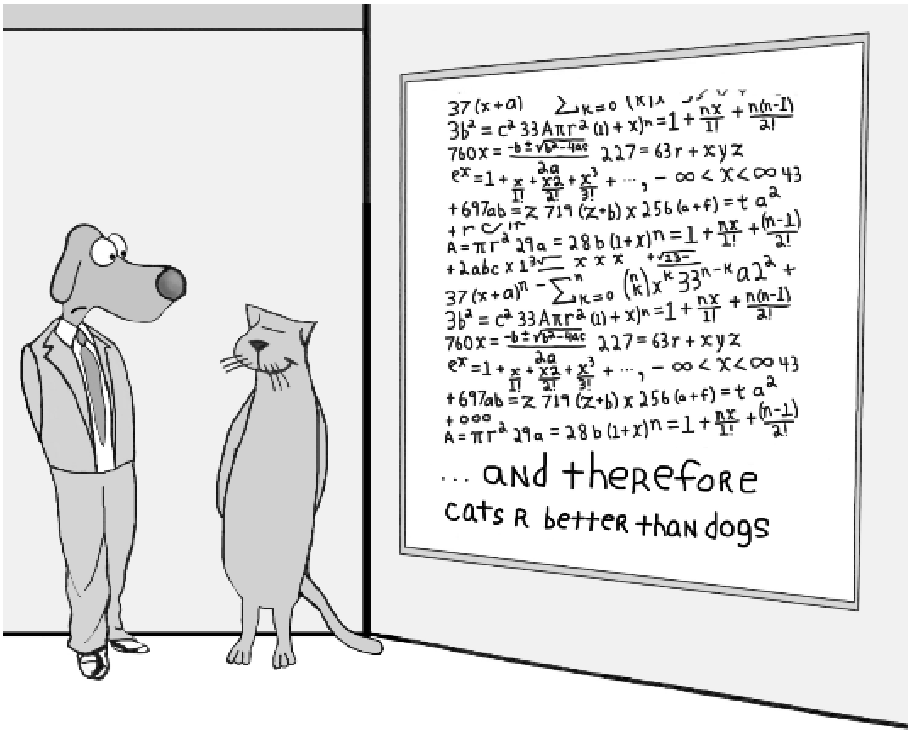
Year 12 Mathematical Methods

Linear Equations and Continuous Random Variables Worksheet

8 Questions on Linear Equations
and
10 questions on Continuous Random Variables
from the Maths C (Maths Methods) national curriculum for Year 12



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Section 1 – Linear Equations

1. Megan is solving simultaneously the three equations:

   $x - 5y - 4z = -5$ .... (i)

   $2x + 13y - z = 68$ .... (ii)

   $5x - 2y + 4z = 19$.......................... (iii)

   Which steps should Megan begin with to eliminate x?

   A: Add two times equation (i) to equation (ii).

   Then add five times equation (i) to equation (iii).

   B: Add thirteen times equation (i) to five times equation (ii).

   Then add twice times equation (i) to five times equation (iii).

   C: Subtract two times equation (i) from equation (ii).

   Then subtract two times equation (ii) from five times equation (iii).

   D: Subtract two times equation (i) from equation (ii).

   Then subtract five times equation (i) from equation (iii).

   Circle the correct answer A, B, C or D.

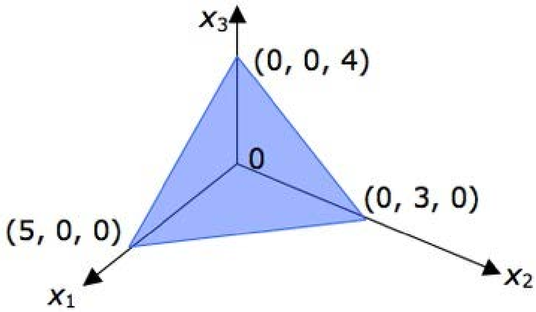
2. $k(x + 3y - 6) + (3x - 3y + 7) = 0$ represents a family of straight lines as k varies from $-\infty$ to $+\infty$.

Which values of k will give you the member of the family that is (i) parallel to the x-axis, (ii) parallel to the y-axis?

a) $k = -3$; (ii) $k = 1$.
b) $k = -1$; (ii) $k = 3$.
c) $k = 3$; (ii) $k = -1$.
d) $k = 1$; (ii) $k = -3$.

Circle the correct answer A, B, C or D.

3. The linear equation $c_1X_1 + c_2X_2 + c_3X_3 = b$ where $c_1 = 12$, $c_2 = 20$, $c_3 = 15$ and $b = 60$ is represented in three dimensions by the plane shown in the diagram.



The system of linear equations:

$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = b_1,$

$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = b_2,$

$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = b_3.$

is a set of equations of three planes in three dimensions.

How many solutions will there be to this system of equations if, in a multi-storey building:

(i) the three planes are the first, second and third floors,
(ii) two of the planes are the first and second floors and the third plane is a ramp leading from the first floor to the second floor,
(iii) one plane is the flat roof and the other planes are the east and north walls of the building.

a) (i) ∞; (ii) ∞; (iii) 1.

b) (i) 0; (ii) 0; (iii) 1.

c) (i) 1; (ii) 1; (iii) 0.

d) (i) 1; (ii) ∞; (iii) ∞.

Circle the correct answer A, B, C or D.

4. $k(4x_1 + 2x_2 - 3) + (3x_1 - 3x_2 + 2) = 0$ for values of k from $-\infty$ to $+\infty$, represents a family of straight lines on the x_1, x_2 - Cartesian plane.

   If P is the point of intersection of these lines and Q is the point (1, 1) find:

   (i) the value of k for the line through P with slope 1

   (ii) the equation of the line PQ.

   a) *(i)* $k = 0$; (ii) $x_1 - 13x_2 + 12 = 0$.
   b) *(i)* $k = 1$; (ii) $-x_1 + 13x_2 - 12 = 0$.
   c) *(i)* $k = 0$; (ii) $x_1 + 13x_2 - 12 = 0$.
   d) *(i)* $k = 1$; (ii) $x_1 - 13x_2 + 12 = 0$.

Circle the correct answer A, B, C or D.

5. Billy used Gaussian elimination to solve the system of equations:
   $x_1 - x_2 + x_3 + 3x_4 = 4$, ... equation (i)
   $2x_1 - x_2 - x_3 = 2$, ... equation (ii)
   $5x_1 + 2x_2 - 2x_3 + 2x_4 = 0$, ... equation (iii)
   $3x_1 + x_2 + 5x_4 = 2$, ... equation (iv)

   His first three steps were:
   Step 1. Replace (ii) with {(ii) – twice (i)}.
   Step 2. Replace (iii) with {(iii) – 5 times (i)}.
   Step 3. Replace (iv) with {(iv) – 3 times (i)}.

   Billy's new equations are:
   $x_1 - x_2 + x_3 + 3x_4 = 4$, ... equation (i)
   $x_2 - 3x_3 - 6x_4 = -6$, ... new equation (ii)
   $7x_2 - 7x_3 - 13x_4 = -20$, ... new equation (iii)
   $4x_2 - 3x_3 - 4x_4 = -10$, ... new equation (iv)

   His next steps were:
   Step 4. Replace (iii) with {(iii) –7 times (ii)}.
   Step 5. Replace (iv) with {(iv) –4 times (ii)}.

   Billy's next equations are:
   $x_1 - x_2 + x_3 + 3x_4 = 4$, ... equation (i)
   $x_2 - 3x_3 - 6x_4 = -6$, ... new equation (ii)
   $14x_3 + 29x_4 = 22$, ... next equation (iii)
   $9x_3 + 20x_4 = 14$, ... next equation (iv)

   What should Billy do in Step 6?

   a) Step 6: Replace (iv) with {–9/14 times (iii)}.

   b) Step 6: Replace (iv) with {(iv) –9/14 times (iii)}.

   c) Step 6: Replace (iv) with {–14/9 times (iii)}.

   d) Step 6: Replace (iv) with {(iv) –14/9 times (iii)}.

Circle the correct answer A, B, C or D.

6. Tammy reduces to row echelon form the augmented matrices for three different systems A, B and C of linear equations in four variables x_1, x_2, x_3, x_4 .

Tammy finds that the augmented matrix

for System A reduces to $\begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 3 & -1 \end{pmatrix}$,

for System B reduces to $\begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

and for System C reduces to, $\begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$.

Which statement is correct?

(i) In System A there is no solution, in System B infinitely many solutions and in System C only one solution.

(ii) In System A there is only one solution, in System B no solution and in System C infinitely many solutions.

(iii) In System A there are infinitely many solutions, in System B only one solution and in System C no solution.

Circle the correct answer (i), (ii) or (iii)

7. $4x_1 - 3x_2 - 3x_3 - 17x_4 = 0$

$-2x_1 + x_2 + x_3 + 9x_4 = 0$

$2x_1 - 3x_2 - 3x_3 - 9x_4 = 0$

By reducing the augmented matrix of the above system of linear equations to row echelon form, find the solution set for this system of linear equations.

a) The solution set is $(a, a, -a, 0)$ for any a from $-\infty$ to $+\infty$.
b) The solution set is $(0, a, a, 0)$ for any a from $-\infty$ to $+\infty$.
c) The solution set is $(a, a, -a, -a)$ for any a from $-\infty$ to $+\infty$.
d) The solution set is $(0, a, -a, 0)$ for any a from $-\infty$ to $+\infty$.

Circle the correct answer A, B, C or D.

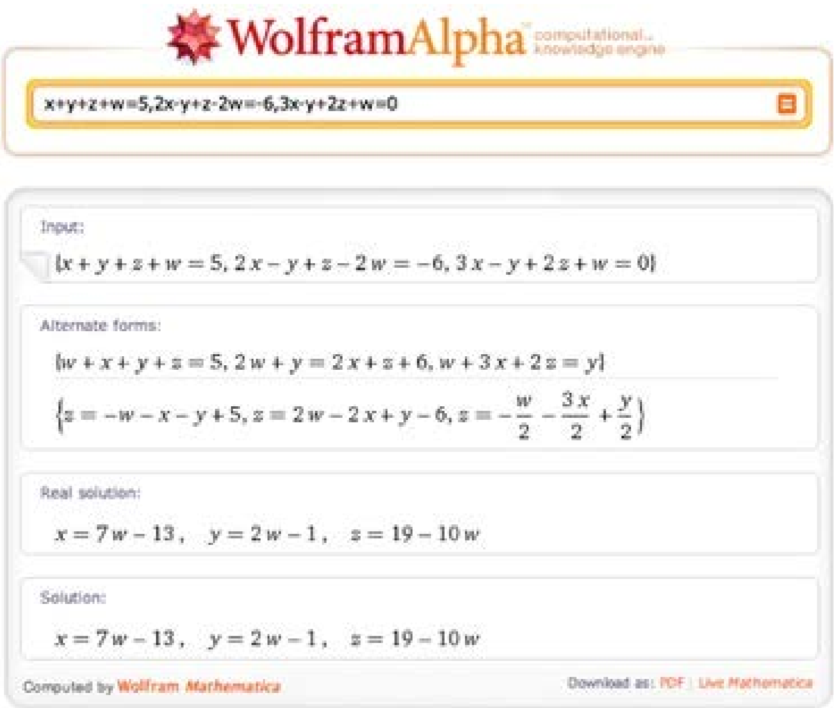
8. Petra enters the three linear equations:

$x + y + z + w = 5$

$2x - y + z - 2w = -6$

$3x - y + 2z + \mathrm{w} = 0$

into the Wolframalpha program on the website http://www.wolframalpha.com/ This program solves the equations simultaneously, as shown in the screen shot below.



Which row vector is **not** in the solution set of these three equations?

(i) (1, 3, –1, 2)

(ii) (–6, 1, 9, 1)

(iii) (–20, –3, 29, –1)

(iv) (–13, –1, 9, 0)

Circle the correct answer (i), (ii), (iii) or (iv).

SECTION 2 – CONTINUOUS RANDOM VARIABLES

Question 1. Discrete and continuous random variables.

A random variable represents, in number form, the possible outcomes which could occur for some random experiment.

A discrete random variable X has possible exact values for example, $x = 1, 2, 3, 4\ldots$

Examples of discrete random variables include:

· number of broken eggs in boxes sold by the supermarket

· number of students with blue eyes in a classroom

· number of separate keyboards sold by the computer store

A continuous random variable is one which takes an infinite number of possible values. Continuous random variables are usually measurements.

Examples of continuous random variables include:

· the level of water in a dam over a month

· the heights of flight attendants (cm), X where $160 < x < 185$

· Pamela's weight over a month when she was dieting

Choose the continuous random variables from the following list:

(i) your mark out of 100 in a Mathematics test

(ii) marketable length of trout

(iii) the number of scales on a fish

(iv) the time it takes to run 100 metres

(v) yield in kg of pears from a tree each season in an orchard.

Answer: ________________________________

Question 2. Continuous random variables.

Continuous random variables can take any value over a certain domain. They can be categorised in various ways such as uniform, exponential, or normal.

Read the following on uniform and exponential continuous random variables.

(a) The values of a uniform continuous random variable are uniformly distributed over an interval. If there is a train every 20 minutes, if you do not know the timetable, the number of minutes that you wait for your train is uniformly distributed over the interval [0, 20] minutes.

(b) An exponential random variable is often concerned with the amount of time until some specific event occurs. Examples include the length, in minutes, of long distance business telephone calls, and the amount of time a car battery lasts. It can be shown, too, that the amount of change that you have in your pocket or purse follows an exponential distribution. There are fewer large values and more small values. For example, the amount of money customers spend in one trip to the supermarket follows an exponential distribution. There are more people that spend less money and fewer people that spend large amounts of money.

The exponential distribution is widely used in the field of reliability. Reliability deals with the amount of time a product lasts or the accumulation of faults over time.

Select which of the following statements are true.

(i) The length of time light globes last is exponential.

(ii) The angle at which a pointer stops when it is spun is uniform.

(iii) The length of time you wait to be served in the bank is exponential.

(iv) The time spent waiting for a bus if a bus comes every 20 minutes is exponential.

Answer: ____________________________________

Question 3. Continuous random variables: The probability density function

A continuous distribution is a sample space divided into an infinite number of samples. The probability of the entire sample space that equals 1 must be divided among such a great number of samples, the probability for any one sample is infinitesimally small. No point has a probability in a continuous distribution, only regions have a probability. There is no probability per sample, only a probability per region.

Regions contain a proportion of the entire sample space, but points do not. No equation can provide the probability at a given point. Instead, the continuous probability density function specifies the probability per unit region. You must add up the probabilities of many smaller regions within it to calculate the probability. This is done by integrating the area under the function's curve for the region required.

A probability density function is a function f defined on an interval [a, b] and having the following properties:

$f(x) \geq 0$ for all x

$\int_a^b f(x)dx = 1$

For any specific value, the probability is equal to 0. That is

$\int_c^c f(x)\, dx = 0$

P($a \leq X \leq c$) = $\int_a^c (f)x\ dx \quad a \leq c \leq b$ This calculates the probability that the random variable X will occur in the region between a and c.

Look at this simple uniform example. A spinner is spun on a circular plate and can land on any of the 360°. The function describing this continuous random variable is

$f(x) = \frac{1}{360-0} = \frac{1}{360} \qquad$ for $0^0 \leq x \leq 360^0$

Choose all correct statements:

(i) $\int_0^{360} \frac{1}{360} dx = 1$

(ii) $\int_{30}^{30} \frac{1}{360} dx = 0$

(iii) The probability that the spinner stops between 20° and 30° is represented by

P($20 \leq X \leq 30$) $\int_{20}^{30} \frac{1}{360} dx = \frac{1}{36}$

A. (ii) only
B. (i), (ii), (iii)
C. (i) only
D. (iii) only

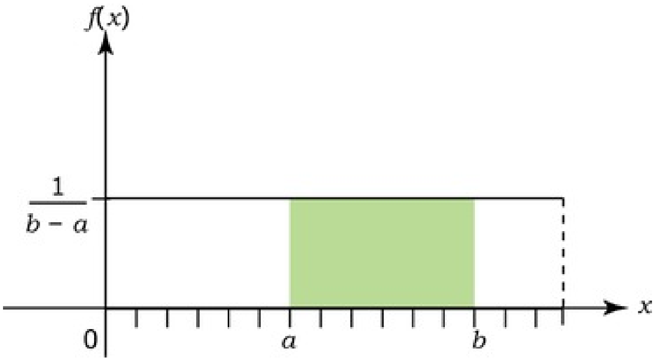
Circle the correct answer: A, B, C or D

Question 4. Uniform continuous random variables

A uniform density function, is a density function that is constant. A uniform density function on the interval [a,b] is the constant function defined by

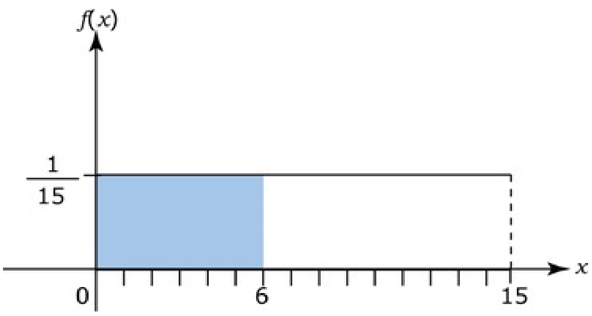
$f(x) = 1/(b-a)$

Its graph is the horizontal line.



Example: On the northwest train line in Sydney, the time a person waits for a train is uniformly distributed between 0 and 15 minutes. What is the probability that a person waits less than 6 minutes?

Suppose X is the number of minutes the person waits for the train. Then $a = 0$ and $b = 15$.
The probability density function is $f(X) = 1/(15 - 0) = 1/15$ for $0 < X < 15$
$P(X < 6)$ = base x height = $(6 - 0) \times 1/15 = 6/15 = 2/5$



This is the area under the line between 0 and 6.

Calculate the probability of waiting between 7 and 10 minutes for a train.

A. 1/5

B. 2/5

C. 7/10

D. 14/25

Circle the correct answer A, B, C or D.

Question 5. Exponential random variables

If a probability density function is given for $X > 0$ by $f(X) = me^{-mx}$ then X is an exponential random variable with parameter m.

In the boom before the recession, many restaurants started up in the city. During the recession years it was noticed that these newly set up restaurants failed continuously at the rate of 8% of those remaining per year. What is the probability that one of these restaurant will fail between 1 to 3 years from now?

The probability density function is given by

$f(X) = 0.08e^{-0.08x}$ $X \geq 0$

$$P(1 < X < 3) = \int_1^3 0.08e^{-0.08x}dx = \left[-e^{-0.08x}\right]_1^3 = -e^{-0.24} - (-e^{-0.08}) = -e^{-0.24} + e^{-0.08}$$

$\approx -0.7866 + 0.9231$

≈ 0.1365

Calculate the probability that a restaurant will fail between 3 and 5 years from now. Write your answer correct to 4 decimal places here:

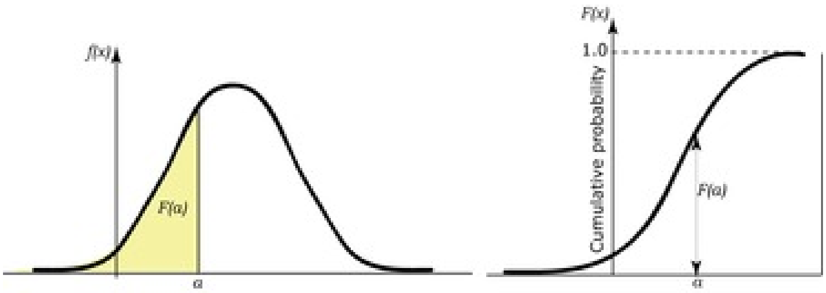
Answer: ______________________________

Question 6. The cumulative distribution function for continuous random variables
All random variables (discrete and continuous) have a cumulative distribution function. It is a function giving the probability that the random variable X is less than or equal to x, for every value x.
The function representing the cumulative frequency is usually written with upper case F.
The probability density function is written with lower case f.

For all real numbers $(-\infty < x < +\infty)$, let $F(x) = P(X \leq x)$.

For a continuous random variable, the cumulative distribution function is the integral of its probability density function.

To find $F(a)$, you integrate from negative infinity to a. This is the probability that the continuous random variable X will take on a value equal to or less than a. You are summing the infinite number of vertical density lines between negative infinity and a. An ogive is the graphical representation of a continuous cumulative distribution. Because the total area (probability) under the continuous probability distribution is 1.0,$F(x)$ rises continuously to the right to a maximum probability of 1.0



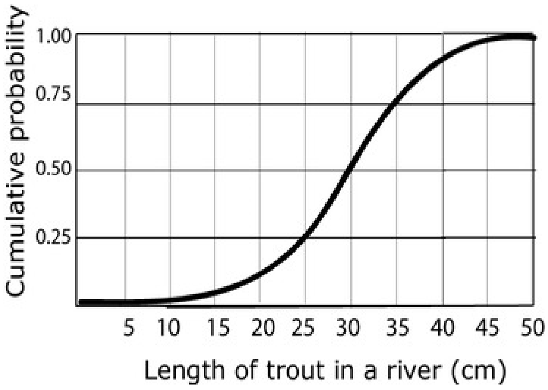
It is always true for continuous random variables that $P(X=x)=0$ and therefore, that $P(a<X\leq b) = P(a\leq X\leq b) = P(a<X<b) = P(a\leq X<b)$.
Formerly, for any continuous variable cumulative distribution function $F(x)$, given any two numbers a and b such that $a<b$, then
$P(a<X\leq b) = P(a\leq X\leq b) = P(a<X<b) = P(a\leq X<b) = F(b) - F(a)$

For all real numbers $(-\infty < x < +\infty)$, $F(x) = P(X \leq x)$.
You find $P(X \leq a)$ by calculating

$$F(a) = \int_{-\infty}^{a} f(x)dx$$

Using the cumulative frequency curve for continuous random variables

Look at the ogive below.



Reading from the 0.5 line, you can see that 50% of the trout are 30 cm or less and 50% of the trout are 30 cm or more. The median length is 30 cm.

Calculate the interquartile range for this data and put your answer here.

Answer: ______________________________

Question 7. Continuous random variables: mean, variance and standard deviation

You should already be familiar with calculating the mean for discrete random variables. The formulas are similar, but for continuous random variables, you use the integral sign in your calculation of the mean or the expected value.

Mean = expected value = $E(X) = \int_{-\infty}^{\infty} xf(x)\, dx$

Calculate the expected value of waiting for a train if a train comes every 15 minutes. That is, you wait between 0 and 15 minutes.

$f(x) = \frac{1}{15-0} = \frac{1}{15}$

Express your answer as a decimal and put your answer here: ____________________

Question 8. Continuous random variables: variance

Variance is used as a measure of how far a set of numbers are spread out from each other. It describes how far the numbers lie from the mean or expected value. It forms part of a systematic approach to distinguishing between probability distributions.

$\mathrm{Var}(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

μ (expected value) = $\int_{-\infty}^{\infty} xf(x)dx$

This can be simplified if you square $(x-\mu)$ and express as separate integrals.

$$\int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - 2\mu \int_{-\infty}^{\infty} xf(x)dx + \mu^2 \int_{-\infty}^{\infty} f(x)dx$$

but $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\int_{-\infty}^{\infty} xf(x)dx = \mu$

$Var(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

Determine the variance if the function is defined as follows:

$f(x) = \begin{cases} 2-2x \text{ for } 0 < x < 1 \\ 0 \text{ otherwise} \end{cases}$ Answer: ______________________________

Question 9. Random continuous variables: standard deviation

Standard deviation shows how much variation there is from the mean or expected value. A low standard deviation indicates that the data points tend to be very close to the mean, whereas high standard deviation indicates that the data are spread out over a large range of values.

The standard deviation is the square root of the variance.

Trains come every 20 minutes. A person, not knowing the timetable, can wait between 0 and 20 minutes.

$$f(x) = \frac{1}{20-0} = \frac{1}{20}$$

Calculate the standard deviation of this distribution. Express it as a decimal correct to 2 decimal places and write your answer here:

____

Question 10. The normal distribution

The normal distribution, in continuous random variables, describes random variables that cluster around a single mean value. The graph of the probability density function is known as a bell curve because of its shape and is given by the function

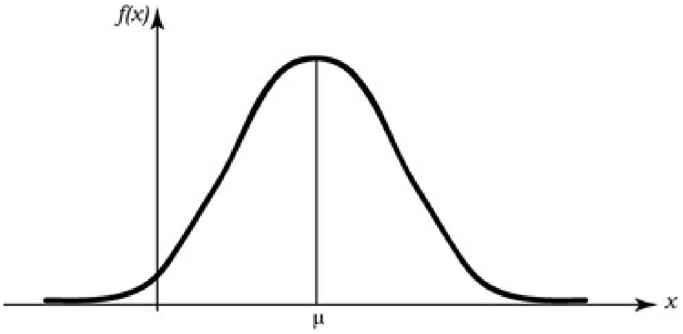
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad \text{for } -\infty < x < \infty$$

These probability density function curves are symmetric about the vertical line $x = \mu$

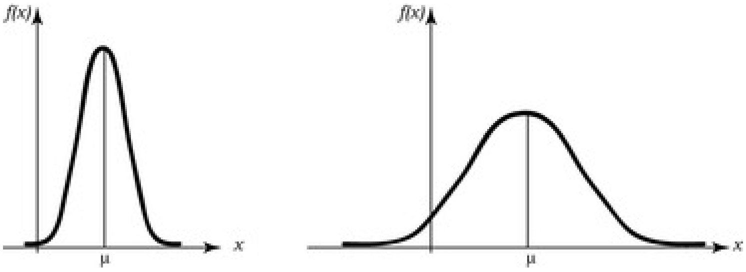
As $x \to \pm\infty$ the curve approaches the x-axis

The area under the curve $\int_{-\infty}^{\infty} f(x)\,dx = 1$

The curve is bell shaped because more scores are distributed closer to the mean.

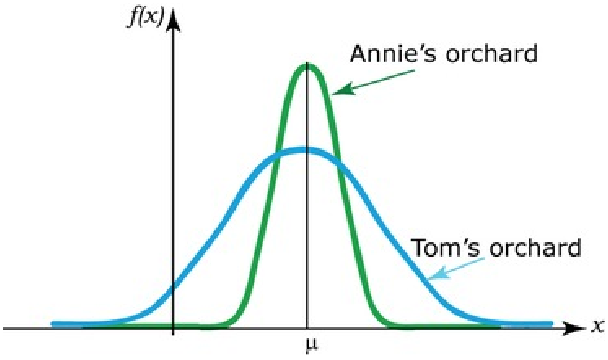


The shape of the bell curve is determined by the variance. If the variance is small, this means the values cluster close to the mean or expected value. In the diagram below, the graph on the left is an example of small variance.



If the variance is large, the curve is spread further to the right and left of the mean, like that seen in the graph on the right hand side.

Tom and Annie own orchards where they grow pears. They compare their outputs and find that the mean output per tree in kilograms is the same for both orchards. The bell curve for each orchard is shown below.



Circle all correct statements:

(i) Tom has a greater range in output from his trees compared to Annie.

(ii) Annie's total production is greater than Tom's.

(iii) The difference between the production per tree is less in Annie's orchard than in Tom's orchard.

The Answers.

Hey! No peeking until you've finished…

SECTION 1 – LINEAR EQUATIONS

Question 1

Answer: D

The coefficient of x in equation (i) is 1, in equation (ii) is 2 and in equation (iii) is 5.

Subtracting 2 times equation (i) from equation (ii) will eliminate x.

Subtracting 5 times equation (i) from equation (iii)will eliminate x.

This will give Megan two equations in two unknowns y and z.

To complete the solution Megan would eliminate y and get one equation in the one unknown, z.

By back substituting she can find the values of y and x.

Question 2

Answer: A

In $k(x + 3y – 6) + (3x – 3y + 7) = 0$ the coefficient of x is $k + 3$.

The coefficient of x is zero when $k = –3$.

Then $k(x + 3y – 6) + (3x – 3y + 7) = 0$ is $–3(x + 3y – 6) + (3x – 3y + 7) = 0$. This simplifies to $y = 25/12$ which is a line parallel to the x-axis.

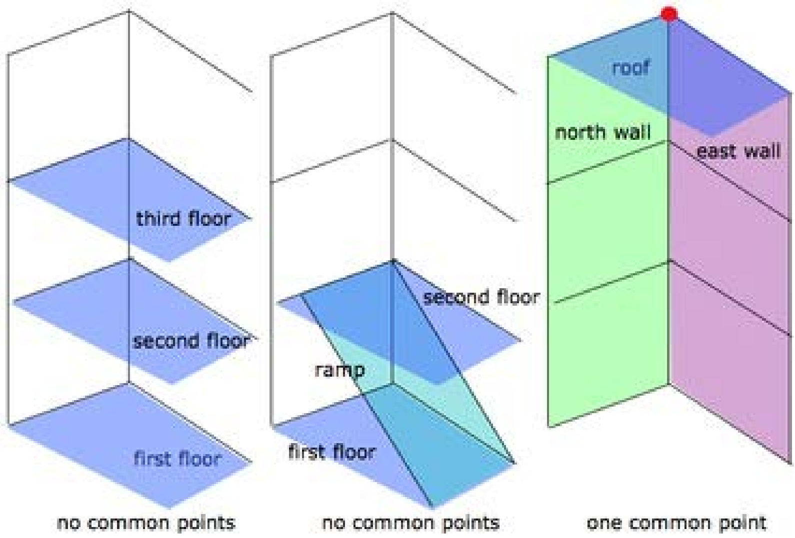
In $k(x + 3y – 6) + (3x – 3y + 7) = 0$ the coefficient of y is $3k – 3$.

The coefficient of y is zero when $k = 1$.

Then $k(x + 3y – 6) + (3x – 3y + 7) = 0$ is $(x + 3y – 6) + (3x – 3y+ 7) = 0$. This simplifies to $x = –1/4$ which is a line parallel to the y-axis.

Question 3

Answer: B



The floors in the building are parallel and have no points in common.

The east and north walls meet each other and the roof in straight lines which all pass through the top, north east corner of the building.

Question 4

Answer: A

(i) $k(4x_1 + 2x_2 – 3) + (3x_1 – 3x_2 + 2) = 0$ can be written as:

$(2k – 3)x_2 = –(4k + 3)x_1 + 3k – 2$ which has slope $–(4k + 3)/(2k– 3)$.

If the slope = 1, $–(4k + 3) = (2k – 3)$ and $6k = 0$.

Therefore $k = 0$.

The line through P with slope one is $3x_1 – 3x_2 + 2 = 0$.

(ii) If Q (1, 1) lies on $k(4x_1 + 2x_2 – 3) + (3x_1 – 3x_2 + 2) = 0$ then

$k(4 + 2 – 3) + (3 – 3 + 2) = 0$ and $3k = –2$.

The line PQ has equation $(–2/3)(4x_1 + 2x_2 – 3) + (3x_1 – 3x_2 + 2) = 0$, which simplifies to $–8x_1 – 4x_2 + 6 + 9x_1 – 9x_2 + 6 = 0$ which is $x_1 – 13x_2 + 12 = 0$.

Question 5

Answer: B

In Step 6 the aim is to eliminate x_3.

This can be achieved by subtracting a suitable multiple of the third equation from the fourth equation.

The suitable multiple to subtract is 9/14.

This will give an equation in only one variable and hence the value of x_4.

The other unknowns can then be found by back substitution.

Question 6

Answer: i

The solution for system A is $x_1 = 4$, $x_2 = 2$, $x_3 + 3x_4 = 7$ and $x_3 + 3x_4 = -1$. These last two equations are inconsistent and no solution exists for System A.

The solution for system B is $x_1 = 4$, $x_2 = 2$, $x_3 + 3x_4 = 7$.

x_4 can be given any value from $-\infty$ to $+\infty$ and x_3 calculated as $x_3 = 7 - 3x_4$. There are infinitely many solutions for System B.

The solution for system C is $x_1 = 4$, $x_2 = 2$, $x_3 = 7$ and $x_4 = -1$. This is the unique solution for System C.

Question 7

Answer: D

At each step multiples of one row have been added or subtracted from another and the resulting rows multiplied or divided to give ones.

$$\begin{pmatrix} 4 & -3 & -3 & -17 & 0 \\ -2 & 1 & 1 & 9 & 0 \\ 2 & -3 & -3 & -9 & 0 \end{pmatrix} \qquad \begin{pmatrix} 4 & -3 & -3 & -17 & 0 \\ -2 & 1 & 1 & 9 & 0 \\ 0 & -2 & -2 & 0 & 0 \end{pmatrix}$$

Original matrix After Step 1.

$$\begin{pmatrix} 0 & -1 & -1 & 1 & 0 \\ -2 & 1 & 1 & 9 & 0 \\ 0 & -2 & -2 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & -1 & 1 & 0 \\ -2 & 1 & 1 & 9 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

After Step 2. After Step 3.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

After Step 4. After Step 5.

The solution set is (0, *a*, -*a*, 0) for any *a* from –∞ to +∞.

Question 8

Answer: (iv)

The Wolframalpha program returns the Solution:

$x = 7w - 13$, $y = 2w - 1$, $z = 19 - 10w$.
Any value may be chosen for w and the corresponding values found for x, y and z.

When w = 0, x = 0 – 13 = –13, y = 0 – 1 = –1, z = 19 – 0 = 19.

Therefore (–13, –1, **19**, 0) is a solution but (iv) (–13, –1, **9**, 0) is not a solution.
Check that (i), (ii) and (iii) are solutions by substituting w = 2, 1 and –1 in the Wolframalpha Solution.

SECTION 2

Question 1. Answer: ii, iv, v

Discrete random variables take an exact value, often a whole number. Continuous random variables can take a range of values and sometimes are expressed as ranging between two possible limits.

(i) your mark out of 100 in a Mathematics test. This is an exact number. This is a discrete random variable.

(ii) marketable length of trout. This is usually 30 to 40 cm. **This is a continuous random variable**.

(iii) the number of scales on a fish. This is an exact, countable number. This is a discrete random variable.

(iv) the time it takes to run 100 metres. **This is continuous** since different people take different times. For really good men athletes, the race can take between 9.58 and 10.8 seconds. For women athletes, it can take between 10.49 and 11 seconds. It would take the rest of the world a lot longer to run.

(v) yield in kg of pears from a tree each season in an orchard. This varies between 87 to 88 kg for a healthy tree. It is a **continuous random variable**.

Question 2. Answer: i, ii, iii

Remember, exponential will have some large values and many smaller values. Uniform random values are uniformly distributed over a range of values.

i) The length of time a light globe lasts is exponential. **Correct**.

ii) The angle at which a pointer stops when it is spun is uniform. **Correct**. The pointer can stop anywhere between 0 and 360 and the positions are uniformly distributed.

iii) The length of time a person waits to be served in the bank is exponential. **Correct**. This is occasionally long but mostly short.

iv) The time spent waiting for a bus if a bus comes every 20 minutes is exponential. Incorrect. This is uniformly distributed over the 20 minutes and therefore it is uniform.

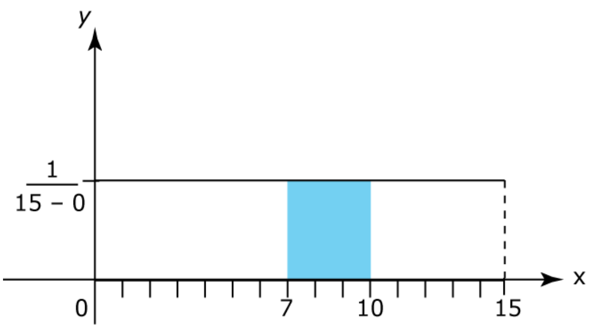
Question 3. Answer: B. (i, ii, iii)

(i) This represents the probability that the spinner will stop on one of the 360 possible degrees. It must have the probability of 1. As you have learnt from looking at tree diagrams for discrete random variables such as tossing a coin, the sum of the probabilities at the end of the branches must add up to 1. This is true for continuous random variables as well that the sum of all the possibilities is 1.

(ii) From the integral you can see that this must equal zero. The probability for any one sample is infinitesimally small. No point has a probability in a continuous distribution, only **regions** have a probability. There is no probability per sample, only a probability per region.

(iii) $P(20 \leq X \leq 30) = \int_{20}^{30} \frac{1}{360} dx = \frac{x}{360}\Big|_{20}^{30} = \frac{30}{360} - \frac{20}{360} = \frac{10}{360} = \frac{1}{36}$ This means that the probability that the spinner stops between 20º and 30º is $\frac{1}{36}$.

Question 4. Answer: A. 1/5



P(7 < X <10) = base x height = (10 - 7) x 1/15 = 3/15 = 1/5

This is the area of the rectangle.

This can also be done by integration, of course.

$$P(7 < X < 10) = \int_{7}^{10} \frac{1}{15} dx = \left[\frac{x}{15}\right]_{7}^{10} = \frac{10}{15} - \frac{7}{15} = \frac{3}{15} = \frac{1}{5}$$

Question 5. Answer: 0.1163

$$P(3 < X < 5) = \int_3^5 0.08e^{-0.08x}dx = [-e^{-0.08x}]_3^5 = -e^{-0.4} + e^{-0.24}$$
$$\approx -0.67032 + 0.78663$$
$$\approx 0.11631$$

There is a probability of approximately 0.1163 that a restaurant will fail.

Question 6. Answer: 10

The interquartile range is the difference between 25% and 75%, that is, between 0.25 and 0.75.

25% of the trout are 25 cm or less and 75% of the trout are 35 cm or less. The difference between these two is 10 cm.

Question 7. Answer: the expected value is 7.5 minutes.

$$E(X) = \int_0^{15} x.\frac{1}{15}\ dx = \int_0^{15}\frac{x}{15}dx = \frac{x^2}{30}]_0^{15} = \frac{225}{30} - \frac{0}{30} = \frac{15}{2} = 7\frac{1}{2}$$

Question 8. Answer: the variance is 1/18

$$\text{Var}(x) = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$$

You must find the expected value or mean first.

$$\mu = \int_0^1 x(2-2x)dx = \int_0^1(2x-2x^2)dx = x^2 - \frac{2x^3}{3}]_0^1 = \frac{1}{3}$$

$$Var(X) = \int_0^1(x-\frac{1}{3})^2(2-2x)dx = 2\int_0^1(x^2-\frac{2x}{3}+\frac{1}{9})(1-x)dx$$

$$= 2\int_0^1(x^2 - x^3 - \frac{2x}{3} + \frac{2x^2}{3} + \frac{1}{9} - \frac{x}{9})dx = 2\int_0^1(\frac{1}{9} - \frac{7x}{9} + \frac{15x^2}{9} - x^3)dx$$

$$= \left|\frac{2x}{9} - \frac{7x^2}{9} + \frac{10x^3}{9} - \frac{x^4}{2}\right|_0^1$$

$$= \frac{1}{18}$$

Or you can use the simplified formula.

$$Var(X) == \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

$$= \int_0^1 x^2(2-2x\ )dx - \frac{1}{9}$$

$$= 2\int_0^1(x^2-x^3)dx - \frac{1}{9} = 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 - \frac{1}{9} = \frac{2}{3} - \frac{1}{2} - \frac{1}{9} = \frac{1}{18}$$

Question 9. Answer: Standard deviation= 10√3/3≈ 5.77

$$\mu = \int_2^5 x.\frac{1}{3}dx = \int_2^5 \frac{x}{3}dx = \left[\frac{x^2}{6}\right]_2^5 = \frac{25}{6} - \frac{4}{6} = \frac{21}{6}$$

$$Var(X) = \int_2^5 x^2.\frac{1}{3}dx - \mu^2 = \int_2^5 \frac{x^2}{3}dx - \mu^2 = \left[\frac{x^3}{9}\right]_2^5 - \frac{441}{36} = \frac{125}{9} - \frac{8}{9} - \frac{441}{36} = \frac{3}{4}$$

$$\text{Standard deviation} = \sigma = \sqrt{Var(X)} = \frac{\sqrt{3}}{2}$$

Question 10. Answer: i and iii

(i) Tom has a greater range in output from his trees compared to Annie. **Correct**.
(ii) Annie's total production is greater than Tom's. Incorrect. The graph tells you nothing about their total output.
(iii) The difference between the production per tree is less in Annie's orchard than in Tom's orchard. **Correct**. The variance and standard deviation are less for Annie's trees, so each tree produces an output which is closer to the mean.