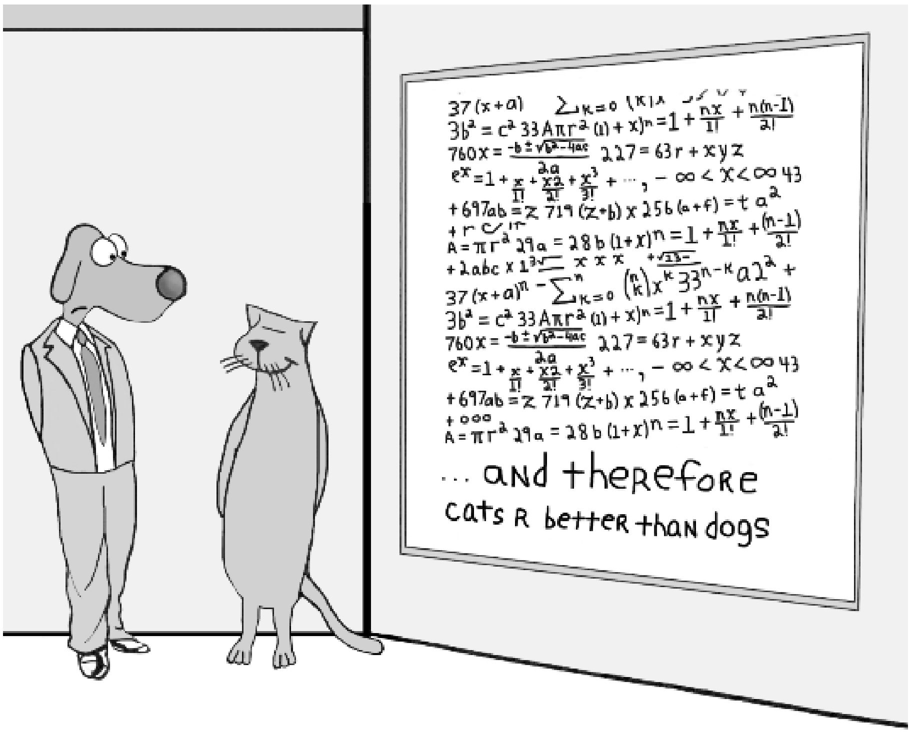
Year 12 Calculus Worksheet

16 questions on Calculus from the Maths C (Maths Methods) national curriculum for Year 12.



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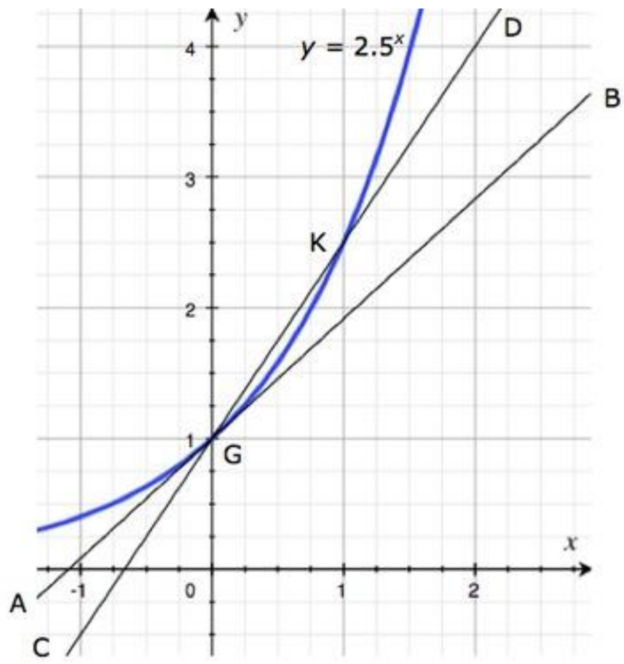
Questions

1. Look carefully at the graph and at the definition of *m*.



$$m = \lim_{h \to 0} \frac{2.5^h - 1}{h}.$$

Which of the following are true?

(i) The slope of the curve $y = 2.5^x$ at K is m.

(ii) The slope of the curve $y = 2.5^x$ at G is m.

(iii) The slope of the line CD is m.

(iv) The slope of the line AB is m.

a) Both (i) and (iii).

b) Both (ii) and (iv).

c) Both (ii) and (iii)

d) Both (iii) and (iv).

Answer: ____________________

2. Terry used a spread sheet to investigate the value of *a* for which the limit,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1.$$

To represent changing values of *h*, Terry entered some values for *x* in column A of the spreadsheet, decreasing from *x* = 1 in cell A2 to *x* = 0.0001 in cell A10. To try different values of *a* from 2 to 3, in cells B2, C2, D2 and E2, Terry entered the formulas:

= (2^A2 – 1) / A2

= (2.7^A2 – 1) / A2

= (2.8^A2 – 1) / A2

= (3^A2 – 1) / A2.

Terry then copied columns B, C, D and E from row 2 down to row 10.

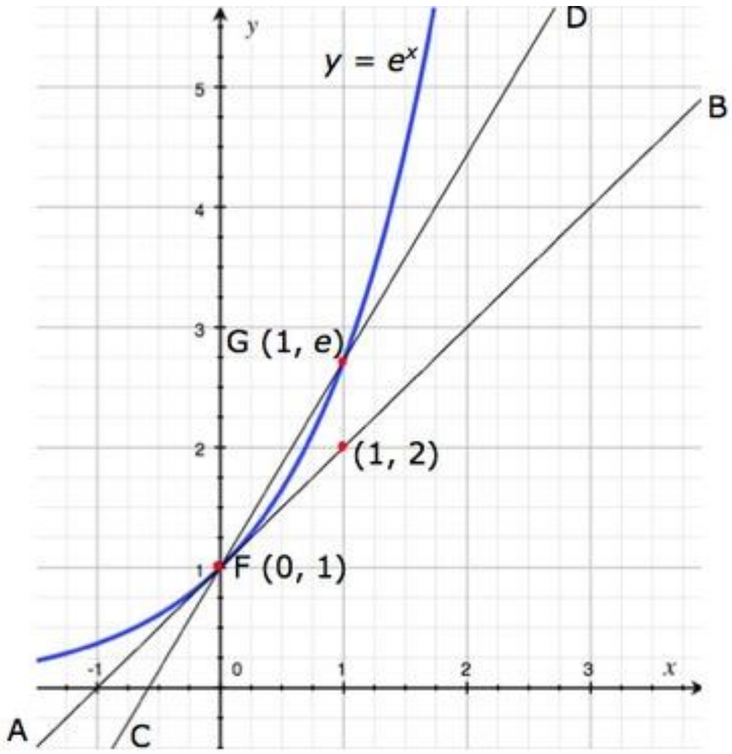
	A	B	C	D	E
1	**x values**	**a = 2**	**a = 2.7**	**a = 2.8**	**a = 3**
2	1	1	1.7	1.8	2
3	0.5	0.82842712	1.28633535	1.34664011	1.46410162
4	0.1	0.71773463	1.04425375	1.08449223	1.16123174
5	0.05	0.70529848	1.01832891	1.05658303	1.12934617
6	0.01	0.69555501	0.99820089	1.03493824	1.10466919
7	0.005	0.6943497	0.99572223	1.03227426	1.10163519
8	0.001	0.69338746	0.99374521	1.03014966	1.09921598
9	0.0005	0.69326731	0.99349845	1.02988449	1.09891408
10	0.0001	0.6931712	0.9933011	1.02967242	1.09867264

What reasonable hypothesis could Terry form from his investigation about the value of *a*?

a) 2.7 < *a* < 2.8
b) 0.0001 < *a* < 1
c) 1.7 < *a* < 1.8
d) *a* = 0.6931712

Answer: ____________________

3. Look carefully at the graph.



(i) What is the equation of the tangent to $y = e^x$ at the point F (0, 1)?

(ii) What is the equation of the tangent to $y = e^x$ at the point G (1, e)?

a) (i) $y = x + 1$;
(ii) $y = ex$.

b) (i) $y = (e - 1)x + 1$;
(ii) $y = ex + 1$.

c) (i) $y = (e - 1)x + 1$;
(ii) $y = ex - 1$.

d) (i) $y = e^x + 1$;
(ii) $y = x + 1$.

Answer: ____________________

4. Phyllis is looking for an infinite power series that is equal to its derived function.

She writes $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$ for t his power series.

Then $\frac{dy}{dx} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$

Also, after differentiating the series for y,

$$\frac{dy}{dx} = 0 + a_1 + 2a_2x^1 + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots$$

(i) If $y = 1$ when $x = 0$, is it **True** or **False** that $y = e^x$.

Phyllis equates coefficients of like powers of x to get a recurrence relation for the coefficients.

If $y = 1$ when $x = 0$, find:

(ii) the recursive step that relates a_n to a_{n-1}

(iii) a formula for a_n in terms of n

(iv) the series for y when $x = 1$.

a) (i) False;
(ii) $na_n = a_{n-1}$;
(iii) $a_n = n!$;
(iv) $\frac{dy}{dx} = 0 + a_1 + 2a_2x^1 + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots$

b) (i) False;
(ii) $a_n = na_{n-1;}$
(iii) $a_n = 1/n!$
(iv) $y = 1 + x + x^2/2! + x^3/3! + \ldots$

c) (i) True;
(ii) $na_n = a_{n-1;}$

(iii) $a_n = \frac{1}{n!}$

(iv) $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$

d) (i) True;

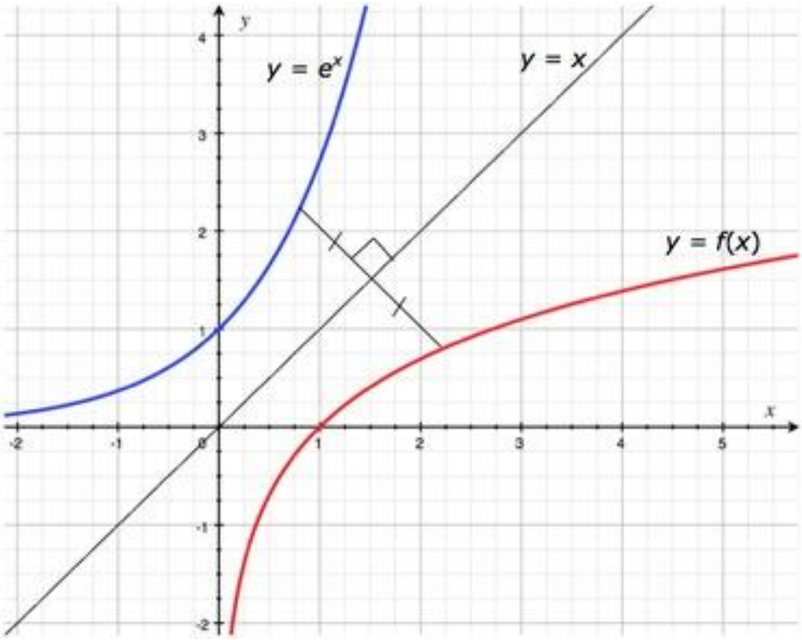
(ii) $a_n = na_{n-1}$;

(iii) $a_n = n!$;

(iv) $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$

Answer: ____________________

5. The red and blue graphs are mirror images under reflection in the line $y = x$.



The equation of the blue graph is $y = e^x$.

The equation of the red graph is $y = f(x)$.

What does $f(x)$ equal?

a) $f(x) = \log_x e$
b) $f(x) = -e^{-}x$
c) $f(x) = e^{-}x$
d) $f(x) = \log_e x$

Answer: ____________________

6. A calculator gives the value of the irrational number e as 2.718 281 828 correct to 9 decimal places.

Ben and Cassie researched and found two different formulas for e.

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots$$

Ben and Cassie entered these formulas into a spreadsheet hoping to find the value of e correct to 12 decimal places.

In column A, they entered values for n, at first incremented by 1, from 1 to 20, and then multiplied by 100 from 20 to 20 x 100^5 = 200 000 000 000, (two hundred billion).

In cell B1 Ben and Cassie entered the formula = (1 + 1/A1)^A1.

In cell C1 they entered the formula =1+1/FACT(A1).

In C2 they entered = C1+1/FACT(A2).

They copied down column B from B1 to B25 and copied down column C from C2 to C16.

	A	B	C
2	2	2.25000000000000	2.50000000000000
3	3	2.37037037037037	2.66666666666667
4	4	2.44140625000000	2.70833333333333
5	5	2.48832000000000	2.71666666666667
6	6	2.52162637174211	2.71805555555556
7	7	2.54649969704071	2.71825396825397
8	8	2.56578451395035	2.71827876984127
9	9	2.58117479171320	2.71828152557319
10	10	2.59374246010000	2.71828180114638
11	11	2.60419901189753	2.71828182619849
12	12	2.61303529022468	2.71828182828617
13	13	2.62060088788573	2.71828182844676
14	14	2.62715155630087	2.71828182845823
15	15	2.63287871772792	2.71828182845899
16	16	2.63792849736660	2.71828182845904
17	17	2.64241437518311	
18	18	2.64642582109769	
19	19	2.65003432664044	
20	20	2.65329770514442	
21	2000	2.71760256932299	
22	200000	2.71827503279984	
23	20000000	2.71828175433185	
24	2000000000	2.71828203081451	
25	200000000000	2.71828205336391	

(i) What is the value of *e* correct to 12 decimal places?

(ii) Which formula converged more rapidly?

a) *e* = 2.718 282 053 364 to 12 decimal places.
The second formula converges more rapidly.

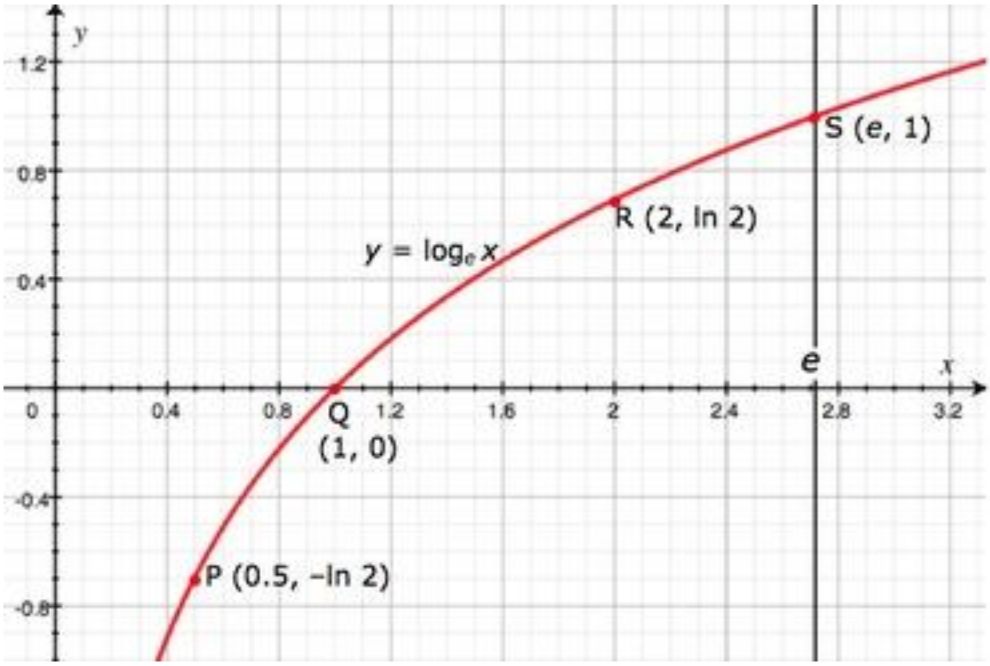
b) *e* = 2.718 282 053 364 to 12 decimal places.
The first formula converges more rapidly.

c) *e* = 2.718 281 828 459 to 12 decimal places.
The second formula converges more rapidly.

d) *e* = 2.718 282 828 459 to 12 decimal places.
The first formula converges more rapidly.

Answer: ____________________

7. At which point, P, Q, R, or S, is the slope of the tangent to $y = \log_e x$equal to 1?



Answer with a single letter, P, Q, R or S.

Answer: ____________________

8. Which of the following formulas are **incorrect**, (*c* is any constant)?

(i) $\frac{d}{dx}(e^x + c) = e^x$

(ii) $\frac{d}{dx}e^x = (e^x + c)$

(iii) $\int e^x \, dx = (e^x + c)$

(iv) $\int (e^x + c) \, dx = e^x$

a) Both (i) and (iii).

b) Both (ii) and (iv).

c) Both (ii) and (iii).

d) Both (i) and (iv).

Answer: ____________________

9. Which of the following formulas are **correct**, (c is any constant)?

(i) $\frac{d}{dx}\ln x = \frac{1}{x}$

(ii) $\frac{d}{dx}\frac{1}{x} = \ln x$

(iii) $\int \log_e x \, dx = \frac{1}{x} + c$

(iv) $\int \frac{1}{x} dx = \log_e x + c$

(v) $\frac{d}{dx}\frac{1}{f(x)} = \ln f(x)$

(vi) $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

(vii) $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$

(viii) $\int \ln f(x) \, dx = \frac{f'(x)}{f(x)}$

a) All of (i), (iv), (vi), (vii).

b) All of (i), (v), (vi), (viii).

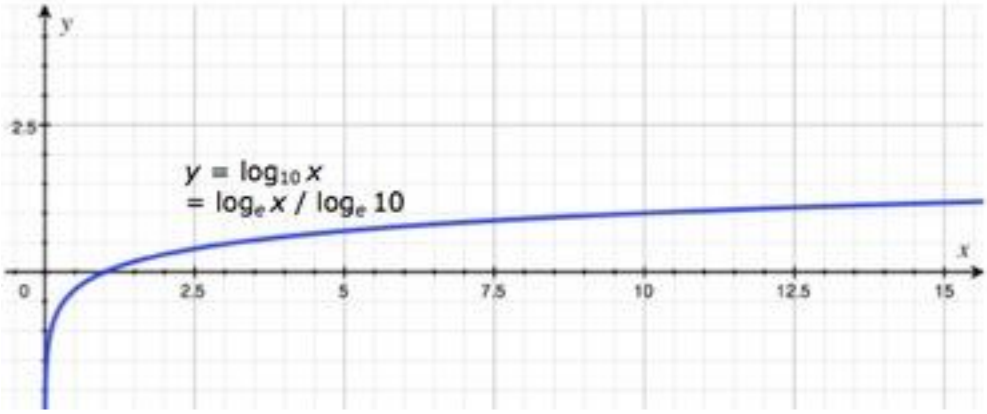
c) All of (ii), (iii), (v), (viii).

d) All of (ii), (iii), (vi), (vii).

Answer: ____________________

10. Brett is asked to find the value of x for which the derivative of $y = \log_{10}x$ with respect to x is equal to 1.

He changes the base of the logarithm from 10 to e and uses a computer drawing program to draw the graph.



(i) If Brett used the graph to estimate the required value of x, what would be a reasonable estimate for him to make?

(ii) What is the exact value of x for which the derivative of $y = \log_{10}x$with respect to x is equal to 1?

a) (i) $x \approx -0.4$; (ii) $x = \log e\ 10$.
b) (i) $x \approx 0.4$; (ii) $x = \log_e 10$.
c) (i) $x \approx -0.4$; (ii) $x = \log_{10} e$.
d) (i) $x \approx 0.4$; (ii) $x = \log_{10} e$.

Answer: ____________________

11. Use the Chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, to differentiate with respect to x:

(i) u^{10} where $u = x^3 - 5x^2 + 7$

(ii) e^{x^2}

(iii) $\ln(3 - \sqrt{x})$

(iv) $(5 - u^2)^4 + (5 - u^2)^{-1}$ where $u = e^x$.

a) (i) $10(3x^2 - 10x)(x^3 - 5x^2 + 7)^9$;

(ii) $2x e^{x^2}$; (iii) $\frac{1}{2x - 6\sqrt{x}}$;

(iv) $4(5 - e^2x)^3 - (5 - e^2x)^{-2}$.

b) (i) $10(3x^2 - 10x)(x^3 - 5x^2 + 7)^9$;

(ii) $\frac{1}{2x - 6\sqrt{x}}$; (iii) $2x e^{x^2}$;

(iv) $4(5 - e^2x)^3 - (5 + e^2x)^{-2}$.

c) (i) $10(3x^2 - 10x)(x^3 - 5x^2 + 7)^9$;

(ii) $2x e^{x^2}$; (iii) $\frac{1}{2x - 6\sqrt{x}}$;

(iv) $4(5 - e^2x)^2 - (5 - e^2x)^2$.

d) (i) $10(x^3 - 5x^2 + 7)^9$;

(ii) $2x e^{x^2}$; (iii) $\frac{1}{2x - 6\sqrt{x}}$;

(iv) $4(5 - e^2x)^3 - (5 - e^2x)^{-2}$.

Answer: ____________________

12. Use the Product Rule, $\frac{d}{dx}\,u(x).v(x) = u'(x).v(x) + u(x).v'(x)$, to differentiate:

(i) xe^x

(ii) $(x^2 - 1).(x + 1)^{-1}$

(iii) $\ln x^x$

(iv) $\frac{u(x)}{v(x)}$.

a) (i) $e^x(x - 1)$; (ii) 0; (iii) $1 + \ln x$;

(iv) $$\frac{v(x).u'(x) - u(x).v'(x)}{[v(x)]^2}$$

b) (i) $e^x(x - 1)$; (ii) 1; (iii) $1 + \ln x$;

(iv) $$\frac{v(x).u'(x) - u(x).v'(x)}{[v(x)]^2}$$

c) (i) $e^x(x + 1)$; (ii) 0; (iii) $1 - \ln x$;

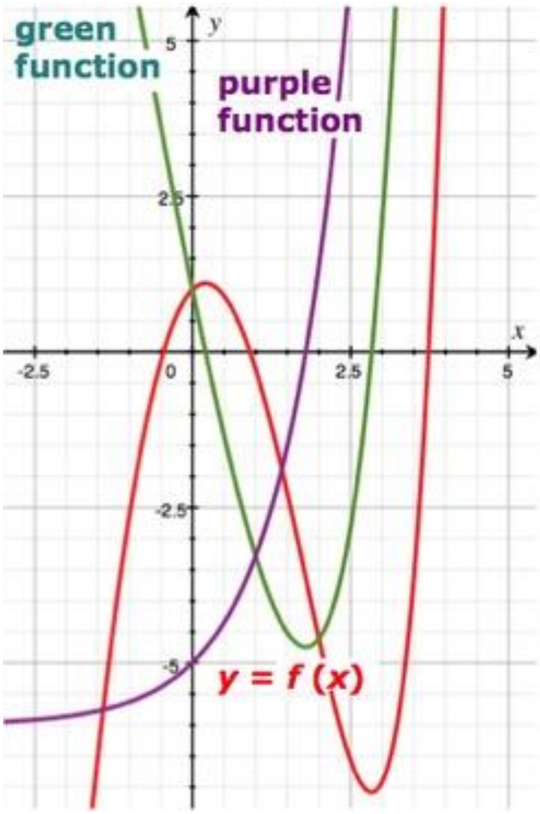
(iv) $$\frac{v(x).u'(x) - u(x).v'(x)}{[v(x)]^2}$$

d) (i) $e^x(x + 1)$; (ii) 1; (iii) $1 + \ln x$;

(iv) $$\frac{v(x).u'(x) - u(x).v'(x)}{[v(x)]^2}$$

Answer: ____________________

13. Jed has drawn three graphs. The red graph is of the function $f(x) = (e^x - 3x^2)$.
The other two graphs are of the first derivative $f'(x)$ and the second derivative $f''(x)$.
Jed has left labels off the green graph and the purple graph.

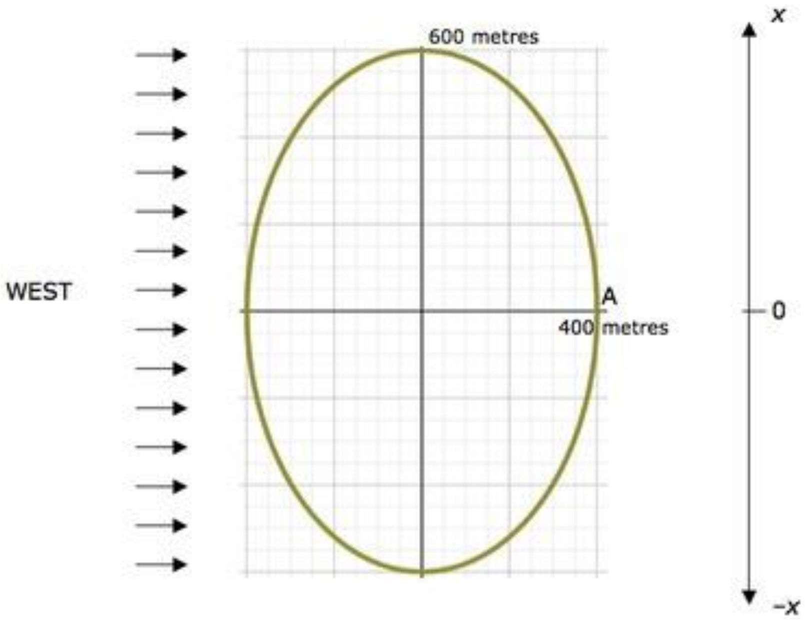


What is the colour of the graph of $y = f''(x)$.

Answer GREEN or PURPLE

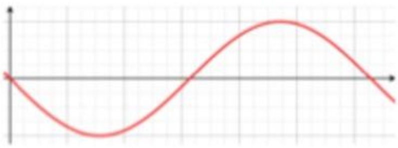
Answer: ____________________

14. A racehorse is galloping anti-clockwise at a constant linear speed around the elliptical track shown in the diagram.
When time, t = 0, the horse is passing the point A.

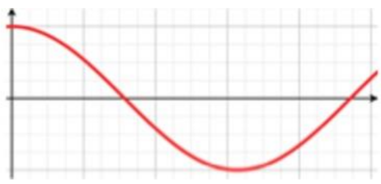


The Sun, setting in the west, casts a shadow of the horse on a straight fence running north-south (shown by the x number line).
Which graph could show the acceleration of the horse's shadow plotted against time?

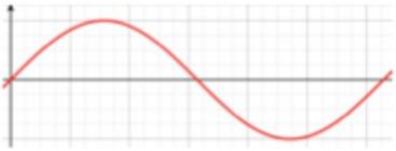
GRAPH A



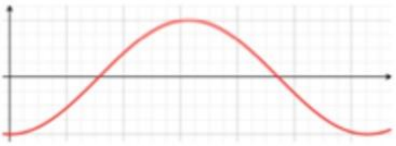
GRAPH B



GRAPH C



GRAPH D



Answer A, B, C or D.

Hint: Consider when the velocity of the shadow will be close to constant, when the velocity will be decreasing and when it will be increasing.

Answer: ____________________

15. Which of the following statements are correct?

On the graph of $y = f(x)$:

(i) If $f(a) \geq f(x)$ for all x in the domain of $f(x)$, then $f(a)$ is an absolute maximum.

(ii) If $f'(a) = 0$ and $f''(a) > 0$ then $f(a)$ is a local minimum.

(iii) If $f'(a) = 0$ and $f''(a) = 0$ then $(a, f(a))$ must be a point of inflection.

(iv) If $f'(a) > 0$ and $f''(a) = 0$ then $f(x)$ is increasing at $x = a$ and $(a,f(a))$ is a point of inflection.

(v) $y = x^4 - x^3 + 1$ has a local minimum at $x = 0$.

a) (i), (ii), (iii)

b) (i), (iii), (iv)

c) (ii), (iv), (v)

d) (i), (ii), (iv)

Answer: ____________________

16. Find the coordinates of all local maxima, minima and points of inflection on the graph of $y = e^{-x^2}$.

a) Local maximum (0,1)
   Inflections $(\frac{1}{\sqrt{e}}, \frac{-1}{\sqrt{2}}), (\frac{-1}{\sqrt{e}}, \frac{1}{\sqrt{2}})$

b) Local maximum (0,1)
   Inflections $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{e}})$

c) Local minimum (0,1)
   Inflections $(\frac{1}{\sqrt{e}}, \frac{1}{\sqrt{e}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

d) Local maximum (0,1)
   Inflections $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{e}})$

Answer: ____________________

The Answers.

Hey! No peeking until you've finished…

Question 1

Answer: B

If H is the point (h, 2.5^h) on $y = 2.5^x$, the gradient of GH is $(2.5^h - 1)/(h - 0)$.
In the limit as h goes to zero, GH becomes the tangent to $y = 2.5^x$ at G

and the slope of the tangent, and of the curve, at G is

$$m = \lim_{h \to 0} \frac{2.5^h - 1}{h}.$$

The straight line AB which touches the curve at G is the tangent.

Question 2

Answer: A

In column C, where $a = 2.7$, by the sixth row the value (2.7^A6 – 1) / A6, has fallen below 1 and is getting smaller.

In column D, where $a = 2.8$, by the tenth row the value (2.8^A10 – 1) / A10, is getting smaller but is still above 1.

It is reasonable to assume that the required value for a is between 2.7 and 2.8.

NOTE: This process can only be continued on a spreadsheet, (or a calculator), for a finite number of values of x.

You could investigate and see how far Terry could have continued before the spreadsheet rounded the decreasing $x$values to zero.

At that point the spreadsheet calculations would become unreliable and a division by zero error would occur.

Question 3

Answer: A

The derivative of e^x is e^x.

(i) The slope of the tangent at F (0, 1) is $e^0 = 1$.

The equation of the tangent at F is $(y – 1) = 1(x – 0)$ which is $y = x + 1$.

(ii) The slope of the tangent at G (1, e) is $e^1 = e$.

The equation of the tangent at G is $(y – e) = e(x – 1)$ which is $y = ex$.

Question 4

Answer: C

(i) It is True that the power series equals e^x for all values of x.

They both pass through (0, 1) and have the same gradient for all values of x

(ii) Equating the constant terms $a_0 = a_1$.

Equating coefficients of x, $a_0 = 2a_2$.

…

Equating coefficients of x_n, $a_{n-1} = na_n$.

(iii) $a_n = a_{n-1} / n$

$= a_{n-2} / n(n – 1)$

$= a_{n-3} / n(n – 1)(n – 2)$

…

$= a_0 / n(n – 1)(n – 2) … 3.2.1$

$= 1/n!$, since the graph passes through (0, 1), $a_0 = 1$.

(iv) When $x = 1$,

$$y = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$$

Question 5

Answer: D

The exponential function and logarithmic functions can be defined as follows:

e^x is the function which equals its own derivative and $\log_e x$ is the inverse of e^x.

or

$\log_e x$ can be defined as the integral of x^{-1} and e^x as the inverse of $\log_e x$.

In either case the red graph is $y = \log_e x$

Question 6

Answer: C

Column B, even with n = two hundred billion agrees with the calculator value rounded off only to a maximum of 6 decimal places.

It has not yet reached e to 12 decimal places.

Column C, with $n = 12$ agrees with the calculator value for $e$rounded off to 9 decimal places.

Column C has the same value for e rounded off to 12 decimal places for $n = 15$ and $n = 16$.

Question 7

Answer: Q

The derivative of $\log_e x$ is x^{-1}. is x^{-1} and is equal to 1 when $x = 1$.

Q is the point on the graph where $x = 1$.

Question 8

Answer: B
(i) and (iii) are correct.

Question 9

Answer: A

(i) is correct.

It follows that (iv), (vi) and (vii) are also correct.

Question 10

Answer: D

(i) A line with slope = 1 just touches the curve at approximately (0.4, –0.4).

(ii) The derivative of $y = \log_{10} x = \log_e x / \log_e 10$ is equal to $1/(x \log_e 10)$.

$1/(x \log_e 10) = 1$.

Then $x = 1/\log_e 10 = \log_e e / \log_e 10 = \log_{10} e$.

Question 11

Answer: A

The derivative of $x^n = nx^{n-1}$ for all values of n.

The derivative of $e^x = e^x$.

The derivative of $\ln x = 1/x$.

Question 12

Answer: D

(i) $u(x) = x$, $u'(x) = 1$, $v(x) = e^x$, $v'(x) = e^x$.

(ii) $u(x) = (x^2 - 1)$, $u'(x) = 2x$, $v(x) = (x + 1)^{-1}$, $v'(x) = -(x + 1)^{-2}$.

(iii) $u(x) = x$, $u'(x) = 1$, $v(x) = \ln x$, $v'(x) = x^{-1}$.

(iv) Let $V(x) = \{v(x)\}^{-1}$, $V'(x) = -1\{v(x)\}^{-2}.\ v'(x)$.

Substitute these expressions into the Product Rule and simplify where necessary.

Question 13

Answer: PURPLE

The stationary points of the red graph have the same x values as the zeros of the Green graph. The Green graph is $y = f'(x)$.

The stationary point of the Green graph has the same x value as the zero of the Purple graph. The Purple graph is $y = f''(x)$.

Question 14

Answer: A

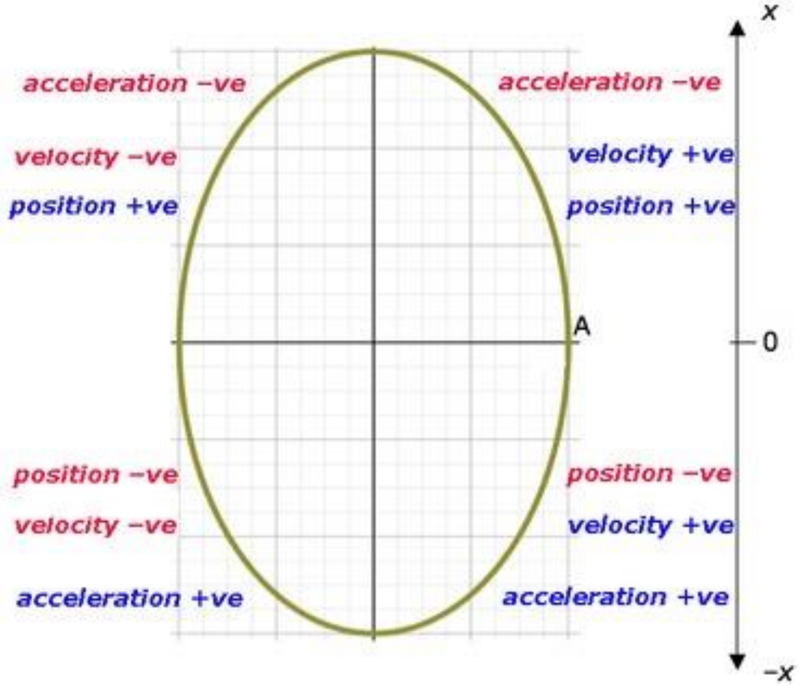
As the horse gallops around the track the shadow moves north and south along the fence $-x0x$.

When the section of the track is nearly parallel to the fence the shadow's speed will be greater. When the section of the track is nearly perpendicular to the fence the shadow's speed will be less.

The acceleration is positive when the velocity (but not necessarily the speed) is increasing positively.

The acceleration is negative when the velocity (but not necessarily the speed) is decreasing.

The diagram shows the signs of x-position, velocity and acceleration of the shadow when the horse is on different quarters of the track.



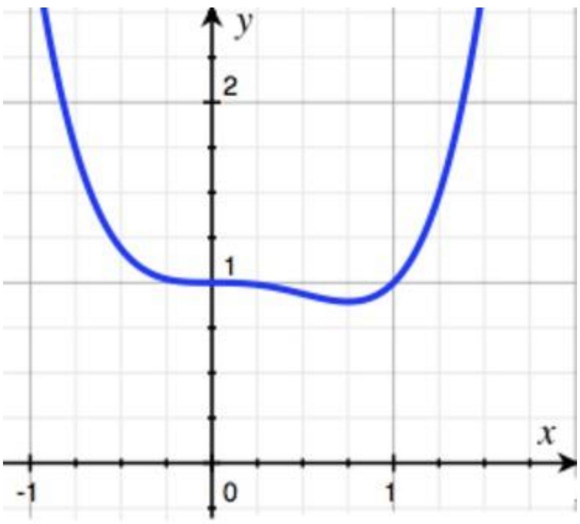
Question 15

Answer: B

(iii) is false because if $f''(a) = 0$ and $f'(a) = 0$, there may be a stationary point of inflection, or a local maximum or a local minimum at $x = a$.

(v) is false. Look at the graph of $y = x^4 - x^3 + 1$ below.

There is a stationary point of inflection at (0, 1).



Question 16

Answer: B

$$\frac{d}{dx}e^{-x^2} = -\mathbf{2x}e^{-x^2},$$

$$\frac{d^2}{dx^2}e^{-x^2} = (4x^2 - 2)e^{-x^2}.$$

When $x = 0$ the first derivative is zero and the second derivative is negative.

(0, 1) is a local maximum.

When $x = \pm 1/\sqrt{2}$ the second derivative is zero and the first derivative is not zero.

($\pm 1/\sqrt{2}$, $1/\sqrt{e}$) are points of inflection, as shown on the graph.