Sadlier School
PROFESSIONAL DEVELOPMENT SERIES

Research Basis for *Sadlier Math*

Grades K–6

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TABLE OF CONTENTS

INTRODUCTION

The purpose of this document is to summarize our current research-based knowledge of mathematics teaching and learning (e.g., how students learn, best instructional approaches), and contrast it with components of *Sadlier Math* 2019. By doing so, it will highlight the strengths of *Sadlier Math* and help teachers, curriculum coordinators, professional development staff, and parents understand the program better, take advantage of the materials fully, and support student learning. For each section that follows, the summary of current research knowledge of the headlined topic is discussed first, highlighting the elements of effective math programs, leading to specific components of *Sadlier Math*, exemplifying how the elements are supported in the program.

CORE MATHEMATICS TOPICS FOR ELEMENTARY SCHOOLS AND LEARNING PROGRESSIONS

2.1. STANDARDS-BASED INSTRUCTION

Content standards are the roadmap for student learning, outlining how different mathematics topics may be connected and sequenced reasonably to support natural progression of student knowledge development through instruction. While different states and districts may follow different local standards (e.g., Common Core State Standards, 2010), these standards share key characteristics: they present mathematics topics to be emphasized at each grade level, with some information on how to teach the topics and details about student thinking in learning the topics, and also relate grade-level topics so that students can use the developing knowledge of one topic (e.g., decomposition) in learning another topic (e.g., fraction).

The standards also sequence the learning experiences across grade levels so that what they learn in one grade will be reviewed, reinforced, and used to learn a more complex topic in later grades. Beyond making the connections between concepts for their development, the standards also coherently outline how particular topics connect with deeper structures inherent in the disciplines as students learn more in upper grades (Schmidt, Houang, & Cogan, 2002).

2.2. LEARNING PROGRESSIONS

Learning progressions are learning pathways students can take in learning certain mathematics topic in a domain (Battista, 2011; Clements and Sarama, 2004; Murata, et al., 2017; Sztajn, Confrey, Wilson, & Edgington, 2012). All major content standards are structured in alignment with the progressions, thus good standard-based mathematics programs, by nature, guide student learning through the progressions. These progressions are informed by empirical research findings on how students learn mathematics, and they are often outlined as student thinking patterns or solution strategies in the standards. For example, in the case of adding and subtracting numbers with totals within 20 in first grade, CCSS-M (2010) outlines student strategies as follows (1.OA6):

> Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on, making ten (e.g., 8+6=8+2+4=10+4=14), decomposing a number leading to a ten (e.g., 13-4=13-3-1=10-1=9), using the relationship between addition and subtraction (e.g., knowing that 8+4=12, one knows 12-8=4), and creating equivalent but easier or known sums (e.g., adding 6+7 by creating the known equivalent 6+6+2=12+1=13).

These strategies outlined in the standards are derived from previous empirical research, to help teachers understand how students typically think in working with these types of problems. In the example above, these ten-based strategies are later extended and used when students work on multi-digit addition and subtraction, becoming a basis for accurate execution of standard algorithms. In these ways, different student strategies illuminate their positions in the overall learning progression.

Standards-based instruction does not merely mean that teachers teach based on each standard outlined for each grade level discretely. Teachers need to understand how students develop a coherent mathematical knowledge base across grade levels, by making conceptual connections between what they have learned in previous grades and new topics. In other words, teachers help students navigate their paths along the learning progressions when they incorporate standards-based instruction.

2.3. SADLIER MATH AND STANDARDS-BASED INSTRUCTION (AND LEARNING PROGRESSIONS)

As the *Sadlier Math* Scope and Sequence illustrates (see attached), the program is closely aligned with the mathematics domains outlined in the standards. The full coverage of the standards guarantees students will be exposed to key mathematics topics in Sadlier classrooms. It is also important to note that for some key mathematics content topics, *Sadlier Math* is organized so that the topics are addressed at multiple times, either within the same grade level or over two grade levels.

For example, representing and solving addition and subtraction problems, one of the key learning milestones as students develop number sense, is addressed across multiple chapters in grades 1 and 2 (Grade1, Chapters 8 & 9 and Grade 2, Chapters 1 & 2). Using common representations, the chapters show how ten can be composed and decomposed as students devise solutions to problems with totals in the teens. *Sadlier Math* also includes pages to help teachers understand learning progressions. For example, in Grade1 Chapters 8 & 9, strategies typically used by students to add and subtract numbers by making tens are explained, so that teachers can readily identify these strategies as they appear in student work. The focused and effective use of ten-frame representations in these chapters helps students and teachers see tens as numbers that are added and subtracted.

Learning progressions also create opportunities for differentiation as they place each student in possible learning trajectories, allowing them to make their own progress toward mastery. *Sadlier Math* provides teachers with guidelines for differentiation by providing different activities and practice ideas for struggling learners and early finishers for each lesson. Examples of these will be found in Section 4.3.

3.1. SUPPORTING THE DEVELOPMENT OF CONCEPTUAL UNDERSTANDING

The National Research Council prescribes that the learning of mathematics includes the development of five interrelated strands that, together, constitute mathematical proficiency (2001).

These five strands are:

1. Conceptual Understanding
2. Procedural Fluency
3. Strategic Competence
4. Adaptive Reasoning
5. Productive Disposition

Among the five strands, we commonly emphasize the importance of conceptual understanding in the mathematics education field the most, partly because it is the foundation of all mathematics learning and also because it has historically not been taught well in U.S. mathematics classrooms (Hiebert, et al., 1996; Schmidt, 1999). Conceptual understanding means the comprehension and connections of concepts, operations, and relationships, which helps establish the foundation for students' mathematical proficiency. Conceptual understanding is necessary for developing procedural fluency (i.e., the meaningful and flexible use of procedures to solve problems; NCTM, 2014). Thus, it is an essential foundation on which students build mathematical practices and reasoning. Without conceptual understanding, problem solving and execution of procedures can become fragile manipulations of superficial ideas.

In order to develop a deeper level of conceptual understanding, students must be given opportunities to solve authentic mathematics problems. Using the knowledge that they have acquired, they revise and articulate the solution processes to make sense of the problems and evaluate and analyze mathematical ideas and connections among different topics. For example, fourth graders may attempt to solve a problem involving finding the number of buses they need for a field trip for 182 students. They search on the Internet to find out that the maximum capacity of a school bus is 72 students, while it is recommended that it not be fully loaded due to safety concerns. Students may use repeated addition to find out that they need at least three buses (72 + 72 = 144; 144 + 72 = 216). Other students may multiply (72 x 3 = 216) or divide 182 ÷ 72 = 2, R38). While discussing and making sense of the commonalities and differences among these solutions and apparently different answers they produced, students develop firmer connections among operations (in this case, addition, multiplication, and division) and learn how these different solution approaches make sense in the same problem context. They decide to reserve three buses and divide the students into three groups (of 60, 61, and 61), allowing extra space for safety reasons. This way of learning is very different from teachers giving students key ideas to memorize and procedures to follow in problem solving. Students are taking ownership of learning processes, developing their own understanding of how mathematics works in problem contexts, and at the same time, building a solid identity as mathematics learners.

3.2. EFFECTIVE USE OF REPRESENTATIONS

Representations are tools to make concepts and relationships visible in mathematics classrooms. These should be planned and used specifically to support conceptual development. While mathematics education research emphasized the importance of using multiple representations for decades (e.g., Brenner, et al. 1997), recent research cautions us against using too many unrelated representations, as this could potentially lead to unnecessary confusion (e.g., Ainsworth, et al., 2002). Murata (2008) describes how a set of representations (linear quantity models) may be used purposefully, through all elementary grade levels, to teach mathematics while students develop conceptual connections in Japanese classrooms. For each mathematics topic, there are representations that are more or less suited to represent the topic, and we must carefully choose which ones will most effectively help students develop meanings. Also, consistent use of one or several similar representations will implicitly guide students to make visual connections among different content topics. Therefore, we should carefully select and coordinate uses of representations. For example, using tape diagrams to show a problem context can easily highlight the relationship among different quantities used in the problem as well as how different operations may work to express the relationships among the quantities. In section 3.5, we will illustrate an example in *Sadlier Math*.

3.3. OPEN-ENDED PROBLEM SOLVING

Open-ended problems, in contrast to closed-ended problems, are given to students without an indication of the solution methods to be used, and students must rely on their current mathematics knowledge to solve the problems. They often invite multiple entry and exit points, meaning that students of different performance levels could attempt to solve the problems using what they know (differentiation), while there may also be more than one correct answer. Problem solving of this kind is very important as students critically examine the problem context, analyze what is known and unknown in the problem, apply solution strategies they have learned, change the strategies as new challenges arise, and find possible answers. Hiebert and Wearne (1993) found strong positive relationships between instructional tasks and how students learned, explaining that lesson tasks supporting productive student thinking (such as with open-ended problem solving) help students think more. This requires a high level of cognitive engagement and mathematical thinking from students. Stein and colleagues investigated how lesson tasks with a high level of cognitive demand produce increased student learning (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996). When high-level cognitive demand is maintained with lesson tasks, research found students of different performance levels earned higher scores with exams (Zohar & Dori, 2003). Putting it together, research agrees that open-ended problems increase the level of cognitive engagement in students, and that leads to students learning more mathematics.

3.4. MATHEMATICS DISCUSSIONS

When high cognitive engagement is driven by lesson tasks, student understanding of the concepts is further achieved through well-facilitated discussions (Henning, et al., 2012; Murata, et al., 2017; NCTM, 2014). Mathematical practices (this will be discussed in Section 8) are typically embedded in student discussions of mathematics, in which students share ideas, ask questions, receive answers, articulate thinking, understand others' ideas, revise and modify solutions, and construct new understanding together. Student reasoning is the core essence of such discussions. Where classroom culture supports productive learning, which can include making mistakes, taking time to think, helping one another, and sharing the experiences, students are able to share thinking in progress (includes correct and incorrect solution paths). Thus, teachers should be familiar with typical student thinking, the new content they are learning, common mistakes students may make (and why), and effective facilitation practices to help students make conceptual connections. In order to support teachers with this work, instructional programs can provide conceptually sound lesson materials, highlight and explain key content, outline typical student problem solving strategies, and help teachers make sense of the lesson process before, during, and after teaching. In the section that follows, an example from *Sadlier Math* will illustrate the process.

3.5. SADLIER MATH ADDRESSING CONCEPTUAL UNDERSTANDING

At the beginning of each chapter in the Teachers' Edition, *Sadlier Math* explains the importance of key concepts in the chapter and strategies that students may employ in classrooms, and also clearly outlines learning progressions from the previous grade to the current grade to the next grade. This short section is critically important for teachers, as it prepares them to observe student thinking in the lessons in situ and to be ready to help students make conceptual connections. These few pages in the Teachers' Edition are an ideal focal point for lesson study groups as they consider curriculum and research before examining the lesson details.

While every *Sadlier Math* chapter exemplifies a focus on supporting students' conceptual understanding, in this section, I will focus my discussion on the analysis of Grade 2, Chapter 14 (Equal Shares) and Grade 3, Chapter 4 (Multiplication and Division Concepts).

Grade 2, Chapter 14 (Equal Shares):

Students are introduced to the foundational concept of fractions, specifically that each part is equal in size and the equal parts can sometimes be visually different. As suggested by research literature, the chapter focuses its use of representations on dot patterns, mainly using rectangles. This consistent use of the representation threads student conceptual development throughout the chapter as the equal part changes from halves to thirds to fourths. The student understanding is further supported by activities with geoboards in classrooms as well as virtual manipulatives available for students (dot pattern). In Lesson 14-1, students become comfortable with the partitioning of an area, using their developing knowledge of arrays. They then explore how similar representations can be partitioned in halves, thirds, and fourths in Lessons 14-2, 14-3, and 14-4 respectively. In 14-5, students are given problems to critically assess how to partition a rectangle in halves, thirds, and fourths (homework). The worked-out problems, presented in this manner, have been known to positively support student understanding as they critically analyze and evaluate each solution process (Mayer, Sims, & Tajika, 1995). *Sadlier Math* clearly takes advantage of the approach, which can also become the basis for productive student discussions.

Grade 3, Chapter 4 (Multiplication and Division Concepts):

Multiplicative reasoning is one of the key mathematics topics learned in third grade, and students use their developing understanding of number sense, which is largely derived from addition and subtraction operations at this point, and extend it to see how numerical relationships can be maintained but expressed differently when the original unit is more than one. In this chapter, examples are used to guide students through the problem situations, such as in Lessons 4-1 and 4-5, where the problems help connect multiplication with repeated addition, and division with repeated subtraction, respectively, using pictorial representations and numbers/symbols. With a brief bridge with number line representation (to show repeated addition and subtraction in Lessons 4-2 and 4-6), the rest of the chapter uses array representations consistently and meaningfully to show multiplicative relationships of the quantities presented throughout. When pictorial representations are used, they are also arranged in arrays, emphasizing the multiplicative nature of the situations. This deliberate choice of representation use is highly effective as students gradually develop understanding of multiplicative relationships. The arrays may be seen as semi-concrete representation or mathematically accessible representation, which helps ground student understanding as they learn to use abstract representations (e.g., numbers and symbols; Fuson, Murata, & Abrahamson, 2015).

4.1. SUPPORTING THE DEVELOPMENT OF DEEPER PROCEDURAL FLUENCY

Procedural fluency means meaningful and flexible use of procedures to solve problems (NCTM, 2014). It requires thorough knowledge of how procedural steps work, guiding students to reflect, plan, and modify the rules to suit the problem situation. This fluency can be misinterpreted at times when students appear to be able to solve problems using appropriate procedures when the procedures are taught without conceptual connections (NCTM, 2014). When teachers "drill" students by focusing solely on the execution of procedural steps and disregarding concepts, students could develop superficial procedural fluency, which only works for routine problems. International studies repeatedly find U.S. students performing relatively well with routine problems but not deeper conceptual problems (e.g., PISA, TIMSS). This is because typical mathematics instruction in the United States has historically focused on developing procedures without concepts (Hiebert, et al., 1996). Thus, it is critically important that conceptual connections be made as students develop procedural fluency in classrooms. Some U.S. mathematics education researchers emphasize that at a deeper level, both conceptual and procedural knowledge must co-exist as students learn mathematics (Baroody, 2003; Baroody, et al., 2007; Murata, in preparation; Star, 2005). These two types of knowledge are interdependent and cannot exist without each other at a deeper level.

4.2. MULTIPLE TYPES OF PRACTICE

One effective way to support the development of deep procedural fluency is by providing students with multiple opportunities to practice using procedures in different problem contexts. Practice was traditionally associated with the solving of multiple routine problems in workbooks. We now consider practice to be multi-faceted, and students learn best when they engage in different forms of practice, such as playing games, solving authentic problems, discussing ideas in peer groups, etc. For example, Tens Go Fish (Bay-Williams & Kling, 2014) can be an engaging practice for Grade 1 and 2 students to learn to make ten. Using a regular deck of cards numbered 1-10 (put other ones aside), students will take turns asking for a card to make a ten with the one in their decks. To further strengthen conceptual connections, teachers may facilitate student discussions after playing the game, having students explain their strategies for making tens. When using a traditional workbook or worksheet for practice, teachers may pick a few problems from the existing problem set and ask students to explain their thinking in class in order to generate conceptual discussions. As always, the deliberate use of representations (e.g., ten frames, arrays) to show the connections among student strategies in the discussions will be further helpful.

4.3. SADLIER MATH ADDRESSING PROCEDURAL FLUENCY

Sadlier Math uses multiple modes of fluency practices. Student books incorporate multiple practice opportunities, such as "practice," "more practice," and "fluency practice.". Additional fluency practice is included in the program in activity forms, and these activities are outlined only in the Teachers' Edition, while Fluency activities and Subitizing activities are available via their web site. In examining multi-layered practice structure, I will focus my analysis on Grade 2, Chapter 9 (Subtraction Facts Within 20).

Grade 2, Chapter 9 (Subtraction Facts Within 20) focuses on helping students fluently use subtraction strategies by regrouping numbers in tens, which requires an understanding of number relationships and base ten place value. The Teachers' Edition outlines four skill-based learning milestones for students in this chapter: Make 10 to Subtract, Fact Families, True and False Equations, and Missing Part of an Equation. As mentioned, each lesson has practice, more practice, homework (similar in form to practice and more practice), and fluency practice. While students work on these routine practice items in the textbook, teachers facilitate fluency-building activities to help connect conceptual understanding with procedures. For example, in Lesson 9-1, for the "develop concepts" activity, two students identify a number of objects in the bag (11-20), then discuss how many objects they need to remove in order to make a ten. Students then write a subtraction equation based on the activity (fluency connection to practice problems in the book). This is an example of an activity that meaningfully combines fluency practice with conceptual development. Also, in the same lesson, the activities for "Struggling Learners" and "Early Finishers" are designed specifically to help make conceptual connections with procedures. For the Struggling Learners activity, in order to solve 15-8, students are provided with cubes and have opportunities to discuss the process of the operation.

For Early Finishers, they play a game with two spinners, one with numbers in the teens and another with single-digit numbers, and students use the two numbers spun to create a subtraction problem and solve it. In the Sadlier classrooms where these conceptually connecting activities are purposely coupled with textbook practices, students receive support in developing fluency with procedures.

5.1. CHANGING FOCI OF PROBLEM SOLVING IN MATHEMATICS EDUCATION

Problem solving is at the heart of mathematics, though it has historically been viewed differently in mathematics education (Schoenfeld, 1992), and students have suffered from this lack of consistency (Boaler, 2008). As a means of moving away from our traditional approach of having problem solving chapters at the end of math textbooks, math educators suggest well-designed problems embedded throughout the curriculum to invite students to explore their world mathematically, make sense of how it works, and help build new mathematical understanding in the process. Schroeder and Lester (1989) explain that there are three different approaches to teaching problem solving:

1. Teaching for problem solving, in which content is taught first then used in problem solving later.
2. Teaching about problem solving, in which heuristic strategies are taught for solving particular problems.
3. Teaching through problem solving, in which content is taught through non-routine problems.

Considering that students need different experiences in developing fluency with any mathematical process, all three approaches can play roles in curriculum and teaching.

5.2. PROBLEM SOLVING MODELS AND INSTRUCTION

Lesh and colleagues discuss problem solving as a model-eliciting activity (Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh & Zawojewsky, 2008), and argue that, unlike traditional word problems requiring short answers to questions about pre-mathematized situations, students should make symbolic descriptions of situations (models) with problems, which are critical components providing conceptual tools to the process. Mathematization occurs when one considers a real-life situation, generates a mathematical problem from it, and uses mathematics to solve the problem (Freudenthal, 1978, 1991). After reflecting and constructing a model of the situation mathematically, the problem can be solved using mathematical tools and processes (e.g., numbers and formulas).

Polya explained four principles of problem solving in his classic, How to Solve It (1945), as:

1 Understand the problem
2 Devise a plan
3 Carry out the plan
4 Review and extend

These principles were quickly adapted and have been used in many curricula and in classrooms worldwide in the last several decades. As a first step, students examine the problem situation and what the problem is asking. They identify critical information in the problem and additional information they need. As the second step, students make a plan for how to solve the problem at hand, using the knowledge they already have. Next, as the third step, students execute the plan, evaluating each step carefully. In the fourth and final step, students evaluate the answer, determine how reasonable it is, and go back and restart the process all over again if necessary. These four principles map out nicely with the model-eliciting activity Lesh and colleagues discuss in their research and mathematization process, for which creating a model helps students understand the problem and mathematize the situation (Principle 1). Focusing on Principle 1 helps to ground students' mathematical learning in a problem context, preparing them to delve further into the problem solving process.

5.3. SADLIER MATH ADDRESSING PROBLEM SOLVING

Each Sadlier student book starts with the Problem Solving Math Handbook, which explains the four-step problem solving process discussed above, with examples appropriate for each grade level. Used as an introduction to math learning for the school year, the handbook makes it clear that problem solving will be a focus in the mathematics classrooms.

Each chapter in *Sadlier Math* is embedded with multiple layers of problem solving activities with varied levels of openness, to support gradual development of student competence in formulating solution strategies. Each lesson typically starts with a problem of the day, with detailed teacher guidance. There are also a few "problem solving" sections found throughout the lesson that address similar problems for students to solve, along with a "write about it" problem, which are deliberately open-ended and require students to decide on a problem solving process and explain their thinking. Problem solving at the end of each lesson extends what students learned in the chapter. The Problem Solving lesson at the end of the chapter mainly repeats the routine problems introduced in earlier lessons in the chapter, along with more open-ended "write about it" problems and homework problems. For the more open-ended problems, students are asked to explain their thinking using their own words, which will require deeper thinking and reasoning, generating higher cognitive engagement.

The problem of the day (the first problem in each lesson) offers great potential for students to generate strategies and discuss their thinking, following the 4-step problem solving process outlined earlier. The Teachers' Edition explains a few possible student responses, typical student mistakes and reasons for them (error alert), and possible teacher questions for facilitating student discussions, along with possible student answers. Teachers would need targeted guidance to understand how different problem solving components of the program come together to support student learning in *Sadlier Math*, in order to prevent these well-designed problems from producing unfocused and unsystematic exploration of ideas that may not be centrally important for student learning.

6.1. MATHEMATICAL PRACTICE STANDARDS

Mathematical Practice (MP) Standards are developed along with content standards to present how the content should be learned by students. Moving away from routine memorization and superficial learning, these Mathematical Practices explain how students should deeply engage with content and make sense of math concepts and how they connect with one another. In the Common Core State Standards – Mathematics (CCSS-M, 2010), there are eight MPs:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

These practices hone in on varieties of expertise that math educators at all levels seek to develop in students (CCSS-M, 2010). It is also important that these do not develop in isolation but rather interdependently. For example, when students attempt to reason abstractly (#2), they may use mathematical models (#4) and tools strategically (#5), while making use of structure (#7), and persevering in solving the problem (#1). As teachers and students become familiar with these practices, each practice will grow naturally from another. A well-designed curriculum and classroom expectations (which can come from teachers and/or curriculum) are critically important in making this possible.

The MPs often happen when students engage in authentic problem solving and discuss their solution strategies. Thus, problem- and discourse-rich classrooms are essential in supporting the development of student expertise. When cognitively demanding tasks are used in lessons, the MPs will naturally occur in the classroom where students articulate their thinking, modify their solutions as they evaluate their own ideas along with others, use and revise their existing mathematics knowledge, and make connections among different math concepts. This kind of classroom also invites a reasonable level of confusion and anxiety for the students (and teachers) because of the unpredictable nature of the solution processes generated in live classrooms (Smith and Stein, 1996). Research found that teachers tend to lower cognitive demand levels after initial set-up in order to reduce the amount of confusion and discomfort among students (and teachers; Henningsen & Stein, 1007; Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Stein, Grover, & Henningsen, 1996). If the curriculum can outline typical student thinking strategies and expected errors students make and why (typically based on less-structured mathematical thinking students were familiar with prior to learning the new concept), and also organize these in a way that allows teachers to use the information to facilitate student discussions and MPs in lessons, the reduction of the cognitive demand levels can be avoided. Focused PD to guide teachers through typical lesson pathways while outlining and explaining how MPs are supported through authentic problem solving would also be helpful in making deep student learning possible.

6.2. SADLIER MATH ADDRESSING MATHEMATICAL PRACTICE STANDARDS

The Problem Solving Math Handbook at the beginning of each student book not only outlines the problem solving steps but also addresses the Mathematical Practice Standards with relevant examples. While the depth and thoroughness of the MP will depend on how the problem solving process is facilitated by each teacher (Walshaw & Anthony, 2008), the program provides the foundational capacity for students and teachers with MPs. Because the MPs primarily develop through student discussions, it is impossible to fully anticipate the execution of MPs in each classroom. *Sadlier Math* provides multiple supports to prepare teachers to facilitate the MPs.

7.1. DIFFERENT TYPES AND LEVELS OF ASSESSMENTS IN MATHEMATICS CLASSROOMS

The term "assessment" was traditionally associated with tests and exams that came at the end of a chapter or a school year, to evaluate how much students had learned and compare their achievement to one another to assign grades. Mathematics educators at different levels now agree on the importance of formative assessments (as opposed to summative assessments), which can support student learning through the process and inform teachers about how students are learning and how to adjust their instruction to meet their needs (NCTM, 1995, 2014). Formative assessment can help determine the next steps needed for all students to succeed. It usually takes place at some point in a chapter to ascertain what is being learned and to determine whether review is needed or how to take the next steps (Laud, 2011). Many well-designed math programs come embedded with layers of formative assessments (e.g., pre-chapter assessments, exit tickets), but the key to successfully taking advantage of these assessments is for teachers to take time to understand what the assessments are saying, and to use that to inform their teaching. Teachers often do not need to formally analyze the pre-assessment data, but it is helpful to review the data to obtain a rough sense of where students are in terms of their prerequisite knowledge and skills so that some of the concepts can be reviewed or further reinforced as needed. Teachers may also choose a few students (typically of different performance levels) and conduct a brief, informal interview with each of them at the beginning and/or in the middle of the unit to get a sense of their overall learning. This will provide a broader picture of student learning, informing teachers about the need for further differentiation or more-focused whole-class heterogeneous group learning. The formative assessments help inform ongoing instruction, so the teachers can make sure each student is moving along the expected learning progressions.

Involving students in their own assessment process is also a wonderful way of both helping teachers understand student learning processes and providing students with opportunities to reflect on their own learning, take ownership of the process, and develop stronger learner identities (Reeves, 2007). This can be achieved by asking students to talk about their experiences in particular learning activities and how they think they are doing regarding learning specific math topics. Teachers may also provide a graded homework or quiz, and ask students to talk about their own learning challenges (mistakes made) and make plans for the next steps. This kind of assessment not only gives teachers concrete ideas as to where students are but also about students' own perceptions of their learning. For example, the students may identify different learning difficulties that the written quiz may not readily reveal. Putting together, assessment data obtained by teachers and students' own insights can together inform next instructional step meaningfully, not to mention it will also help students take firmer ownership for their learning.

7.2. SADLIER MATH'S ASSESSMENTS

Different types and levels of assessments are embedded in *Sadlier Math*. They are divided into three types: Diagnostic, Formative, and Summative.

1. Each chapter comes with a diagnostic pre-test to examine the level of prerequisite skills, as well as a beginning-of-year test to assess what students know from previous grade levels. The information teachers gain from these diagnostic assessments help create foundations for teachers' understanding of what students would bring into future learning experiences. Teachers may determine what additional reviews are needed to prepare students ready for the new learning, and/or how students do not need any preparation for the new challenges.

2. As formative assessments, there are Exit Ticket exercises focusing on each lesson's objectives, Check Your Progress for learning objectives of a cluster of related lessons, and Chapter Review to gain the sense of how students are learning in the chapter. These formative assessments inform teachers in lessons-in-progress, helping them pace instruction (faster or slower) to meet the specific needs of the students. When the assessments identify particular challenges among many students in the classroom, teachers may decide to provide additional learning experiences in the whole class. The assessments may also inform teachers to provide targeted practices for a small group of students who need additional help for certain aspects of the topics. As with diagnostic assessments above, it is recommended that teachers communicate with students how formative assessments are given to inform teachers, to reduce the level of potential performance anxiety.

3. For summative assessments, the program has chapter tests, cumulative tests (every three chapters), and an end-of-year test. Not only these formative assessments inform teachers about the mastery level of the students (from the chapter, year), they also provide insight for the overall student learning experiences (e.g., how they are making conceptual connections across topics) and the effectiveness of the program. They also have performance assessments to address learning of multiple related chapters, which help students integrate what they have been learning in problem solving. These assessments are available both in print and digital formats, to help teachers and students monitor student learning in flexible ways.

SUMMARY AND CONCLUSIONS

This document summarizes our current research-based knowledge in the field of mathematics education, focusing on key aspects of the teaching and learning of mathematics, and illustrates how these aspects are reflected in *Sadlier Math*. As with most effective programs, *Sadlier Math* combines these key aspects to create coherent learning experiences for elementary school students. Mathematics problems are often designed to support conceptual development while students gain fluency with procedures, and mathematical practices are used effectively as students share, articulate, and evaluate their ideas in problem solving processes. Various assessments help teachers and students monitor student learning along the learning progressions in standards-based instructional trajectories. Overall, *Sadlier Math* effectively supports students' mathematics learning in elementary schools.

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