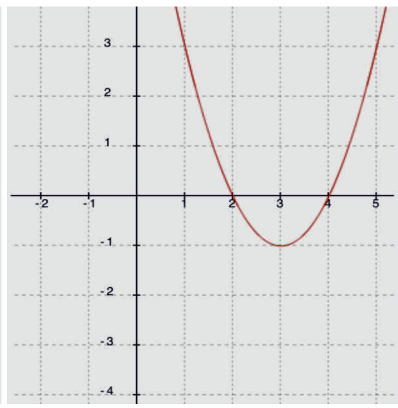
Name ______________________________ Date ______________

1. Label each graph with the function it represents; choose from those listed below.

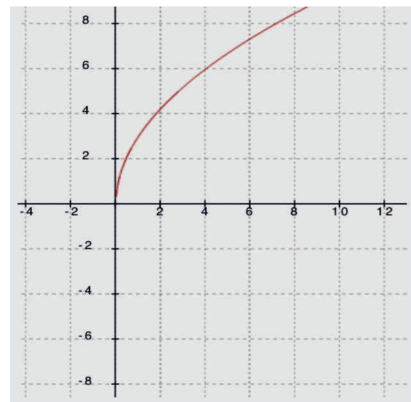
$f(x) = 3\sqrt{x}$ $\quad$ $g(x) = \frac{1}{2}\sqrt[3]{x}$ $\quad$ $h(x) = -5x^2$

$k(x) = \sqrt{x+2} - 1$ $\quad$ $m(x) = \sqrt[3]{x} + 2$ $\quad$ $n(x) = (x-3)^2 - 1$

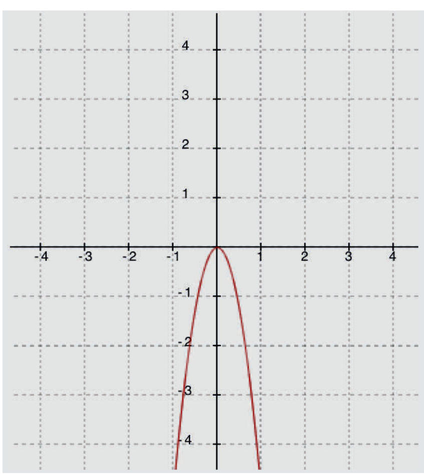
Function ___m___



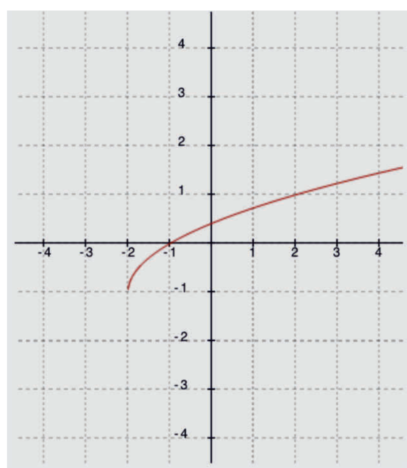
Function ___n___



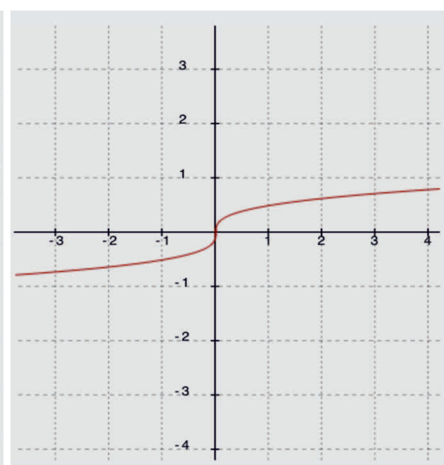
Function ___f___



Function ___h___



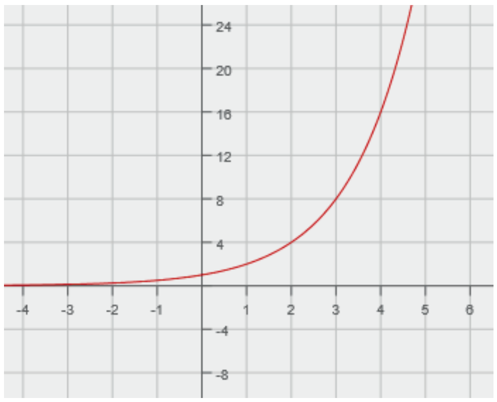
Function ___k___



Function ___g___

2. Compare the following three functions.

   i. A function f is represented by the graph below.



Note: $f(x) = 2^x$

   ii. Function g is represented by the following equation.

$$g(x) = (x - 6)^2 - 36$$

Note: $h(x) = 2x + 12$

   iii. Linear function h is represented by the following table.

x	-1	1	3	5	7
$h(x)$	10	14	18	22	26

For each of the following, evaluate the three expressions given, and identify which expression has the largest value and which has the smallest value. Show your work.

a. $f(0)$, $g(0)$, $h(0)$

$f(0) = 1$, $g(0) = 0$, $h(0) = 12$, so
$g(0)$ has the smallest value, and $h(0)$ has the largest value.

b. $\frac{f(4)-f(2)}{4-2}, \frac{g(4)-g(2)}{4-2}, \frac{h(4)-h(2)}{4-2}$

$f(4) = 16$ $f(2) = 4$: $g(4) = -32$ $g(2) = -20$: $h(4) = 20$ $h(2) = 16$

So, the values for the average rate of change over the interval [2, 4] for each function are as follows:

f: 6   g: −6   h: 2

The rate of change of g has the smallest value, -6, meaning it is decreasing relatively quickly; the rate of change of f has the largest value, 6; f is increasing at the same rate that g is decreasing; the rate of change of h is slowest.

c. $f(1000), g(1000), h(1000)$

$f(1000) = 2^{1000} = 1.1 \times 10^{301}$; $g(1000) = 994^2 - 36 = 988{,}000$; $h(1000) = 2012$

h(1000) has the smallest value, and f(1000) has the largest value.

3. An arrow is shot into the air. A function representing the relationship between the number of seconds it is in the air, t, and the height of the arrow in meters, h, is given by

$$h(t) = -4.9t^2 + 29.4t + 2.5.$$

a. Complete the square for this function. Show all work.

$h(t) = -4.9t^2 + 29.4t + 2.5 = -4.9(t^2 + 6t + \quad) + 2.5$ (factoring out the −4.9 from the two t terms and leaving the constant outside the parentheses)

$= -4.9(t^2 - 6t + 9) + 2.5 + 44.1$ (completing the square inside the parentheses)

$= -4.9(t - 3)^2 + 46.6$

b. What is the maximum height of the arrow? Explain how you know.

46.6 m—this is the value of the function at its vertex, so it is the highest the arrow will reach before it begins its descent.

c. How long does it take the arrow to reach its maximum height? Explain how you know.

3 sec., because that is the t-value (time in seconds) at which the arrow reached its highest point.

d. What is the average rate of change for the interval from $t = 1$ to $t = 2$ seconds? Compare your answer to the average rate of change for the interval from $t = 2$ to $t = 3$ seconds, and explain the difference in the context of the problem.

$h(2) = -4.9(4) + 29.4(2) + 2.5 = 41.7$
$h(1) = -4.9(1) + 29.4(1) + 2.5 = 27$

The average rate of change for the interval [1, 2] is $\frac{41.7 - 27}{2 - 1} = 14.7$.

$h(2) = 41.7$
$h(3) = -4.9(9) + 29.4(3) + 2.5 = 46.6$

The average rate of change for the interval [2, 3] is $\frac{46.6 - 41.7}{3 - 2} = 4.9$.

The average rate of change for the interval from $t = 1$ to $t = 2$ seconds is 14.7 meters per second. The average rate of change for the interval from $t = 2$ to $t = 3$ seconds is 4.9 meters per second. Comparing the average rate of change for the intervals [1, 2] and [2, 3] shows that the arrow is moving faster in the first interval than during the second interval. As the arrow moves upward, the rate slows until it finally turns and begins its downward motion.

e. How long does it take the arrow to hit the ground? Show your work, or explain your answer.

Since the zeros for the function are at −0.08 and 6.08 seconds, the arrow was in flight from 0–6.08 seconds.

f. What does the constant term in the original equation tell you about the arrow's flight?

The constant (2.5) represents the height when $t = 0$ or $h(0)$. That is the initial height of the arrow when it was shot, 2.5 m. (Note: 2.5 m is approximately 8 ft. 2 in. Since a bow and arrow at the ready is held a full arm's length above the head, this would suggest that the person shooting the arrow was around 6 ft. tall.)

g. What do the coefficients on the second and first degree terms in the original equation tell you about the arrow's flight?

−4.9: This is half of the local gravitational constant, -9.8 m/s^2.
29.4: The initial velocity of the arrow as it was shot upward was 29.4 m/s (approximately 66 mph).

4. Rewrite each expression below in expanded (standard) form:

a. $\left(x+\sqrt{3}\right)^2$

$x^2 + 2\sqrt{3}\,x + (\sqrt{3})^2$

$= x^2 + 2\sqrt{3}\,x + 3$

b. $\left(x-2\sqrt{5}\right)\left(x-3\sqrt{5}\right)$

$x^2 - 3\sqrt{5}\,x - 2\sqrt{5}\,x + (2\sqrt{5})(3\sqrt{5})$

$= x^2 - 5\sqrt{5}\,x + 30$

c. Explain why, in these two examples, the coefficients of the linear terms are irrational and the constants are rational.

When two irrational numbers are added (unless they are additive inverses), the result is irrational. Therefore, the linear term will be irrational in both of these cases. When a square root is squared or multiplied by itself, the result is rational.

Factor each expression below by treating it as the difference of squares:

d. $q^2 - 8$

$(q + \sqrt{8})(q - \sqrt{8})$

$(q + 2\sqrt{2})(q - 2\sqrt{2})$

e. $t - 16$

$(\sqrt{t}+4)(\sqrt{t}-4)$ or

$(-\sqrt{t}+4)(-\sqrt{t}-4)$

5. Solve the following equations for r. Show your method and work. If no solution is possible, explain how you know.

a. $r^2 + 12r + 18 = 7$

$r^2 + 12r + 18 = 7$
$r^2 + 12r + 11 = 0$
$(r + 1)(r + 11) = 0$
$r = -1$ or -11

b. $r^2 + 2r - 3 = 4$

$r^2 + 2r - 3 = 4$
$r^2 + 2r - 7 = 0$
Completing the square:
$r^2 + 2r + 1 = 7 + 1$
$(r + 1)^2 = 8$
$r + 1 = \pm 2\sqrt{2}$
$r = -1 \pm 2\sqrt{2}$

Note: Students may opt to use the quadratic formula to solve this equation.

c. $r^2 + 18r + 73 = -9$

$r^2 + 18r + 73 = -9$
$r^2 + 18r + 82 = 0$
Discriminant:
$18^2 - 4(1)(82) =$
$324 - 328 = -4$

There are no real solutions since the discriminant is negative.

6. Consider the equation $x^2 - 2x - 6 = y + 2x + 15$ and the function $f(x) = 4x^2 - 16x - 84$ in the following questions:

a. Show that the graph of the equation $x^2 - 2x - 6 = y + 2x + 15$ has x-intercepts at $x = -3$ and 7.

Substituting -3 for x and 0 for y:
$9 + 6 - 6 = 0 - 6 + 15$
$9 = 9$
This true statement shows that $(-3, 0)$ is an x-intercept on the graph of this equation.

Substituting 7 for x and 0 for y:
$49 - 14 - 6 = 0 + 14 + 15$
$29 = 29$
This true statement shows that $(7, 0)$ is an x-intercept on the graph of this equation.

b. Show that the zeros of the function $f(x) = 4x^2 - 16x - 84$ are the same as the x-values of the x-intercepts for the graph of the equation in part (a) (i.e., $x = -3$ and 7).

Substituting $x = -3$ and $f(x) = 0$:

$$4(9) - 16(-3) - 84 = 0$$
$$36 + 48 - 84 = 0$$
$$0 = 0$$

Substituting $x = 7$ and $f(x) = 0$:

$$4(49) - 16(7) - 84 = 0$$
$$196 - 112 - 84 = 0$$
$$0 = 0$$

c. Explain how this function is different from the equation in part (a).

The graph of the function would be a vertical stretch, with a scale factor of 4, of the graph of the equation. Or, you could say that the graph of the equation is a vertical shrink, with a scale factor of $\frac{1}{4}$ of the graph of f. You know this because the formula for the function f is related to the equation. Solving the equation for y, you get $y = x^2 - 4x - 21$. So, $y = \frac{1}{4} f(x)$.

d. Identify the vertex of the graphs of each by rewriting the equation and function in the completed-square form, $a(x-h)^2 + k$. Show your work. What is the same about the two vertices? How are they different? Explain why there is a difference.

$$x^2 - 2x - 6 = y + 2x + 15$$
$$y = x^2 - 4x - 21$$
$$y = (x^2 - 4x \quad + 4\ ) - 21 - 4$$
$$y = (x-2)^2 - 25$$

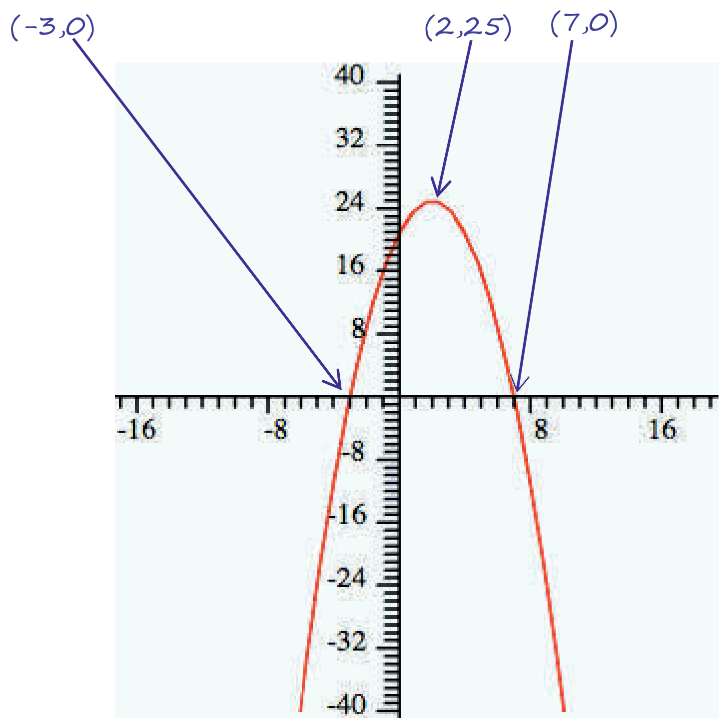
Vertex $(2, -25)$

$$4x^2 - 16x - 84 = f(x)$$
$$f(x) = 4(x^2 - 4x + \ 4\ ) - 84 \ - 16$$
$$f(x) = 4(x-2)^2 - 100$$

Vertex $(2, -100)$

The two vertices have the same x-coordinate (the same axis of symmetry), but the y-coordinate for the vertex of the graph of the function is 4 times the y-coordinate of the vertex of the graph of the two-variable equation because the graph of the function would be a vertical stretch (with a scale factor of 4) of the graph of the equation.

e. Write a new quadratic function with the same zeros but with a maximum rather than a minimum. Sketch a graph of your function, indicating the scale on the axes and the key features of the graph.



Notes: Factored form is easiest to use with the zeros as the given information. We just need to have a negative leading coefficient. Any negative number will work. This example uses $a = -1$:

$f(x) = -(x + 3)(x - 7)$.

To graph this function, plot the zeros/intercepts $(-3, 0)$ and $(7, 0)$. The vertex will be on the axis of symmetry ($x = 2$). Evaluate the equation for $x = 2$ to find the vertex, $(2, 25)$, and sketch.

Results for the graphs may be wider or narrower, and the vertex may be higher or lower. However, all should open down, pass through the points $(-3, 0)$ and $(7, 0)$, and have 2 as the x-coordinate for the vertex.

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